ON INTEGRAL EQUATIONS INVOLVING WHITTAKER'S FUNCTION

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1. Recently some inversion integrals for integral equations involving Legendre, Chebyshev, Gegenbauer and Laguerre polynomials in the kernel have been obtained [**1, 2, 3, 5, 6**]. In this note, two inversion integrals for integral equations involving Whittaker's function in the kernel are obtained. We shall make use of the following known integral [**4,** p. 402]

$$\int_0^1 x^{\mu-\frac{1}{2}}(1-x)^{\nu-\frac{1}{2}}M_{k,\mu}(xy)M_{\lambda,\nu}\{y(1-x)\}\,dx = B(2\mu+1, 2\nu+1)M_{k+\lambda,\mu+\nu+\frac{1}{2}}(y)$$
$$(\mu > -\tfrac{1}{2},\ \nu > -\tfrac{1}{2}). \qquad (1)$$

The results of this note are based on the following two integrals, which are derived from (1) by writing $u-t=(v-t)x$.

$$\int_t^v (u-t)^{\frac{1}{2}(m-1)-\nu}(v-u)^{\nu-\frac{1}{2}}M_{n+\frac{1}{2}m-\lambda+1}(u-t)M_{\lambda,\nu}(v-u)\,du$$
$$= B(m-2\nu+1, 2\nu+1)(v-t)^{m+1}e^{-\frac{1}{2}(v-t)}(n!)L_n^{m+1}(v-t)$$
$$= B(m-2\nu+1, 2\nu+1)e^{\frac{1}{2}(v+t)}\left\{\frac{d}{dv}\right\}^n\{(v-t)^{n+m+1}e^{-v}\}, \qquad (2)$$

for $m+1 > 2\nu > -1$;

$$\int_t^v (u-t)^{\frac{1}{2}(m-1)-\nu}(v-u)^{\nu-\frac{1}{2}}M_{-\frac{1}{2}m-\lambda-1,\frac{1}{2}m-\nu}(u-t)M_{\lambda,\nu}(v-u)\,du$$
$$= B(m-2\nu+1, 2\nu+1)(v-t)^{m+1}e^{\frac{1}{2}(v-t)} \qquad (3)$$

for $m+1 > 2\nu > -1$.

2. The operator $\mathscr{F}_n$ and its properties. The operator $\mathscr{F}_n$ occurring in this note is defined by the formula

$$\mathscr{F}_n\{F(v)\} = \frac{1}{\Gamma(n)}\int_v^1 (y-v)^{n-1}F(y)\,dy, \qquad (4)$$

where n is a positive integer. Since n is a positive integer, by explicit computation we have

$$\frac{d}{dv}\{\mathscr{F}_n\{F(v)\}\} = -\mathscr{F}_{n-1}\{F(v)\}, \qquad (5)$$

$$\left(\frac{d}{dv}\right)^n\{\mathscr{F}_n\{F(v)\}\} = (-)^n F(v), \qquad (6)$$

and

$$\mathscr{F}_n\{F(v)\} = 0 \quad \text{for} \quad v = 1. \qquad (7)$$

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3. Integral equations and their solutions. Consider the integral equation

$$\int_t^1 (u-t)^{\frac{1}{2}(m-1)-\nu} M_{n+\frac{1}{2}m-\lambda+1,\frac{1}{2}m-\nu}(u-t)y(u)\,du = f(t) \quad (t \in I), \tag{8}$$

where $I = \{t : c \leqq t \leqq 1\}$, c is a positive constant and $f(t)$ is defined on I. The integral is taken in the Riemann sense. It is assumed that (a) $m+1 > 2\nu > -1$, where m is a non-negative integer and n is a positive integer, (b) $f^{(k)}(1) = 0$ for $0 \leqq k \leqq n+m+1$, and (c) $(d/dt)^{n+m+2}\{e^{-\frac{1}{2}t}f(t)\}$ is piecewise continuous on I. If these conditions are satisfied, then the solution of (8) is

$$y(u) = -[B(m-2\nu+1, 2\nu+1)\Gamma(m+n+2)]^{-1}\int_u^1 (v-u)^{\nu-\frac{1}{2}} M_{\lambda,\nu}(v-u)e^{-\frac{1}{2}v}\mathscr{F}_n\{F(v)\}\,dv, \tag{9}$$

where
$$F(v) = e^v\left\{-\frac{d}{dv}\right\}^{n+m+2}\{e^{-\frac{1}{2}v}f(v)\}.$$

Next consider the integral equation

$$\int_t^1 (u-t)^{\frac{1}{2}(m-1)-\nu} M_{-\frac{1}{2}m-\lambda-1,\frac{1}{2}m-\nu}(u-t)z(u)\,du = g(t) \quad (t \in I), \tag{10}$$

where $g(t)$ is defined on I. The integral is taken in the Riemann sense. It is assumed that (a) $m+1 > 2\nu > -1$, where m is non-negative integer, (b) $g^{(k)}(1) = 0$ for $0 \leqq k \leqq m+1$, and (c) $\{d/dv\}^{m+2}\{e^{\frac{1}{2}v}g(v)\}$ is piecewise continuous on I. If these conditions are satisfied, then the solution of (10) is

$$z(u) = -[\Gamma(2\nu+1)\,.\,\Gamma(m-2\nu+1)]^{-1}\int_u^1 (v-u)^{\nu-\frac{1}{2}} M_{\lambda,\nu}(v-u)e^{-\frac{1}{2}v}\left\{-\frac{d}{dv}\right\}^{m+2}\{e^{\frac{1}{2}v}g(v)\}\,dv. \tag{11}$$

4. Proof of the dual relation (8) **and** (9). Substituting the value of $y(u)$ from (9) into the left-hand side of (8) and proceeding exactly in the same way as in [5], after using (2), we obtain the expression

$$J = -\frac{e^{\frac{1}{2}t}}{\Gamma(n+m+2)}\int_t^1 \frac{d^n}{dv^n}\{(v-t)^{n+m+1}e^{-v}\}\mathscr{F}_n\{F(v)\}\,dv.$$

Successive integrations by parts and the application of the operational relations (5), (6) and (7) then yield

$$J = -\frac{e^{\frac{1}{2}t}}{\Gamma(n+m+2)}\int_t^1 (v-t)^{n+m+1}\left\{-\frac{d}{dv}\right\}^{n+m+2}\{e^{-\frac{1}{2}v}f(v)\}\,dv.$$

Further successive integrations by parts and the application of the conditions $f^{(k)}(1) = 0$, $0 \leqq k \leqq n+m+1$ finally yield

$$J = f(t).$$

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5. Proof of the dual relations (10) **and** (11). Substituting the value of $z(u)$ from (11) into the left-hand side of (10) and proceeding as above, after using (3), we obtain the expression

$$J_1 = -\frac{e^{-\frac{1}{2}t}}{\Gamma(m+2)} \int_t^1 (v-t)^{m+1} \left\{-\frac{d}{dt}\right\}^{m+2} \{e^{\frac{1}{2}v} g(v)\}\, dv.$$

Successive integrations by parts and the application of the conditions $g^{(k)}(1) = 0, 0 \leqq k \leqq m+1$, yield

$$J_1 = g(t).$$

REFERENCES

1. R. G. Buschman, An inversion integral for a Legendre transformation, *Amer. Math. Monthly* **69** (1962), 288–289.

2. R. G. Buschman, An inversion integral, *Proc. Amer. Math. Soc.* **13** (1962), 675–677.

3. A. Erdélyi, An integral equation involving Legendre's polynomial, *Amer. Math. Monthly* **70** (1963), 651–652.

4. A. Erdélyi, *Tables of integral transforms*, Vol. II (New York, 1954).

5. Ta Li, A new class of integral transforms, *Proc. Amer. Math. Soc.* **11** (1960), 290–298.

6. D. V. Widder, The inversion of a convolution transform whose kernel is a Laguerre polynomial, *Amer. Math. Monthly* **70** (1963), 291–293.

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