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Parameter spaces for curves on surfaces and enumeration of rational curves

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Abstract. Let S be a smooth, minimal rational surface. The geometry of the Severi variety parametrising irreducible, rational curves in a given linear system on S is studied. The results obtained are applied to enumerative geometry, in combination with ideas from Quantum Cohomology. Formulas enumerating rational curves are found, some of which generalise Kontsevich's formula for plane curves.

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1. Introduction

In this paper we investigate the geometry of families of rational curves on a nonsingular, rational surface S. All varieties are assumed to be projective over $\mathbb{C}$.

Let D be an effective divisor class in S and let $|D|$ be the set of all effective divisors linearly equivalent to D; this is a projective space whose dimension we denote by $r(D)$. Inside $|D|$, we consider the locus of rational curves: we let

$$\tilde{V}(D) = \{[X] \in |D| \text{ such that } X \text{ is an irreducible, rational curve}\}.$$

This is a locally closed subset of $|D|$; we let $V(D) \subset |D|$ be its closure. We call $V(D)$ the *Severi variety* of rational curves associated to the divisor class D, and we denote its dimension by $r_0(D)$. We have in general $r_0(D) \geqslant r(D) - p_a(D)$ with equality holding in all the cases that we shall study.

The particular aspect of the geometry of $V(D)$ of concern to us here is its degree, which we denote by $N(D)$. This can be characterized directly: it is the number of irreducible rational curves that are linearly equivalent to D and that pass through $r_0(D)$ general points of S. The principal results of this paper is the computation of $N(D)$ in some cases. For simplicity, we define $N(D)$ to be zero if $V(D)$ is empty.

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There are various approaches to the calculation of degrees of Severi varieties (see [CH] for a different technique). We call the one we take here the 'cross-ratio' method; it is based on ideas of Kontsevich and Manin, expressed in the 'First Reconstruction Theorem' of [KM]. In [KM] they describe a formula discovered by Kontsevich, for the number of plane rational curves of given degree passing through the appropriate number of points (the first proofs of it appear in in [RT] and in [K] – [KM]). These ideas have also been used to give formulas for the degrees of genus 0 Severi varieties on certain rational surfaces; see [CM], [DI] and [KP].

For an illustration of how the cross-ratio method can be used to give a rather simple proof of Kontsevich's formula see [C] (see also [CH] for an even simpler proof).

Let $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ be a Hirzebruch surface. The Picard group of $\mathbb{F}_n$ has rank 2, and we choose generators as follows

$$\text{Pic}(\mathbb{F}_n) = \mathbb{Z} \cdot C \oplus \mathbb{Z} \cdot F,$$

where $C^2 = n, F^2 = 0$ and $F \cdot C = 1$. We denote by E the unique curve of negative self intersection, so that $E^2 = -n$ and $E \sim C - nF$.

Let D be any divisor class on the surface $S = \mathbb{F}_n$ other than E, and let $\underline{m} := (m_1, m_2, \dots, m_k)$ be any sequence of positive integers with $\sum m_i = (D \cdot E)$. We define the locally closed subvariety $\tilde{V}_{\underline{m}}(D) \subset V(D)$ to be the locus of irreducible rational curves X such that, if $\nu \colon \mathbb{P}^1 \to X$ is the normalization of X, the pullback divisor

$$\nu^*(E) = \sum m_i \cdot q_i,$$

for some collection of distinct points $q_1, \dots, q_k \in \mathbb{P}^1$, and we let $V_{\underline{m}}(D)$ be its closure; for example, as we will see, if $\underline{m} = (1, 1, \dots, 1)$, then $V_{\underline{m}}(D) = V(D)$. When $\underline{m}$ contains a single integer i greater than 1 (i.e. $\underline{m} = (i, 1, 1, \dots, 1)$), we denote these by $\tilde{V}_i(D)$ and $V_i(D)$ respectively. We set

$$r_0^i(D) = \dim(V_i(D))$$

and

$$N_i(D) = \deg V_i(D).$$

We have $V(D)$ for $V_1(D)$, $N(D) = N_1(D)$ and $r_0(D) = r_0^1(D)$. We define $N_i(D)$ to be zero if $V_i(D)$ is empty.

Similarly, let $\Omega = \{p_1, \dots, p_k\} \subset E \subset \mathbb{F}_n$ be any collection of k distinct points. We let $\tilde{W}^{\Omega}_{\underline{m}}(D) \subset V(D)$ be the locus of irreducible rational curves X such that, if $\nu \colon \mathbb{P}^1 \to X$ is the normalization of X, then for some collection of distinct points $q_1, \dots, q_k \in \mathbb{P}^1$ we have

$$\nu(q_i) = p_i$$

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and

$$\nu^*(E) = \sum m_i \cdot q_i$$

and again let $W^{\Omega}_{\underline{m}}(D) \subset V(D)$ be its closure.

Now we give a list of some formulas including all the ones that we prove in this paper. They are very similar from a formal point of view. We state them in a way that highlights the analogies.

Fix two curves C_3 and C_4 on S. For any pair of divisor classes D_1 and D_2 we introduce the function

$$\begin{aligned}\gamma(D_1, D_2) :=& \\ & N(D_1)N(D_2)\left[\binom{r_0(D)-3}{r_0(D_1)-1}(D_1\cdot C_3)(D_2\cdot C_4)\right. \\ & \left. - \binom{r_0(D)-3}{r_0(D_1)-2}(D_2\cdot C_3)(D_2\cdot C_4)\right].\end{aligned}$$

Using this notation, we state the following results

Recursion for $\mathbb{P}^2$ ([KM]) *Let* C_3 *and* C_4 *be two fixed lines in the plane, then*

$$N(D) = \sum_{D_1+D_2=D} \gamma(D_1, D_2)(D_1 \cdot D_2).$$

Recursion for $\mathbb{P}^1 \times \mathbb{P}^1$ ([KM], [DI], [KP]) *Let* C_3 *and* C_4 *be two fixed elements of the two distinct rulings, then*

$$N(D) = \sum_{D_1+D_2=D} \gamma(D_1, D_2)(D_1 \cdot D_2).$$

The first new result of this paper is a recursion formula for the degrees of Severi varieties of rational curves on $\mathbb{F}_2$. The recursion contains now a new term which is due to the contribution of degenerate curves containing E.

Recursion for $\mathbb{F}_2$ (Theorem 3.2) *Let* C_3 *and* C_4 *be two fixed elements of the class* C*, then*

$$\begin{aligned}2N(D) =& \sum_{D_1+D_2=D} \gamma(D_1, D_2)(D_1 \cdot D_2) \\ & + 2 \sum_{D_1+D_2=D-E} \gamma(D_1, D_2)(D_1 \cdot E)(D_2 \cdot E).\end{aligned}$$

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Finally, on $\mathbb{F}_n$, the general reducible curves $X = \cup X_i \in |D|$ that are limits of irreducible rational curves and contain E have the property that each component X_j may have a point of tangency of order i_j with E – that is, will belong to $V_{i_j}(D_j)$, where D_j is the divisor class of X_j. Accordingly, we shall define later (Section 3.4) a generalized version of the number $\gamma(D_1, D_2)$; this will be a function $\gamma_{i_1,i_2,\ldots,i_t}(D_1, D_2, \ldots, D_t)$ depending recursively on the degrees $N_{i_j}(D_j)$. In these terms, we give a formula expressing the degree $N(D)$ of $V(D)$ on $\mathbb{F}_n$ in terms of the degrees of the tangential Severi varieties of smaller divisor classes.

A sample formula for $\mathbb{F}_n$ (Theorem 3.4)

$$\begin{aligned} nN(D) = & \sum_{D_1+D_2=D} (D_1 \cdot D_2)\gamma_{1,1}(D_1, D_2) \\ & + \sum_{t=2}^{n} \sum_{D_1+D_2+\cdots+D_t=D-E} \sum_{i_1,\ldots,i_t} \\ & \times \prod_{j:i_j=1} (E \cdot D_j)\gamma_{i_1,\ldots,i_t}(D_1, \ldots, D_t). \end{aligned}$$

The difference here is that in case $n \geqslant 3$ this does not give a complete recursion: to be able to enumerate rational curves on such surfaces, we would need formulas for the degrees of the 'tangential' Severi varieties as well, that is, we need formulas for $N_i(D)$. The first case for which this occurs is that of $\mathbb{F}_3$. Very possibly a complete recursion could still be obtained using the cross-ratio method, although the level of difficulty seems to us to get very high. Instead we found a different technique that we successfully applied in a few cases; for example, we obtained a complete set of recursions for the surface $\mathbb{F}_3$. This different method is the subject of another paper of ours (cf. [CH]); it also is heavily based on the deformation theory results that are developed in the second chapter of this paper.

Finally, we obtain a closed formula for the class $2C$ on any ruled surface $\mathbb{F}_n$.

Closed formula for 2C **on** $\mathbb{F}_n$ (Theorem 3.3)

$$N(2C) = \sum_{k=0}^{n-1} (n-k)^2 \binom{2n+2}{k}.$$

2. Degenerations of rational curves

2.1. The basic set-up

We start with the complete linear system $|D|$ associated to a divisor class D on the ruled surface $S = \mathbb{F}_n$, and with the Severi variety $V(D) \subset |D|$. We then choose $r_0(D) - 1$ general points $q_1, \ldots, q_{r_o(D)-1} \in S$, and let Γ be the intersection of $V(D)$ with the linear subspace of curves in $|D|$ passing through $q_1, \ldots, q_{r_o(D)-1}$;

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we let $\mathcal{X} \subset \Gamma \times S$ be the corresponding family of curves over Γ.

Next, we let $\Gamma^\nu \to \Gamma$ be the normalization of the base Γ, and $\mathcal{X}^\nu = (\mathcal{X} \times_\Gamma \Gamma^\nu)^\nu \longrightarrow \Gamma^\nu$ the normalization of the pullback of the family to Γ^ν, so that $\mathcal{X}^\nu \to \Gamma^\nu$ is a family whose general fiber is a smooth rational curve. If X is a fiber of $\mathcal{X} \to \Gamma$, the notation X^ν will be used for a corresponding fiber of the family $\mathcal{X}^\nu \to \Gamma^\nu$, which may differ from the normalization of X.

Then we fix two curves C_3 and C_4 in $\mathbb{F}_n$, which will be linearly equivalent to C. We need to make a further base change $B \to \Gamma^\nu$, so that the points of intersection of the curves in our family with C_3 and C_4 become rational over the base. We thus let $B \to \Gamma^\nu$ be any finite cover, unramified at the points $b \in \Gamma^\nu$ with X_b^ν singular, and let $\mathcal{X}' \to B$ be the pullback of the family $\mathcal{X}^\nu \to \Gamma^\nu$ to B. (By Propositions 2.1 and 2.5, the map $B \to \Gamma^\nu$ introduced in Chapters 1 and 3 in order to define the sections p_i will indeed be unramified at the points of B corresponding to the singular fibers of $\mathcal{X}^\nu \to B$.) Because the results of this chapter are all local in the base of our family, however, we will not need to introduce this extra step in the construction. For the remainder of this chapter, accordingly, we will take $B = \Gamma^\nu$; and all of the results of the chapter describing the map $\mathcal{X}^\nu \to \Gamma^\nu$ will still hold after the base change $B \to \Gamma^\nu$.

Next we introduce the *nodal reduction* of the family $\mathcal{X}' \to B$. That is to say, after making a base change $\tilde{B} \to B$ and blowing up the pullback family $\mathcal{X}' \times_B \tilde{B} \to \tilde{B}$, we arrive at a family $\mathcal{Y} \to \tilde{B}$ such that

(1) $\mathcal{Y} \to \tilde{B}$ is a family all of whose fibers of $\mathcal{Y} \to \tilde{B}$ are reduced curves having only nodes as singularities;
(2) the total space $\mathcal{Y}$ is smooth;
(3) $\mathcal{Y}$ admits a regular birational map $\mathcal{Y} \to \mathcal{X}' \times_B \tilde{B}$ over $\tilde{B}$.

In fact, most of our concerns with this definition will turn out in the end to be unnecessary: we will see below as a corollary of Propositions 2.6 and 2.7 that in fact $\mathcal{X}' \to B$ is already a family of nodal curves. Thus, in practice, we will not have to make a base change at all at this stage, and $\mathcal{Y}$ will be simply the minimal desingularization of $\mathcal{X}'$. For this reason (and because B is itself already an arbitrary finite cover of the normalization Γ^ν of our original base Γ) we will abuse notation slightly and omit the tilde in $\tilde{B}$, that is, we will speak of the family $\mathcal{Y} \to B$.

One further remark: in the applications we will have four sections of the family $\mathcal{Y} \to B$ and will correspondingly want to consider this as a family of four-pointed nodal curves. For this reason, we may want to make further blow-ups at points where these sections cross. By Propositions 2.1 and 2.5, however, the sections in question will cross only at smooth fibers of $\mathcal{Y} \to B$ and so this will not affect our descriptions of the singular fibers of the family.

The final construction is one that we will use only in the following chapter, but we mention it here just to have all the definitions in one place. After arriving as above at a family $\mathcal{Y} \to B$ of nodal curves with four disjoint sections p_i, we may then proceed to blow down 'extraneous' components of fibers Y of $\mathcal{Y} \to B$:

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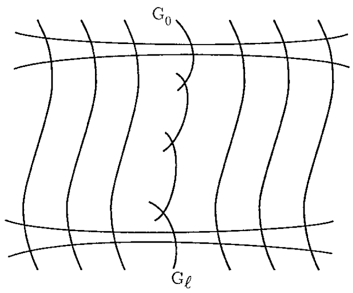
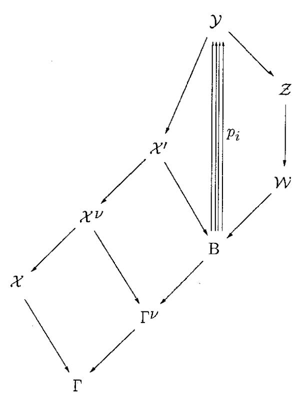
that is, any component of Y that meets the other components of Y in only one point, and that meets at most one of the sections p_i. Iterating this process until there are no extraneous components left, we arrive at what we will call the *minimal smooth semistable model* of our family: that is, a family $\mathcal{Z} \to B$ such that $\mathcal{Z}$ is smooth, the fibers are nodal, the sections p_i are disjoint and $\mathcal{Z} \to B$ is minimal with respect to these properties. Note that the special fiber Z of $\mathcal{Z}$ must be a chain of rational curves $G_0, \ldots, G_\ell$ with two of the sections meeting each of the two end components



(the case $\ell = 0$ is simply the case where Z is irreducible). Finally, we can blow down the intermediate components $G_1, \ldots, G_{\ell-1}$ in this chain to arrive at a family $\mathcal{W} \to B$ of 4-pointed stable curves, called the *stable model* of our family. The special fiber of this family will have just two components (or one, if $\ell = 0$), with a singularity of type A_ℓ at the point of their intersection.

In sum, we have the diagram of families and maps.



2.2. The main results from deformation theory

We give here a summary of the main results to be proved in this chapter.

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- The first is Proposition 2.1 in which we consider the Severi varieties $V(D)$ and $V_{\underline{m}}(D)$, compute their dimension and describe the geometry of their general point. In particular, we characterize the general fiber of the family $\mathcal{X} \to \Gamma$. The results are unsurprising: for example, the general point $[X]$ of $V(D)$ corresponds to a curve X with only nodes as singularities; general points $[X]$, $[X']$ of, respectively, $V(D)$ and $V(D')$ correspond to curves X, X' that intersect transversely.
- Then, in Proposition 2.5, we study the geometry of the general point of the boundary of $V(D)$. We do that by listing all types of reducible fibers that occur in the family $\mathcal{X} \to \Gamma$. This result is not predictable on the basis of a simple dimension count; in most linear systems $|D|$ on $\mathbb{F}_n$ the subvariety corresponding to reducible rational curves containing E is larger-dimensional than $V(D)$; so the question of which points of the former lie in the closure of the latter does not have an immediate answer.
- The third result is Proposition 2.6, which is specifically about the family $\mathcal{X} \to \Gamma$. We describe the geometry of the base Γ in a neighborhood of each point $[X] \in \Gamma$ corresponding to a degenerate fiber X. In particular, we say how many branches Γ has at $[X]$ and say how the nodes of the nearby irreducible fibers approach the singularities of X as we approach $[X]$ along each branch of Γ.
- Finally we have Proposition 2.7, describing the singularities of the total space of the families $\mathcal{X} \to \Gamma$ and $\mathcal{X}^\nu \to \Gamma^\nu$. This will be a crucial ingredient in calculating the multiplicities of zeroes of the cross-ratio function on the base of our family.

One word of warning is in order. Many of both the statements and proofs of these propositions are just routine verifications of statements easily guessed on the basis of naive dimension counts. At the same time, mixed in with these largely predictable statements are some interesting phenomena. These are described in the second parts of Propositions 2.5, 2.6 and 2.7, in which we describe the geometry of the one-parameter families $X \to \Gamma$ and $\mathcal{X}^\nu \to \Gamma^\nu$ in a neighborhood of the reducible fibers containing E. Near such a curve, the local geometry of the universal family over the Severi variety is, to us, somewhat surprising.

2.3. Geometry of Severi varieties

Here is the first result about the varieties $V_{\underline{m}}(D)$ defined in the introduction.

PROPOSITION 2.1. *Let $|D|$ and $|D'| \neq |E|$ be any linear series on the surface $S = \mathbb{F}_n$; let $G \subset S$ be any fixed curve not containing E and let $P_1, P_2, \ldots \in S$ be any given finite collection of points. Let $\underline{m} = (m_1, m_2, \ldots)$ be any collection of positive integers with $\sum m_i = (D \cdot E)$.*

(1) *If $V_{\underline{m}}(D)$ is nonempty, then it has pure dimension*

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$$\dim(V_{\underline{m}}(D)) = -(K_S \cdot D) - 1 - \sum(m_i - 1).$$

(2) *A general point* $[X]$ *of any component of* $V_{\underline{m}}(D)$ *corresponds to a curve* $X \subset S$ *having only nodes as singularities, smooth everywhere along* E, *intersecting* G *transversely and not containing* P_i *for any* i.

(3) *If* $[X]$ *and* $[X']$ *are general points of irreducible components of* $V_{\underline{m}}(D)$ *and* $V_{\underline{m}'}(D')$ *respectively, then* X *and* X' *intersect transversely, and none of their points of intersection lie on* G *or* E.

REMARK. Many of the techniques necessary to prove this statement are in [H]. In fact, many of these assertions are proved there, but unfortunately with slightly different hypotheses: they are proved first on a general rational surface S, but only for $V(D)$, that is, without the tangency condition (Proposition (2.1) of [H]); and then with a single tangency condition, but only with respect to a line in the plane (Lemma (2.4) of [H]).

Proof. We start with the dimension statement. The assertion that the dimension of $V(D)$ is everywhere equal to $-(K_S \cdot D) - 1$ is standard deformation theory (and is well known; c.f. [K]). To see it, observe first that if $[X] \in \tilde{V}(D)$ is any point and $\nu: X^\nu \to X \subset S$ the normalization of the corresponding curve, the first-order deformations of the map ν are given by sections of the pullback $\nu^*(T_S)$ of the tangent bundle to S. Now, the tangent bundle to the ruled surface $S = \mathbb{F}_n$ is generated by its global sections everywhere except along E; since X doesn't contain E, it will likewise be true that the pullback $\nu^*(T_S)$ will be generically generated by its global sections. Since $X^\nu \cong \mathbb{P}^1$, it follows in turn that $h^1(X^\nu, \nu^*(T_S)) = 0$. The deformations of the map ν are thus unobstructed, from which it follows that the space of such deformations is smooth of dimension

$$\begin{aligned} h^0(X^\nu, \nu^*(T_S)) &= \deg(\nu^*(T_S)) + 2 \\ &= -(K_S \cdot D) + 2. \end{aligned}$$

If we mod out by automorphisms of the domain $\mathbb{P}^1$, we see that the space of deformations of the image curve $X \subset S$ as a rational curve has dimension

$$h^0(\mathbb{P}^1, \nu^*(T_S)) - 3 = -(K_S \cdot D) - 1,$$

which is the same as the dimension of $T_{[X]}V(D)$.

We next establish the

CLAIM. *The dimension of* $\tilde{V}_{\underline{m}}(D)$, *and hence of* $V_{\underline{m}}(D)$, *is everywhere at least* $r_0(D) - \sum(m_i - 1)$.

To see this, set $l = (D \cdot E)$. Let $[X] \in \tilde{V}(D)$ be any point, U an analytic neighborhood of $[X]$ in $\tilde{V}(D)$, $\mathcal{X} \subset U \times S \to U$ the universal family of curves over U, and $\mathcal{X}^\nu$ and U^ν the normalizations of $\mathcal{X}$ and U; we may assume that the map $\tau: \mathcal{X}^\nu \to U^\nu$ is smooth. Now let $\mathcal{X}^\nu_l$ be the lth symmetric fiber product of $\mathcal{X}^\nu \to U^\nu$. We then have a map

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$$\rho\colon U \to \mathcal{X}_l^{\nu} \qquad [X] \mapsto \psi_{[X]}^* \nu_{[X]}^*(E).$$

Now, inside the symmetric product $\mathcal{X}_l^{\nu}$, the locus $\Gamma_{\underline{m}}$ of divisors having points of multiplicities m_i or more is irreducible of codimension $\sum(m_i - 1)$; since $\tilde{V}_{\underline{m}}(D) \cap U$ is an open subset of the inverse image $\rho^{-1}(\Gamma_{\underline{m}})$, it follows that it must have dimension at least $\dim(V(D)) - \sum(m_i - 1)$ everywhere.

Note that an analytic neighborhood U of any point of $\tilde{V}_{\underline{m}}(D)$ admits a map to E^k, sending $[X] \in U$ to the images $q_i = \nu(p_i)$; the fibers of this map are analytic open sets in the varieties $W_{\underline{m}}^{\Omega}(D)$. In particular, we have

$$\dim(V_{\underline{m}}(D)) \leqslant \dim(W_{\underline{m}}^{\Omega}(D)) + k,$$

so that in order to prove the opposite inequality $\dim(V_{\underline{m}}(D)) \leqslant r_0(D) - \sum(m_i - 1)$, it is enough to show that the dimension of the variety $W_{\underline{m}}^{\Omega}(D)$ is equal to $r_0(D) - \sum m_i$ for any subset $\Omega = \{p_1, \ldots, p_k\} \subset E$.

To prove the remaining parts of the Proposition we first identify the projective tangent space to the space of deformations of a given reduced curve X preserving the geometric genus of X; and then the subspaces corresponding to deformations that also preserve singularities other than nodes and/or tangencies with fixed curves. This is the part that is in common with [H], and for the most part we simply recall here the statements of the relevant results (Theorem 2.2 and Lemma 2.3). Then, to apply these, we need to estimate the dimension of these subspaces of $|D|$; this is carried out in Lemma 2.4 and the following argument.

We identify the tangent space to the linear series $|D|$ at $[X]$ with the *characteristic series*

$$H^0(X, \mathcal{O}_X(X)) = \frac{H^0(S, \mathcal{O}_S(X))}{\mathbb{C}\tau},$$

where $\tau \in H^0(S, \mathcal{O}_S(X))$ is the section vanishing along X (this identification is natural up to scalars; more precisely, the tangent space to $\mathbb{P}(H^0(S, \mathcal{O}_S(X)))$ at $[X] = \mathbb{C}\tau$ is

$$\mathrm{Hom}\left(\mathbb{C}\tau, \frac{H^0(S, \mathcal{O}_S(X))}{\mathbb{C}\tau}\right) = \frac{(\mathbb{C}\tau)^* \otimes H^0(S, \mathcal{O}_S(X))}{\mathbb{C}\tau}.$$

Now suppose that we are given any subvariety W of the linear series $|D|$ on S. Let $[X] \in W$ be a general point of W. The following theorem of Zariski ([Z], Theorems 1 and 2) characterizes the tangent space to W at $[X]$.

THEOREM 2.2. (**Zariski's theorem**) *In terms of the identification of the tangent space to the linear series $|\mathcal{O}_S(D)|$ at $[X]$ with the characteristic series $H^0(X, \mathcal{O}_X(X))$,*

(1) The tangent space $T_{[X]}W$ is contained in the subspace $H^0(X, \mathcal{I}(X))$ of $H^0(X, \mathcal{O}_X(X))$, where $\mathcal{I} \subset \mathcal{O}_S$ is the *adjoint ideal* of X;

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(2) If X has any singularities other than nodes, then $T_{[X]}W$ is contained in a subspace $H^0(X, \mathcal{J}(X))$ where $\mathcal{J} \subsetneq \mathcal{I}$ is an ideal strictly contained in the adjoint ideal.

This characterizes the tangent space to $V(D)$ at a general point $[X]$. (If the fact that it does is not clear, it will be after Lemma 2.4 below.) Now, we have to consider the additional information coming from the tangency with E. To express this, note first that, if $\nu: X^\nu \to X$ is the normalization of X and $\mathcal{J} \subset \mathcal{O}_X$ is any ideal contained in the adjoint ideal of X, then the pullback map gives a natural bijection between ideals $\mathcal{J} \subset \mathcal{I} \subset \mathcal{O}_X$ contained in $\mathcal{I}$ and ideals $\nu^*\mathcal{J} \subset \nu^*\mathcal{I} \subset \mathcal{O}_{X^\nu}$. We will invoke this correspondence implicitly in our notation: if $p \in X^\nu$ is any point, and $\mathcal{J} \subset \mathcal{O}_X$ any ideal contained in the adjoint ideal of X, we will write $\mathcal{J}(-mp) \subset \mathcal{O}_X$ to mean the ideal in $\mathcal{O}_X$ whose pullback to X^ν is $\nu^*\mathcal{J} \otimes \mathcal{O}_{X^\nu}(-mp)$. In these terms, we have the following.

LEMMA 2.3. *Let $G \subset S$ be any fixed curve and $p \in G$ a smooth point of G. Let W be any subvariety of $|D|$. If the general point $[X]$ of W satisfies the condition: there is a point $q \in X^\nu$ such that $\nu(q) = p$ and*

$$\mathrm{mult}_q(\nu^*(G)) = m,$$

then the tangent space to W at $[X]$ satisfies

$$T_{[X]}W \subset H^0(X, \mathcal{I}(X)(-mp)).$$

Moreover, if X has any singularities other than nodes, or is singular at the point p, we have

$$T_{[X]}W \subset H^0(X, \mathcal{J}(X)(-mp)),$$

where $\mathcal{J} \subsetneq \mathcal{I}$ is an ideal strictly contained in the adjoint ideal.

Proof. We will prove the Lemma by applying Zariski's theorem to the proper transform of X on the surface $\tilde{S}$ obtained by blowing up $S = \mathbb{F}_n$ a total of m times along the curve E. To carry this out, let $S_1 \to S_0$ be the blow-up of $S_0 = S$ at the point p, $E_1 \subset S_1$ the exceptional divisor of the blow-up and $p_1 \in E_1$ the point of intersection of E_1 with the proper transform of E in S_1. Similarly, let $S_2 \to S_1$ be the blow-up of S_1 at the point p_1, $E_2 \subset S_2$ the exceptional divisor of the blow-up and $p_2 \in E_2$ the point of intersection of E_2 with the proper transform of E in S_1, and so on, until we arrive at the surface $\tilde{S} = S_m$; we will denote by $\pi: \tilde{S} \to S$ the composite of the blow-up maps, by $\tilde{X}$ the proper transform of X in $\tilde{S}$ and by $\tilde{E}_i$ the proper transform of E_i in $\tilde{S}$; so that the pullback to $\tilde{S}$ of the divisor E is given by

$$\pi^*E = \tilde{E} + \sum i \cdot \tilde{E}_i.$$

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We denote by X' the branch of X corresponding to the point $q \in X^\nu$, that is, the image of an analytic neighborhood of q in X^ν, by $\tilde{X}'$ its proper transform in $\tilde{S}$, and by $\tilde{p}$ the point of $\tilde{X}'$ lying over p.

Now, let X_i be the proper transform of X in S_i, and let k_i be the multiplicity of X_{i-1} at the point p_{i-1}; for each $j = 1, \ldots, m$ we will set

$$l_j = k_1 + k_2 + \cdots + k_j.$$

Thus, for example, we have the equality of divisors

$$\pi^* X = \tilde{X} + \sum_{i=1}^{m} l_i \cdot \tilde{E}_i.$$

Similarly, we let X'_i be the proper transform on X' in S_i, k'_i the multiplicity of X'_{i-1} at p_{i-1} and $l'_j = k'_1 + \cdots k'_j$. Note that $l_j \geqslant l'_j$ for each j; and the requirement that X' have intersection multiplicity m with E at p is equivalent to the assertion that

$$\mathrm{mult}_p(X' \cdot E) = (\pi^* X' \cdot \tilde{E}) = l'_m = m,$$

so that we have in particular $l_m \geqslant m$, with equality if and only if (locally) $X = X'$. We can also write the intersection number $m_p(X' \cdot E)$ as

$$\mathrm{mult}_p(X' \cdot E) = \mathrm{mult}_q(\tilde{X}' \cdot \pi^* E) = m_q(\tilde{X}' \cdot (\tilde{E} + \sum j \cdot \tilde{E}_j)),$$

so we see that one of three things occurs: either

- X' is smooth, $k_i = 1$ for all i, and $\tilde{X}'$ meets the last exceptional divisor E_m transversely; or
- $\tilde{X}'$ passes through the point $\tilde{E}_i \cap \tilde{E}_{i-1}$ for some $i < m$; or
- for some $j < m$, $\tilde{X}'$ meets the exceptional divisor $\tilde{E}_j$ at a point other than $\tilde{E}_j \cap \tilde{E}_{j-1}$ or $\tilde{E}_j \cap \tilde{E}_{j+1}$, and has a point of intersection multiplicity $m/j > 1$ with $\tilde{E}_j$.

We now compare the adjoint ideal $\mathcal{I}_X$ of X with that of $\tilde{X}$. The basic fact here is that if $C \subset S$ is any curve on a smooth surface, $p \in C$ a point of multiplicity m, and $\tilde{C} \subset \tilde{S}$ the proper transform of C in the blow-up $\pi\colon \tilde{S} \to S$ of S at p, the adjoint ideals of C and $\tilde{C}$ are related by the formula

$$\pi^* \mathcal{I}_C = \mathcal{I}_{\tilde{C}}(-(m-1)E),$$

where E is the exceptional divisor. Applying this m times to the curve X, we have

$$\pi^* \mathcal{I}_X = \mathcal{I}_{\tilde{X}} \left(-\sum (l_j - j) \tilde{E}_j \right).$$

Now, $[X] \in W$ being general, any deformation of X coming from the family W preserves the multiplicities k_i, and hence the decomposition $\pi^* X = \tilde{X} + \sum l_i \tilde{E}_i$. It also preserves the geometric genus of $\tilde{X}$, so that identifying the space

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$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\tilde{X}))$ of deformations of $\tilde{X} \subset \tilde{S}$ with a subspace of the deformations $H^0(X, \mathcal{O}_X(X))$ of $X \subset S$ via the pullback map, we have

$$\begin{aligned} T_{[X]}W &\subset H^0(\tilde{X}, \mathcal{I}_{\tilde{X}}(\tilde{X})) \\ &= H^0\left(\tilde{X}, (\pi^*\mathcal{I}_{\tilde{X}})\left(\sum (l_j - j)\tilde{E}_j\right)\left(\pi^*X - \sum l_j \tilde{E}_j\right)\right) \\ &= H^0\left(\tilde{X}, (\pi^*\mathcal{I}_X)\left(\pi^*X - \sum j\tilde{E}_j\right)\right) \\ &= H^0(\tilde{X}, \pi^*(\mathcal{I}_X(X))(-l_m q)) \\ &= H^0(X, \mathcal{I}_X(-l_m p)) \\ &\subset H^0(X, \mathcal{I}_X(-mp)). \end{aligned}$$

Note that the inclusion in the last line of the above sequence is proper if $X \neq X'$. Now, suppose that $X = X'$ is not smooth at p. In this case, as we noted $\tilde{X}'$ will either be singular at $\tilde{p}$ or be tangent to $\tilde{E}_i$ there, or else will pass through the point $\tilde{E}_i \cap \tilde{E}_{i-1}$ for some i. In the first case, since $\tilde{X}$ has a unibranch singularity, its deformations correspond to sections of $H^0(\tilde{X}, \mathcal{K}(\tilde{X}))$ for some ideal $\mathcal{K}$ strictly contained in the adjoint ideal $\mathcal{I}_{\tilde{X}}$; while in the latter two cases the deformations correspond to sections of $H^0(\tilde{X}, \mathcal{I}_{\tilde{X}}(\tilde{X}))$ vanishing at q. In either case, the inclusion in the first line of the equation above is strict. Thus $T_{[X]}W \subset H^0(X, \mathcal{I}_X(-(m+1)q))$ unless X is smooth at p, and the remainder of the statement of the Lemma follows. □

To conclude the proof of Proposition 2.1 we need one more fact. To state it, let $X \in |D|$ be any irreducible rational curve, $\nu\colon X^\nu \to X$ the normalization and $p_1, p_2, \ldots \in X^\nu$ any points; suppose that the divisor $\nu^*(E)$ has multiplicity m_i at p_i. Let $\mathcal{I} \subset \mathcal{O}_S$ be the adjoint ideal of X, and set

$$\mathcal{K} = \mathcal{I}\left(-\sum m_i p_i\right) \subset \mathcal{O}_X.$$

Let $\mathcal{K}'$ be any ideal of index 2 or less in $\mathcal{K}$ – that is, any ideal $\mathcal{K}' \subset \mathcal{K}$ with $h^0(\mathcal{K}/\mathcal{K}') \leqslant 2$, or equivalently an ideal of the form

$$\mathcal{K}' = \mathcal{K}(-q - r),$$

for some pair of points $q, r \in X^\nu$. We will need these ideals $\mathcal{K}' \subset \mathcal{K}$ of index 2 in order to see, for example, that a general curve $X \in V(D)$ does not have a node on E. In these terms, our result is the

LEMMA 2.4. *The ideal $\mathcal{K}'$ imposes independent conditions on the linear series $|\mathcal{O}_X(X)|$, i.e.,*

$$h^0(X, \mathcal{K}'(X)) = h^0(X, \mathcal{O}_X(X)) - \dim_{\mathbb{C}}(\mathcal{O}_X/\mathcal{K}').$$

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In particular, $\mathcal{K}$ imposes independent conditions on $|\mathcal{O}_X(X)|$, that is,

$$h^0(X, \mathcal{K}(X)) = r_0(D) - \sum m_i.$$

Proof. By the adjunction formula we have

$$K_{X^\nu} = \nu^*(K_S \otimes \mathcal{O}_S(X) \otimes \mathcal{I}).$$

Thus,

$$\nu^*(\mathcal{O}_S(X) \otimes \mathcal{K}) = K_{X^\nu} \otimes \nu^*(\mathcal{O}_S(-K_S)) \otimes \mathcal{O}_{X^\nu}\left(-\sum m_i p_i\right).$$

Now, $\nu^* E - \sum m_i p_i \geqslant 0$, and on $S = \mathbb{F}_n$, we have

$$K_S = \mathcal{O}_S(-C - E - 2F),$$

so that we have an inequality of divisor classes

$$\nu^*(\mathcal{O}_S(X) \otimes \mathcal{K}) \geqslant K_{X^\nu} \otimes \nu^* \mathcal{O}_S(C + 2F).$$

Moreover, the divisor class $C + 2F$ has intersection number at least 3 with any irreducible curve X not linearly equivalent to either F or E, so it follows that

$$\deg(\nu^*(\mathcal{O}_S(X) \otimes \mathcal{K})) \geqslant -2 + 3 = 1.$$

Thus

$$\deg(\nu^*(\mathcal{O}_S(X) \otimes \mathcal{K}')) \geqslant -1,$$

so that $h^1(X^\nu, \nu^*(\mathcal{O}_S(X) \otimes \mathcal{K}')) = 0$, and the result follows. □

We can now complete the proof of Proposition 2.1. We have already established, in the Claim above, that

$$\dim(\tilde{V}_{\underline{m}}(D)) \geqslant r_0(D) - \sum (m_i - 1);$$

but applying Lemmas 2.3 and 2.4 in turn we see that for any subset $\Omega = \{p_1, \ldots, p_k\} \subset E$,

$$\begin{aligned} \dim(\tilde{W}^{\Omega}_{\underline{m}}(D)) &\leqslant h^0(X, \mathcal{K}(X)) \\ &= r_0(D) - \sum m_i \end{aligned}$$

and hence

$$\begin{aligned} \dim(\tilde{V}_{\underline{m}}(D)) &\leqslant \dim(\tilde{W}^{\Omega}_{\underline{m}}(D)) + k \\ &= r_0(D) - \sum (m_i - 1), \end{aligned}$$

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so that equality must hold. Moreover, if a general point $[X] \in V_{\underline{m}}(D)$ corresponded to a curve X with singularities other than nodes, the second inequality above would be strict; so X must be nodal, and smooth at its points of intersection with E.

We can eliminate all the other possible misbehaviors of our general curve X similarly. If the point $p \in X^\nu$ is mapped to one of the points P_i, we would have

$$\begin{aligned} \dim(\tilde{V}_{\underline{m}}(D)) &\leqslant h^0(X, \mathcal{K}(X)(-p)) \\ &< h^0(X, \mathcal{K}(X)); \end{aligned}$$

and if the multiplicity of the pullback divisor $\nu^*(G)$ at p were $m > 1$ we would have

$$\begin{aligned} \dim(\tilde{V}_{\underline{m}}(D)) &\leqslant h^0(X, \mathcal{K}(X)(-(m-1)p)) \\ &< h^0(X, \mathcal{K}(X)). \end{aligned}$$

Suppose next that X had a node on E, with branches corresponding to a pair of points $q, r \in X^\nu$ and the branch corresponding to r transverse to E. It would follow that

$$h^0(X, \mathcal{K}(X)(-q-r)) = h^0(X, \mathcal{K}(X)) - 1,$$

since a section of $\mathcal{K}(X)$ vanishing at q but not at r would correspond to a deformation of X in $\tilde{V}_{\underline{m}}(D)$ in which the two branches would meet E in distinct points.

Finally, to prove part 3 of Proposition 2.1, we simply let X' be a general member of the family $\tilde{V}_{\underline{m}'}(D')$ and apply the above to $X \in \tilde{V}_{\underline{m}}(D)$, including X' in G and its points of intersection with G and E among the points P_i. □

The next Proposition is stated as a characterization of the reducible elements of the one-parameter family $\mathcal{X}^\nu \to \Gamma$, but in fact it is a characterization of the codimension one components of the boundary $V(D) \setminus \tilde{V}(D)$ of $V(D)$.

PROPOSITION 2.5. *Let $X \subset S$ be any reducible fiber of the family $\mathcal{X} \to \Gamma$.*

(1) *If X does not contain E, then X has exactly two irreducible components X_1 and X_2, with $[X_i] \in V(D_i)$ and $D_1 + D_2 = D$. Moreover $[X_i]$ is a general point in $V(D_i)$.*
(2) *If X does contain E, then X has irreducible components $E, X_1, \ldots, X_k$, with $[X_i] \in V(D_i)$ and $E + D_1 + \cdots + D_k = D$. Moreover each X_i is general in $V_{m_i}(D_i)$ for some collection $m_1, \ldots, m_k$ of positive integers such that $\sum(m_i - 1) = n - k$.*

REMARK. Notice that by Proposition 2.1, the above result says that if X does not contain E, then it has only nodes as singularities. And, if X contains E, away from the k points of tangency of E with the curves X_i, X has only nodes as singularities.

Proof. Assume first that X does not contain E. Write the divisor X as $X = \sum_{i=1}^k a_i \cdot X_i$ where $a_i > 0$ and the X_i are irreducible curves in S. We claim first

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that since $[X] \in V(D)$, all the curves X_i must be rational. To see this, take any one-parameter family $\mathcal{X} \to B$ of irreducible rational curves specializing to X. Proceeding as in 2.1 we arrive at a family $\mathcal{Y} \to B$ of nodal curves, with general fiber $\mathbb{P}^1$, that admits a regular map $\mathcal{Y} \to \mathcal{X}$. Now, since the fibers of $\mathcal{Y} \to B$ are reduced curves of arithmetic genus 0, every component of every fiber of $\mathcal{Y}$ must be a rational curve. Thus every component of X is dominated by a rational curve and so must be itself rational.

Thus $[X_i] \in V(D_i)$, where D_i are divisor classes such that $\sum a_i D_i = D$. On the other hand, since X is a general member of an $(r_0(D) - 1)$-dimensional family, we must have

$$\sum_{i=1}^{k} r_0(D_i) \geqslant r_0(D) - 1$$

$$\sum_{i=1}^{k} (-(K_S \cdot D_i) - 1) \geqslant (-K_S \cdot D) - 2 = \sum_{i=1}^{k} a_i(-K_S \cdot D_i) - 2.$$

Comparing the two sides, we see that

$$2 - k - \sum_{i=1}^{k} (a_i - 1)(-K_S \cdot D_i) \geqslant 0.$$

But $(-K_S \cdot D_i) \geqslant 2$ for any curve D_i on S other than E; so we may conclude that all $a_i = 1$ and that $k \leqslant 2$. Moreover, if $k = 2$ we have equality in the above inequality, which says that the pair of curves (X_1, X_2) is general in $V(D_1) \times V(D_2)$.

We come now to the case where X contains E. The first thing we see here is that the dimension-count argument we used above doesn't work: since

$$(-K_S \cdot (X - aE)) = (-K_S \cdot X) + a(n-2),$$

the sums $\sum a_i X_i$ of rational curves $X_i \in |D_i|$ may well move in a larger-dimensional family than X itself.

The key here is to look at the semistable reduction of a family of curves in $\tilde{V}(D)$ specializing to X. This will allow us to limit the number of points of intersection of the curves X_i with E, that is to say, to show that in fact the X_i belong to $V_{\underline{m}}(D_i)$ for suitable $\underline{m}$. This replaces the naive bound above on the dimension of the family of such curves X with a stronger one, which turns out to be sharp.

Consider then the family $\mathcal{Y} \to B$ obtained from $\mathcal{X} \to \Gamma$ as in Section 2.1. We can thus assume that the total space $\mathcal{Y}$ of the family is smooth and every fiber of $\mathcal{Y}$ is a union of smooth rational curves meeting transversely, and whose dual graph is a tree.

Now, let Y be the special fiber of $\mathcal{Y} \to B$. We decompose Y into two parts: we let Y_E be the union of the irreducible components of Y mapping to E, and

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Y_R the union of the remaining components. Next, we decompose Y_R further into k parts, letting Y_i be the union of the components mapping to X_i. Denote the connected components of Y_E by Z_i, and for each i let α_i be the degree of the map $\mu|_{Y_i}: Z_i \to E$, so that $\sum \alpha_i = a$. Similarly, let $\{Z_{i,j}\}_j$ be the connected components of Y_i and $\alpha_{i,j}$ the degree of the restriction $\mu|_{Z_{i,j}}: Y_{i,j} \to X_i$, so that $\sum_j \alpha_{i,j} = a_i$.

Note that the inverse image of E in $\mathcal{Y}$ is given by $\pi^{-1}(E) = Y_E \cup \Gamma_1 \cup \cdots \cup \Gamma_b$. (where $\pi: \mathcal{Y} \to S$ is the natural map.)

As we indicated, the essential new aspect of the argument in this case is keeping track of the number of points of intersection of the X_i with E. To do this, we note that, over any such point, there will be a point of intersection of a component of Y_i with the inverse image $\pi^{-1}(E)$; which by the expression above for $\pi^{-1}(E)$ will be either a point of intersection of Y_i with Y_E or one of the b points of intersection of the Γ_i with Y.

It thus remains to bound the number ε of points of intersection of Y_E with the remaining parts Y_i of Y. This we can do by using the fact that the dual graph of Y is a tree: this says that the number of pairwise points of intersection of the connected components $Z_{i,j}$ of Y_i and the connected components Z_i of Y_E is equal to the total number of all such connected components, minus one. Thus,

$$\begin{aligned}\varepsilon = \#(Y_R \cap Y_E) \;&=\; \#\{\text{connected components of } Y_E\}\\ &\quad + \sum \#\{\text{connected components of } Y_i\}.\end{aligned}$$

Note that the degree $\alpha_i > 0$ on each component Z_i of Y_E, so that

$$\#\{\text{connected components of } Y_E\} \leqslant a$$

and similarly

$$\#\{\text{connected components of } Y_i\} \leqslant a_i.$$

Thus we can deduce in particular that

$$\varepsilon \leqslant a + \sum a_i - 1.$$

Now, say $X_i \in \tilde{V}_{\underline{m}^i}(D_i)$ for each $i = 1, \ldots, k$. Let $\nu_i: X_i^\nu \to X_i$ be the normalization map. Choose any irreducible component X_i^0 of Y dominating X_i (and hence dominating the normalization X_i^ν), and let $\pi_i: X_i^\nu \to X_i$ be the restriction of π to X_i^ν. Trivially, the total number of points of the pullback $\nu_i^*(E)$ of E to X_i^ν is

$$\#\nu_i^*(E) \leqslant \#\pi_i^*(E) = \#(X_i^0 \cap Y_E)$$

and hence

$$\sum \#\nu_i^*(E) \leqslant \sum \#(X_i^0 \cap Y_E) \leqslant \#(Y_R \cap Y_E) = \varepsilon$$

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with strict inequality if any $a_i > 1$. But the sum of degrees of E on the curves X_i is at least

$$\begin{aligned}\sum \deg(\pi_i^* E) &\geqslant \Big(\Big(\sum X_i\Big) \cdot E\Big)\\ &= \Big(\Big(D - aE - \sum(a_i - 1)D_i\Big) \cdot E\Big)\\ &= (D \cdot E) + an - \sum a_i(D_i \cdot E).\end{aligned}$$

Comparing the number of points of the pullbacks of E to the normalizations X_i^ν with the degrees of these pullbacks, we conclude that there must be multiplicities in these divisors: specifically, the sum $\sum(m_j^i - 1)$ of the multiplicities minus one must be the difference of these numbers, so that

$$\begin{aligned}\sum(m_j^i - 1) &\geqslant \sum \deg \pi_i^*(E) - \varepsilon - (D \cdot E)\\ &\geqslant (D \cdot E) + an - \sum(a_i - 1)(D_i \cdot E)\\ &\quad -a - \sum a_i + 1 - (D \cdot E)\\ &\geqslant a(n-1) - \sum(a_i - 1)(D_i \cdot E) - \sum a_i + 1.\end{aligned}$$

This in turn allows us to bound the number of degrees of freedom of the curves X_i: we have

$$\begin{aligned}\sum \dim \tilde{V}_{\underline{m}^i}(D_i) &= \sum r_0(D_i) - \sum(m_j^i - 1)\\ &= \sum_{i=1}^{k}((-K_S \cdot D_i) - 1) - \sum(m_j^i - 1)\\ &\leqslant \sum(-K_S \cdot D_i) - k - a(n-1)\\ &\quad + \sum(a_i - 1)(D_i \cdot E) + \sum a_i - 1.\end{aligned}$$

On the other hand, this must be at least equal to the dimension of $V(D)$ minus one, that is,

$$\begin{aligned}r_0(D) - 1 &= (-K_S \cdot D) - 2\\ &= a(-K_S \cdot E) + \sum a_i(-K_S \cdot D_i) - 2\\ &= a(n-2) + \sum a_i(-K_S \cdot D_i) - 2.\end{aligned}$$

In the end, then, we must have

$$\begin{aligned}&a(n-2) + \sum a_i(-K_S \cdot D_i) - 2\\ &\quad\leqslant \sum(-K_S \cdot D_i) - k - a(n-1)\\ &\quad + \sum(a_i - 1)(D_i \cdot E) + \sum a_i - 1.\end{aligned}$$

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We can (partially) cancel the $a(n-1)$ and $a(n-2)$ terms, and combine the terms involving $(-K_S \cdot D_i)$ to rewrite this as

$$a + \sum (a_i - 1)(-K_S \cdot D_i) - 1 \leqslant \sum (a_i - 1)(D_i \cdot E) - k + \sum a_i - 1,$$

or, in other words,

$$a + \sum (a_i - 1)[((-K_S - E) \cdot D_i) - 1] - 1 \leqslant 0.$$

Now, we have already observed that $-K_S - E = C + 2F$ meets every curve X_i strictly positively, so that the sum in this last expression is nonnegative. We conclude that $a = 1$, and (since any $a_i > 1$ would have led to strict inequality) that all $a_i = 1$. Next, since there is a unique component of Y mapping to each X_i, each curve X_i will have at most one point of intersection multiplicity $m > 1$ with E. Thus, finally, X_i is a general member of the family $\tilde{V}_m(D_i)$ for some collection of integers $m_1, \ldots, m_k$ with $\sum(m_i - 1) = n - k$, completing the proof of Proposition 2.5. $\square$

Note that we have not said here that every reducible curve satisfying the conditions of the Proposition in fact lies in the closure of the locus of irreducible rational curves. This is true, and is not hard to see in the case of curves of types (1); but for curves of type (2) it is a deeper fact, and we will require the proof of Proposition 2.7 to establish it.

Having characterized as a set the locus Γ of curves in $V(D)$ passing through $q_1, \ldots, q_{r_o(D)-1}$, we now turn to a statement about the local geometry of Γ around each point.

We introduce one bit of terminology here. Let X be a fiber of $\mathcal{X} \to \Gamma$; and, in case Γ is locally reducible at the point $[X] \in \Gamma$, pick a branch of Γ at $[X]$ (that is, a point b of the normalization Γ^ν of Γ lying over $[X]$). Let P be a node of X. We then make the following

DEFINITION. If P is a limit of nodes of fibers of $\mathcal{X} \to \Gamma$ near X in the chosen branch–that is, if (P, b) is in the closure of the singular locus of the map $\mathcal{X} \times_\Gamma (\Gamma^\nu \setminus \{b\}) \to \Gamma^\nu$–we will say that P is an *old* node of X. If (P, b) is an isolated singular point of the map $X \times_\Gamma (\Gamma^\nu \setminus \{b\}) \to \Gamma^\nu$ we will say that P is a *new* node of X.

Equivalently, P is an old node if the fiber X^ν of $\mathcal{X}^\nu \to \Gamma^\nu$ over b is smooth at the (two) points lying over P; if it is a new node, X^ν will have a single point lying over P, which will be a node of X^ν.

Note that if P is a singular point of X other than a node, the situation is not so black-and-white. For example, if P is an m-fold tacnode – that is, if the curve X has two smooth branches at P with contact of order m – then a priori, any number $n \leqslant m$ of nodes of nearby fibers may approach P along any branch of Γ at $[X]$, with the result that the fiber of $\mathcal{X}^\nu \to \Gamma^\nu$ over the corresponding point $b \in \Gamma^\nu$ will

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have an $(m-n)$-fold tacnode over P, or will be smooth over P if $n=m$. (The proof of the relevant case $n=m-1$ will emerge in the proof of Proposition 2.7.)

In these terms, we can state

PROPOSITION 2.6. *Let X be a reducible fiber of the family $\mathcal{X} \to \Gamma$. Keeping the notations and hypotheses of Proposition* 2.5,

(1) *If $X = X_1 \cup X_2$ does not contain E, and X_1 and X_2 meet at $(D_1 \cdot D_2) = \ell$ points $P_1, \dots, P_\ell$, then in a neighborhood of $[X]$ Γ has ℓ smooth branches $\Gamma_1, \dots, \Gamma_\ell$; along Γ_i the point P_i is new, and all other nodes of X are old.*

(2a) *If $X = E \cup X_1 \cup \ldots \cup X_k$, and X_i meets E transversely in $(D_i \cdot E) = \ell_i$ points $P_{i,1}, \dots, P_{i,\ell_i}$, then in a neighborhood of $[X]$ Γ consists of $\prod \ell_i$ smooth branches $\Gamma_\alpha = \Gamma_{(\alpha_1,\dots,\alpha_k)}$. Along Γ_α the points $P_{1,\alpha_1}, \dots, P_{k,\alpha_k}$ are new, and all other nodes of X are old.*

(2b) *If $X = E \cup X_1 \cup \cdots \cup X_k$, and X_i meets E transversely in $(D_i \cdot E) = \ell_i$ points $P_{i,1}, \dots, P_{i,\ell_i}$ for $i = 2, \dots, k$, while D_1 has a point P of intersection multiplicity $m \geqslant 2$ with E, then in a neighborhood of $[X]$ Γ consists of $\prod_{i=2}^k \ell_i$ smooth branches $\Gamma_\alpha = \Gamma_{(2,\dots,\alpha_k)}$. Along Γ_α the points $P_{2,2}, \dots, P_{k,k}$ are new; all other nodes of X are old; and exactly $m-1$ nodes of nearby fibers will tend to P.*

REMARK 1. The proof of this Proposition will not be complete until the end of the following section. More precisely, we will postpone the proof of the existence and smoothness of the branches of Γ. Actually, cases 1 and $2a$ could very well be proved here, but it is more convenient do it later (that is, at the beginning of the proof of Proposition 2.7).

REMARK 2. We believe that an analogous description of the family $\mathcal{X} \to \Gamma$ may be given without the assumption that the components of the curve X other than E have altogether at most one point of tangency with E, and otherwise intersect E transversely in distinct points. The restricted statement above will suffice for our present purposes. We hope to prove the general statement in the future.

REMARK 3. The statement of Proposition 2.6 can also be expressed in terms of the normalized family $\mathcal{X}^\nu \to \Gamma^\nu$, and indeed that is how we will use it in the following chapter. In these terms, the statements are:

(1) If $[X]$ is a point of Γ corresponding to a curve X in our family not containing E, then there will be $(D_1 \cdot D_2) = \ell$ points of Γ^ν lying over $[X]$, corresponding naturally to the nodes of X. The fibers of $\mathcal{X}^\nu \to \Gamma^\nu$ over these points will be the normalizations of X at all the nodes of D_1 and D_2 and at all but one of the ℓ points of intersection of D_1 with D_2.

(2a) If $X = E + D_1 + \cdots + D_k$ contains E and the components D_i intersect E transversely, then the fibers of $\mathcal{X}^\nu \to \Gamma^\nu$ over points lying over $[X] \in \Gamma$ are the curves obtained by normalizing X at all nodes of the D_i, at all the points of pairwise intersection of the D_i, and at all but one of the points of intersection of E

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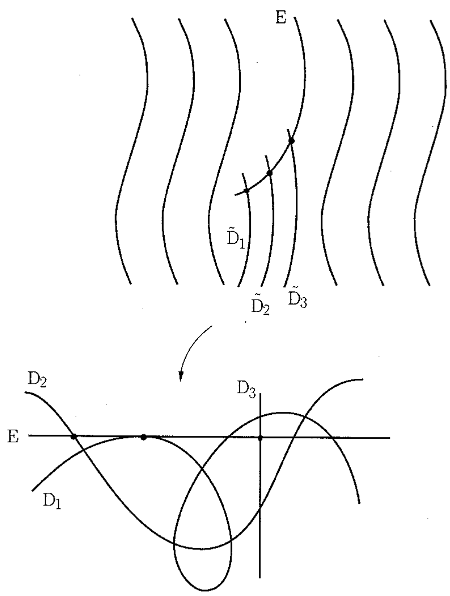
with each of the components D_i. In other words, the fibers consist of the disjoint union of the normalizations $\tilde{D}_i$ of the curves D_i, each attached to E at one point.

(2b) If $X = E + D_1 + \cdots + D_k$ as before and one of the components D_1 of X has a smooth point P of intersection multiplicity $m \geqslant 2$ with E, then the fibers X^ν of $\mathcal{X}^\nu \to \Gamma^\nu$ corresponding to $[X] \in \Gamma$ are the curves obtained by normalizing X at all nodes of the D_i, at all the points of pairwise intersection of the D_i, at all but one of the points of intersection of E with each of the components D_i for $i = 2, \ldots, k$, at all the transverse points of intersection of D_1 with E, and finally taking the partial normalization of X at P having an ordinary node over P. (The fact that each fiber of $\mathcal{X}^\nu \to \Gamma^\nu$ lying over X has an ordinary node over P follows either from the fact that the δ-invariant of the singularity $P \in X$ is m and that, along each branch, $m - 1$ nodes of nearby fibers tend to P; or – what is essentially the same thing – the fact that the arithmetic genus of the fibers of $\mathcal{X}^\nu \to \Gamma^\nu$ are zero. This will be verified independently in the course of the proof of Proposition 2.7.) The picture is therefore similar to the preceding case: the fibers consist of the disjoint union of the normalizations $\tilde{D}_i$ of the curves D_i, each attached to E at one point. The one difference is that, while for $i = 2, \ldots, k$ the point of attachment of the normalizations $\tilde{D}_i$ with E can lie over any of the points of intersection of D_i with E, the point of intersection of the normalization of D_1 with E can only be the point lying over P.

A typical picture of the original curve X and its partial normalization X^ν is this:



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Proof. Consider first of all a reducible curve X in our family that does not contain E. By Proposition 2.5, this must be of the form $X = X_1 \cup X_2$ where X_i is a general member of the family $V(D_i)$ with $D_1 + D_2 = D$. In particular, X_i is an irreducible rational curve with $p_a(D_i)$ nodes, and X_1 and X_2 intersect transversely in $(D_1 \cdot D_2)$ points. Note that

$$p_a(D_i) = \frac{(D_i \cdot D_i) + (D_i \cdot K_S)}{2} + 1,$$

so that the total number of nodes of X will be

$$p_a(D_1) + p_a(D_2) + (D_1 \cdot D_2) = p_a(D) + 1.$$

In other words, along any branch of Γ, all but one of the nodes of X will be limits of nodes of nearby fibers (that is, will be old nodes), while one node of X will be a new node. Note also that not any node of X can be the new node: that must be one of the points of intersection of the two components X_1 and X_2; otherwise the fiber of the normalization $\mathcal{X}^\nu$ would be disconnected.

In case X contains E, the analogous computation yields that X has $p_a(D) + k$ nodes (or $p_a(D) + k - m$ nodes and one tacnode of order m in case $(2b)$); hence X has k new nodes (or, $k-1$ in $(2b)$). Then the analysis in the proof of Proposition 2.5 shows that in the normalization of the total space of the family, the corresponding fiber will consist of a curve $\tilde{E}$ mapping to E, plus the normalizations $\tilde{X}_i$ of the curves X_i, each meeting $\tilde{E}$ in one point and disjoint from each other. In particular, all the nodes of X arising from points of pairwise intersection of the components X_i are old. As for the points of intersection of the components X_i with E, there are two cases. First, if a component X_i has a point of contact of order $m > 1$ with E, that must be the image of the point $\tilde{X}_i \cap \tilde{E} \in \mathcal{X}^\nu$; and all the other points of $X_i \cap E$ will be old nodes of X on any branch. On the other hand, if a component X_i intersects E transversely, any one of its points of intersection with E can be a new node.

2.4. Singularities of the Total Space

We come to the fourth result, in which we describe the singularities of the total space of the normalized family $\mathcal{X}^\nu \to \Gamma^\nu$ along a given fiber X^ν. (Given a fiber X over Γ, we will fix a corresponding fiber X^ν throughout.)

We keep a simplified form of the notation introduced in the statement of Proposition 2.6: we denote by $P_1, \ldots, P_\ell$ the new nodes of X along E, coming from transverse points of intersection of other components of X with E; and by P (if it exists) one double point of X other than a node, coming from a point of contact of order $m \geqslant 2$ of E with another component of X. We recall that the nearby fibers of our family are smooth near P_i, there will be one point p_i of $\mathcal{X}^\nu$ lying over each

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P_i, which will be a node of X^ν, while the nearby fibers have $m-1$ nodes tending to the point P, so that the partial normalization $X^\nu \to X$ will again have one point p lying over P, and that point will be a node of X^ν. With all this said, we have

PROPOSITION 2.7 (1) *If X does not contain E, or if X contains E and the closure of $X \setminus E$ intersects E transversely, then $\mathcal{X}^\nu$ is smooth along X^ν.*

(2) *In case X does contain E and the closure of $X \setminus E$ has a point P of intersection multiplicity $m \geqslant 2$ with E, the point p of X^ν lying over P is a smooth point of $\mathcal{X}^\nu$; the other nodes p_i of X^ν will be singularities of type A_{m-1} of $\mathcal{X}^\nu$.*

Proof. We start with the first statement, which is by far the easier. Recall that by the two previous propositions X, being a general point on a codimension-one locus in $V(D)$, will have $p_a(D)+k$ or $p_a(D)+1$ nodes, depending whether X does or doesn't contain E. Of these, $p_a(D)$ will be old nodes and the remaining ones are new nodes; if E is contained in X, then the new nodes all lie on E. Let $r_1,\dots,r_{p_a(D)}$ be the old nodes of X and let P be any fixed new node. The fiber X^ν of $\mathcal{X}^\nu$ lying over X will be the partial normalization of X at $r_1,\dots,r_{p_a(D)}$, so that $\mathcal{X}^\nu$ will certainly be smooth there, and we need only concern ourselves with the point of $\mathcal{X}^\nu$ lying over P.

Consider, in an analytic neighborhood of $[X]$ in $|D|$, the locus W of curves that pass through the base points $q_1,\dots,q_{r_0(D)-1}$ and that preserve all of the old nodes of X. The projective tangent space to W at $[X]$ will be contained in the sub-linear series of $|D|$ of curves passing through the $p_a(D)$ old nodes of X and through $q_1,\dots,q_{r_0(D)-1}$. This gives a total of $r_0(D)-1+p_a(D)=r(D)-1$ points which, by an argument analogous to the proof of Lemma 2.4, impose independent conditions on the linear series $|D|$. We only exhibit the proof in case E is a component of X, the other case being similar and easier. Let $\mathcal{H}$ be the ideal sheaf of the subscheme of S given by the old nodes $r_1,\dots,r_{p_a(D)}$, and let $\nu\colon \tilde{X} \to X$ be the normalization map. We have to show that $r_1,\dots,r_{p_a(D)}$ impose independent conditions on $|D|$, which will follow (cf. Lemma 2.4) from

$$H^1(\tilde{X}, \nu^*(\mathcal{O}_S(X)\otimes\mathcal{H}))=0.$$

This, by the adjunction formula, is equivalent to

$$H^0(\tilde{X}, \nu^*(K_S\otimes\mathcal{I})\otimes(\nu^*\mathcal{H})^{-1})=0,$$

where $\mathcal{I}$ is the adjoint ideal of X. Now notice that the line bundle $\nu^*(\mathcal{I})\otimes\nu^*(\mathcal{H})^{-1}$ has degree $-k$ on the component of $\tilde{X}$ lying over E, and degree -1 on every other component. Since K_S has degree $n-2=k-2$ on E and negative degree on X_i, the line bundle $\nu^*(K_S\otimes\mathcal{I})\otimes(\nu^*\mathcal{H})^{-1}$ cannot have any sections.

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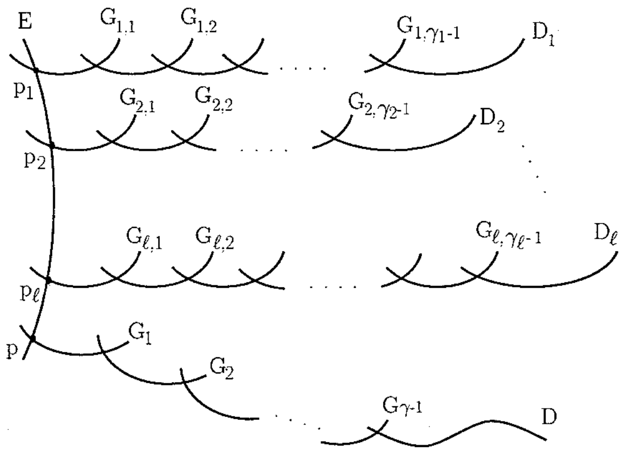
We conclude that W is smooth of dimension 1. Notice that this completes the proof of Proposition 2.6, parts (1) and (2a).

To analyze the total space of $\mathcal{X}^\nu$ we consider the map from W to the versal deformation space of the node (X, P). This has nonzero differential because P is not a base point of the linear series of curves passing through $q_1, \dots, q_{r_0(D)-1}$ and through the $p_a(D)$ old nodes of X (to see this, the argument above applied to the ideal sheaf of the union of the old nodes of X and P will work). Thus the family $\mathcal{X}^\nu \to \Gamma^\nu$ has local equation $xy - t = 0$ near p; in particular, it is smooth at p.

We turn now to the second part, which will occupy us for the remainder of this chapter. We will start by carrying out a global analysis of the family in a neighborhood of the whole fiber X, and then proceed to a local analysis around the point P specifically. From the global picture we will establish that, for some integer γ, the point P will be a singularity of type A_γ and the points P_i all singularities of type $A_{\gamma m}$. The local analysis will then show that in fact we have $\gamma = 1$.

To carry out the global analysis, we use the family $\mathcal{Y} \to \Gamma^\nu$ and the map $\pi: \mathcal{Y} \to \mathbb{F}_n$ (cf. Section 2.1), where $\mathcal{Y}$ is the minimal desingularization of the surface $\mathcal{X}^\nu$. Since the singularities of the fiber of $\mathcal{X}^\nu$ are all nodes, the total space $\mathcal{X}^\nu$ will have singularities of type A_β at each; let us say the point p is an A_γ singularity of $\mathcal{X}^\nu$, and the point p_i an A_{γ_i} singularity. When we resolve the singularity at p we get a chain $G_1, \dots, G_{\gamma-1}$ of smooth rational curves; likewise, p_i is replaced by a chain $G_{i,1}, \dots, G_{i,\gamma_i-1}$ of smooth rational curves. Denoting the component of X meeting E at P_i by D_i and the component meeting E at P by D (we are not assuming here that these are distinct irreducible components of X), we arrive at a picture of the relevant part of the fiber Y of Y.



We now look at the pull-back of E from $\mathbb{F}_n$ to $\mathcal{Y}$. We can write it as

$$\pi^*(E) = k \cdot E + \sum a_i \cdot G_i + \sum a_{i,j} \cdot G_{i,j} + E',$$

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where E' is a curve in $\mathcal{Y}$ that meets the fiber Y only along D_i and D, with $(E' \cdot D_i) = (E \cdot \pi(D_i)) - 1$ and $(E' \cdot D) = (E \cdot \pi(D)) - m$.

We can use what we know about the degree of this divisor on the various components of Y to impose conditions on the coefficients k, a_i and $a_{i,j}$. First, since π maps components G_i and $G_{i,j}$ to points in $\mathbb{F}_n$,

$$\deg_{G_i}(\pi^*(E)) = \deg_{G_{i,j}}(\pi^*(E)) = 0.$$

Now, each of the curves G_i and $G_{i,j}$ has self-intersection -2; so, setting $a_\gamma = a_{i,\gamma_i} = 0$ and $a_0 = a_{i,0} = k$, we get

$$a_{i-1} - 2a_i + a_{i+1} = 0,$$

for each $i = 1, \ldots, \gamma - 1$; and similarly

$$a_{i,j-1} - 2a_{i,j} + a_{i,j+1} = 0,$$

for each $j = 1, \ldots, \gamma_i - 1$–in other words, the sequences $a_0, \ldots, a_\gamma$ and $a_{i,0}, \ldots,$ a_{i,γ_i} are arithmetic progressions. On the other hand, the map π restricted to the component D_i is transverse to E at $P_i = \pi(p_i)$; so the multiplicity at p_i of the restriction to D_i of the divisor $\pi^*(E) - E'$ is one. This says that $a_{i,\gamma_i - 1} = 1$; and similarly $a_{\gamma-1} = m$. Following the arithmetic progression $a_0, \ldots, a_\gamma$ up from D to E, we arrive at $k = \gamma \cdot m$ and hence $\gamma_i = \gamma \cdot m$.

The proof of the Proposition will be completed once we show that $\gamma = 1$, that is, that p is a smooth point of $\mathcal{X}^\nu$.

Note that this part of the analysis did not rely, except notationally, on the hypothesis that all but one point of intersection of E with the remaining components of X are transverse. If the points P_i were points of intersection multiplicity m_i of E with other components D_i of X, we could (always assuming that $m_i - 1$ nodes of the general fiber of our family approach P_i) carry out the same analysis and deduce that for some integer k, the point p_i was a singularity of type A_{k/m_i}– loosely speaking, the singularity of $\mathcal{X}^\nu$ at p_i is 'inversely proportional' to the order of contact of D_i with E at P_i. The remaining question then would be, is the number k as small as possible, that is, the least common multiple of the m_i? That is what we will establish with the following local analysis, which does ultimately rely on the hypothesis that all but one of the m_i are one.

2.4.1. *The versal deformation space of the tacnode.* We now carry out the analysis around the point P. The versal deformation of $P \in X \subset \mathbb{F}_n$ has the vector space $\mathcal{O}_{\mathbb{F}_n,P}/\mathcal{J}$ as base, where $\mathcal{J}$ is the Jacobian ideal of X at p. Choose local coordinates x, y for $\mathbb{F}_n$ centered at P, so that the curve E is given as $y = 0$ and the equation of X is

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$$y(y + x^m) = y^2 + yx^m = 0.$$

The Jacobian ideal of this polynomial is $\mathcal{J} = (2y + x^m, yx^{m-1})$. The monomials $y, xy, x^2y, \ldots, x^{m-2}y$ and $1, x, x^2, \ldots, x^{m-1}$ form a basis for $\mathcal{O}_{\mathbb{F}_n,P}/\mathcal{J}$, so that we can write down explicitly a versal deformation space: the base Δ will be an analytic neighborhood of the origin in affine space $\mathbb{A}^{2m-1}$ with coordinates $\alpha_0, \alpha_1, \ldots, \alpha_{m-2}$ and $\beta_0, \beta_1, \ldots, \beta_{m-1}$, and the deformation space will be the family $\mathcal{S} \to \Delta$, with $\mathcal{S} \subset \Delta \times \mathbb{A}^2$, given by the equation

$$y^2 + yx^m + \alpha_0 y + \alpha_1 xy + \cdots + \alpha_{m-2}x^{m-2}y$$
$$+\beta_0 + \beta_1 x + \beta_2 x^2 + \cdots + \beta_{m-1}x^{m-1} = 0.$$

Inside Δ we look closely at the closures Δ_{m-1} and Δ_m of the loci corresponding to curves with $m-1$ and m nodes, respectively. We have

LEMMA 2.8. (1) Δ_m *is given in* Δ *by the equations* $\beta_0 = \cdots = \beta_{m-1} = 0$*; in particular it is smooth of dimension* $m-1$.

(2) Δ_{m-1} *is irreducible of dimension* m*, with* m *sheets crossing transversely at a general point of* Δ_m.

Proof. We introduce the *discriminant* of the polynomial f above, viewed as a quadratic polynomial in y:

$$\delta = \delta_{\alpha,\beta}(x) = (x^m + \alpha_{m-2}x^{m-2} + \cdots + \alpha_1 x + \alpha_0)^2$$
$$-4(\beta_{m-1}x^{m-1} + \cdots + \beta_1 x + \beta_0)$$

Note that the map $\delta\colon \Delta \to V$ to the space V of monic polynomials of degree $2m$ in x with vanishing x^{2m-1} term is an isomorphism of Δ with a neighborhood of the origin in V: given an equation

$$(x^m + \alpha_{m-2}x^{m-2} + \cdots + \alpha_1 x + \alpha_0)^2 - 4(\beta_{m-1}x^{m-1} + \cdots + \beta_1 x + \beta_0)$$
$$= x^{2m} + c_{2m-2}x^{2m-2} + \cdots + c_1 x + c_0,$$

we can write

$$\alpha_{m-2} = \frac{c_{2m-2}}{2}, \qquad \alpha_{m-3} = \frac{c_{2m-3}}{2},$$

$$\alpha_{m-4} = \frac{c_{2m-4} - \alpha_{m-2}^2}{2} = \frac{4c_{2m-4} - c_{2m-2}^2}{8}$$

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and so on, recursively expressing the coefficients α_i as polynomials in the coefficients $c_{2m-2}, \dots, c_m$. We can then solve for the β_i in terms of the remaining coefficients $c_{m-1}, \cdots, c_0$, thus obtaining a polynomial inverse to the map δ.

Now, since the equation f above for $\mathcal{S}$ is quadratic in y, the fibers of $\mathcal{S} \to \Delta$ are expressed as double covers of the x-line. The discriminant δ is a polynomial of degree $2m$ in x, so that the general fiber of $\mathcal{S} \to \Delta$, viewed as a double cover of the x-axis, will have $2m$ branch points near P. To say that any fiber $S_{\alpha,\beta}$ has m nodes is thus tantamount to saying that $\delta_{\alpha,\beta}(x)$ has m double roots – that $\delta_{\alpha,\beta}(x)$ is the square of a polynomial of degree m. The locus of squares being smooth of dimension $m-1$ in V, we see that Δ_m is smooth of dimension $m-1$; indeed, it is given simply by the vanishing $\beta_0 = \cdots = \beta_{m-1} = 0$.

Similarly, to say that a fiber $S_{\alpha,\beta}$ has $m-1$ nodes amounts to saying that $\delta_{\alpha,\beta}(x)$ has $m-1$ double roots, i.e., that it can be written as a quadratic polynomial in x times the square of a polynomial of degree $m-1$

$$\delta_{\alpha,\beta}(x) = (x^{m-1} + \lambda_{m-2}x^{m-2} + \cdots + \lambda_1 x + \lambda_0)^2(x^2 + \mu_1 x + \mu_0).$$

The Lemma is then proved. □

Now we consider the natural map from a suitable analytic neighborhood W of $[X]$ to Δ. To set this up, let $r_1, \dots, r_k$ be the old nodes of X; since all the singularities of X other than P are nodes, this will consist of b nodes on E and $k-b$ nodes lying off E where $b = (D \cdot E)$. Since $m-1$ nodes of the general curve of our family tend to P, we have $k = p_a(D) - m + 1$. Now consider, in an analytic neighborhood of the point $[X] \in |D|$, the locus W of curves passing through the $r_0(D) - 1$ assigned points $q_1, \dots, q_{r_0(D)-1}$ and preserving the nodes $r_1, \dots, r_k$ of X – that is, such that the restriction of the family of curves $\{D_\lambda\}_{\lambda \in |D|}$ to W is equisingular at each point r_i of X. Since this is a total of $r_0(D) - 1 + p_a(D) - m + 1 = r(D) - m$ points and they impose independent conditions on the linear series $|D|$, we see that W is smooth of dimension m at $[X]$.

We then get a natural map $\phi\colon W \to \Delta$ such that $\phi([X]) = 0$. We will prove that ϕ is an immersion and that the intersection of $\phi(W)$ with Δ_{m-1} is the union of Δ_m with a smooth curve Ψ; moreover Ψ and Δ_m will have contact of order m at the origin. This will conclude the proof of Proposition 2.6; in fact the original family $\mathcal{X} \to \Gamma$ will be the pullback to W of the restriction to Ψ of the versal deformation $\mathcal{S} \to \Delta$.

To illustrate, here is a representation of the simplest case $m = 2$. This does not convey the general picture, because $\phi(W) \cap \Delta_{m-1}$ happens to be proper. Also, the picture is inaccurate in at least one respect: the actual surface Δ_1 in the deformation space of a tacnode is also singular along the locus of curves $S_{\alpha,\beta}$ with cusps.

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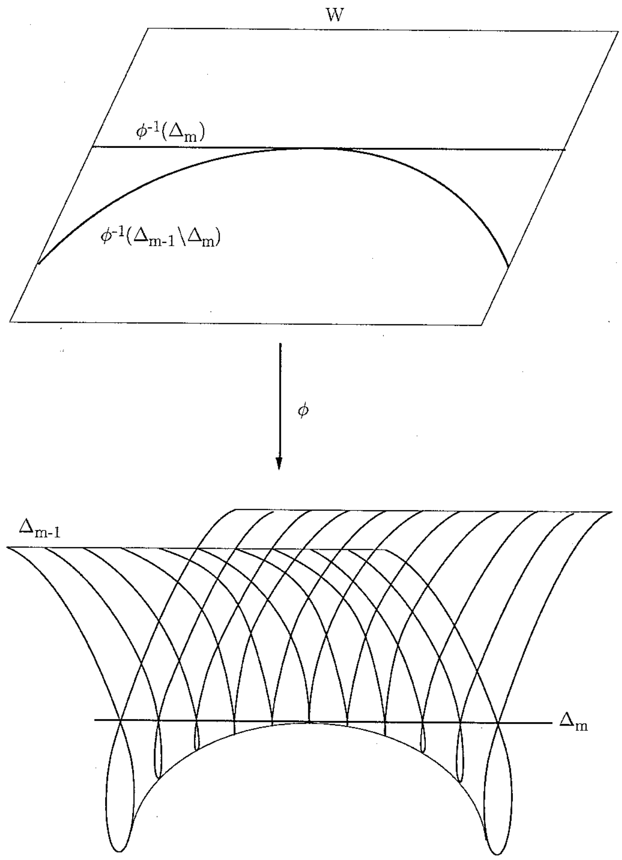

(Note that we see again locally the picture that we have already observed globally in the linear series $|D|$: the closure of the variety $V(D)$ of irreducible rational curves has the expected dimension; but the locus of rational curves has another component of equal or larger dimension.)

Now that we have a basic picture of the deformation space Δ, the crux of our argument will be to describe the pullback of the loci Δ_m and Δ_{m-1} under the map $\phi\colon W \to \Delta$ (or, equivalently, the intersection of $\phi(W)$ with these loci). We start in the following subsection by saying what we can about the geometry of the map ϕ.

2.4.2. *The deformations coming from* $V(D)$. Let $\phi\colon W \to \Delta$ be as before, denote by H the subspace of Δ given by $\beta_0 = 0$. Then we have

LEMMA 2.9. *The map ϕ is an immersion; the tangent space to the image at the origin contains the plane $\beta_0 = \cdots = \beta_{m-1} = 0$ but is not contained in H.*

REMARK. It is important to note here, and throughout the following argument, that while the loci Δ_m and Δ_{m-1} are well-defined subsets of the base Δ of the

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deformation space of our tacnode, H is not; it depends on the choice of coordinates. It is well-defined, however, as a hyperplane in the tangent space $X(\Delta) = \mathcal{O}_{\mathbb{F}_n,P}/\mathcal{J}$ to Δ at the origin: it corresponds to the quotient $\mathbf{m}/\mathcal{J} \subset \mathcal{O}_{\mathbb{F}_n,P}/\mathcal{J}$ of the maximal ideal $\mathbf{m} \subset \mathcal{O}_{\mathbb{F}_n,P}$.

Proof. The projective tangent space to W at the point $[X]$ is the sublinear series of $|D|$ of curves passing through the points $r_1, \dots, r_k$ and $q_1, \dots, q_{r_0(D)-1}$. The kernel of the differential at $[X]$ of the map ϕ is thus the vector space of sections of the line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{F}_n}(D)$ vanishing at $r_1, \dots, r_k$ and $q_1, \dots, q_{r_0(D)-1}$ and lying in the subsheaf $\mathcal{L} \otimes \mathcal{J}$, where $\mathcal{J} \subset \mathcal{O}_{\mathbb{F}_n,P}$ is as before the Jacobian ideal of $[X]$ at P. The zero locus of such a section will be a curve in the linear series $|D|$ containing $r_1, \dots, r_k, q_1, \dots, q_{r_0(D)-1}$ and P and so must contain E, that is, must be of the form $E + G$ with $G \in |D - E|$. Moreover, from the description above of $\mathcal{J}$ we see that G must also have contact of order at least m with E at P as well as pass through the $k - b$ nodes of X lying off E and the assigned points $p_1, p_2, q_3, \dots, q_{r_0(D)-1}$. This represents a total of

$$m + r_0(D) - 1 + p_a(D) - m + 1 - b = r(D) - b = r(D - E) + 1,$$

conditions, so we need to show that they are independent to conclude that no such curve exists. But they are also a subset of the adjoint conditions of X, hence impose independent conditions on the series $|D + K_{\mathbb{F}_n}| = |D - C - E - 2F|$, and hence on the series $|D - E|$.

The remaining statements of the lemma, that the tangent space to the image contains the plane $\beta_0 = \cdots = \beta_{m-1} = 0$ but is not contained in the hyperplane $\beta_0 = 0$, follow from the facts that the image contains the subvariety Δ_m and that not every curve in the linear series $|D|$ containing $r_1, \dots, r_k, q_1, \dots, q_{r_0(D)-1}$ contains P. □

To complete the proof of Proposition 2.7, we thus have to establish the following lemma about the geometry of the deformation space Δ.

LEMMA 2.10. *Let $\Lambda \subset \Delta$ be a smooth, m-dimensional variety such that*

(1) Λ *contains* Δ_m*;*

(2) *the tangent space to* Λ *at the origin is not contained in* H*.*

Then the intersection $\Lambda \cap \Delta_{m-1}$ consists of the union of Δ_m and a smooth curve Ψ having contact of order m with Δ_m at the origin.

Our proof of this Lemma is lengthy and roundabout; it occupies the remainder of this chapter. We give here a summary of the four main steps:

• First, in 2.4.3 we treat a special case. In Lemma 2.11 we prove Lemma 2.10 by direct calculation when Λ is the linear subspace of Δ given by equations $\beta_1 = \cdots = \beta_{m-1} = 0$. The results of 2.4.3 also appear in [R]; we include our proof for the sake of completeness.

• Secondly, in subsection 2.4.4, we introduce the the blow-up $\tilde{\Delta}$ of Δ along Δ_m, and translate Lemma 2.11 into the statement that the proper transform $\tilde{\Delta}_{m-1}$

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of Δ_{m-1} in $\tilde{\Delta}$ is smooth at the point Q, and has contact of order m with the exceptional divisor Z of the blow up there.

• Third, we use the automorphisms of the deformation space Δ (and its blow-up $\tilde{\Delta}$) to deduce that $\tilde{\Delta}_{m-1}$ is smooth everywhere along the open subset $\Phi_0 = \Phi \setminus (\Phi \cap \tilde{H})$ of the fiber Φ of the blow up over the origin, and has contact of order m with the exceptional divisor Z (Lemma 2.15). Lemma 2.14 will say that $\tilde{\Delta}_{m-1}$ contains Φ, and the global analysis will say that the intersection multiplicity of $\tilde{\Delta}_{m-1}$ with Z at a general point of Φ is at least m. Then the special case, together with Lemma 2.13, implies that the intersection multiplicity is at most m, and hence exactly m, at any point of Φ_0; and that $\tilde{\Delta}_{m-1}$ is smooth along Φ_0.

• Finally, for any subvariety $\Lambda \subset \Delta$ satisfying the hypotheses of Lemma 2.10, its proper transform in $\tilde{\Delta}$ will intersect Φ transversely at a point of Φ_0, and the desired result – that the intersection of Δ_{m-1} with Λ consists of the union of Δ_m and a smooth curve having contact of order m with Δ_m at the origin – follows.

2.4.3. *A special case.* We will start by considering the intersection of Δ_{m-1} with the simplest possible variety satisfying the hypotheses of Lemma 2.10, the plane Λ_0 given by $\beta_1 = \cdots = \beta_{m-1} = 0$. We obtain

LEMMA 2.11. *The intersection of Δ_{m-1} with Λ_0 consists of the union of Δ_m with multiplicity m and a smooth curve Ψ having contact of order m with Δ_m at the origin.*

Proof. Restricting to Λ_0, we can rewrite the equation of the family more simply as

$$y^2 + yx^m + \alpha_0 y + \alpha_1 xy + \cdots + \alpha_{m-2}x^{m-2}y + \beta = 0$$

and the discriminant as

$$\delta(x) = (x^m + \alpha_{m-2}x^{m-2} + \cdots + \alpha_1 x + \alpha_0)^2 - 4\beta.$$

We need now to express the condition that δ has $m-1$ double roots. One obviously sufficient condition is that $\beta = 0$, so that δ is a square. If we assume $\beta \neq 0$, however, things get more interesting. To see the locus of $(\alpha_0, \ldots, \alpha_{m-2}, \beta)$ that satisfy this condition, set

$$\nu(x) = x^m + \alpha_{m-2}x^{m-2} + \cdots + \alpha_1 x + \alpha_0$$

and write

$$\delta(x) = \nu(x)^2 - 4 = (\nu(x) + 2\sqrt{\beta}) \cdot (\nu(x) - 2\sqrt{\beta}).$$

Now, if $\beta \neq 0$, the two factors in this last expression have no common factors; so if their product has $m-1$ double roots, each must have a number of double roots itself: $\nu(x) + 2\sqrt{\beta}$ and $\nu(x) - 2\sqrt{\beta}$ are polynomials of degree m with a combined total of $m-1$ double roots. In fact, this uniquely characterizes ν and β up to a one-parameter group of automorphisms of $\mathbb{P}^1$, as we will prove in the following.

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LEMMA 2.12. *Let γ be a nonzero scalar, and let m be a positive integer. There is a polynomial $\nu(x)$ of degree m, monic with no x^{m-1} term, such that*

(1) *if m is odd, the polynomials $\nu(x)+\gamma$ and $\nu(x)-\gamma$ each have $(m-1)/2$ double roots.*
(2) *if m is even, $\nu(x)+\gamma$ has $m/2$ double roots and $\nu(x)-\gamma$ has $(m-2)/2$ double roots.*

In both cases, ν is unique up to replacing $\nu(x)$ by $\nu(\zeta x)$, where ζ is an mth root of unity.

Proof. Suppose that $\nu(x)$ is a polynomial satisfying the conditions of the lemma. Take first the case of $m = 2\ell + 1$ odd, and consider the map $\nu\colon \mathbb{P}^1 \to \mathbb{P}^1$ given by $\nu(x)$. This is a map of degree m, sending the point ∞ to ∞, and totally ramified there. In addition the hypotheses assert that over the points $\pm\gamma$ in the target we have ℓ ramification points. The point is, this accounts for a total of $(m-1)+2(\ell-1) = 2m-2$ ramification points, and these are all a map of degree m from $\mathbb{P}^1$ to $\mathbb{P}^1$ will have. We have thus specified the covering ν up to a finite number of coverings, and our principal claim is that in fact we have described ν uniquely, up to automorphisms of the domain.

This is combinatorial. The monodromy permutation σ around the point ∞ is cyclic, while the the monodromy permutations τ and μ around γ and $-\gamma$ are each products of ℓ disjoint transpositions. Our claim that there is a unique such covering of $\mathbb{P}^1$ by $\mathbb{P}^1$ amounts then to the assertion that, up to the action of the symmetric group $\mathfrak{S}_m$ by conjugation, there is a unique pair of permutations τ and μ, each a product of ℓ disjoint transpositions, whose product $\tau \circ \mu$ is cyclic of order m. This is an easy combinatorial exercise. To complete the proof of Lemma 2.12, consider the effect on ν of automorphisms of the domain. The requirement that $\nu(\infty) = \infty$ – that is, that $\nu(x)$ is a polynomial! – restricts us to the group of automorphisms $x \mapsto ax + b$; the requirement that $\nu(x)$ have no x^{m-1} term limits us to automorphisms of the form $x \mapsto ax$; and the fact that $\nu(x)$ is monic says that a must be an mth root of unity. □

Note that, in case m is odd, by uniqueness we must have $\nu(x) = -\nu(-x)$; that is, ν will be odd. Similarly, in case m is even (where the two branch points $\pm\gamma$ have different multiplicity) we must have $\nu(x) = \nu(-x)$, so that ν will be even. This will not be logically relevant to the following calculation, but will be reflected in the notation.

Back to the proof of Lemma 2.11. Note that if we do not specify the value of γ the polynomial $\nu(x)$ will not be unique; we can replace it with $u^m\nu(x/u)$ for any nonzero scalar u. Now, suppose first that $m = 2\ell$ is even. Choose $\gamma = 1$, and let

$$\nu(x) = x^m + c_{m-2}x^{m-2} + c_{m-4}x^{m-4} + \cdots + c_0,$$

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be the polynomial satisfying the conditions of the lemma. Then any collection $(\alpha_0, \alpha_1, \ldots, \alpha_{m-2}, \beta)$ with $\beta \neq 0$ such that the discriminant

$$\delta(x) = (x^m + \alpha_{m-2}x^{m-2} + \cdots + \alpha_1 x + \alpha_0)^2 - 4\beta,$$

has $m-1$ double roots, must be of the form

$$\alpha_0 = t^\ell \cdot c_0, \qquad \alpha_1 = 0,$$

$$\alpha_2 = t^{\ell-1} \cdot c_2, \qquad \alpha_3 = 0,$$

$$\alpha_4 = t^{\ell-2} \cdot c_4$$

and so on, ending with $\alpha_{m-2} = t \cdot c_{m-2}$; with finally $\beta = t^m/4$. This is then a parametric representation of the closure Ψ of the intersection $\Lambda_0 \cap (\Delta_{m-1} \setminus \Delta_m)$. It is obviously a curve; the fact that it is smooth is visible from the coordinate $\alpha_{m-2} = t \cdot c_{m-2}$; and we see that it has contact of order m with H from the exponent in the expression for β.

Finally, in case $m = 2\ell + 1$ is even we get a similar expression. Let

$$\nu(x) = x^m + c_{m-2}x^{m-2} + c_{m-4}x^{m-4} + \cdots + c_1 x,$$

be the polynomial satisfying the conditions of the lemma for $\gamma = 1$. Then any collection $(\alpha_0, \alpha_1, \ldots, \alpha_{m-2}, \beta)$ with $\beta \neq 0$ such that the discriminant

$$\delta(x) = (x^m + \alpha_{m-2}x^{m-2} + \cdots + \alpha_1 x + \alpha_0)^2 - 4\beta,$$

has $m-1$ double roots must be of the form

$$\alpha_0 = 0, \qquad \alpha_1 = t^\ell \cdot c_1,$$

$$\alpha_2 = 0, \qquad \alpha_3 = t^{\ell-1} \cdot c_3,$$

ending with $\alpha_{m-2} = t \cdot c_{m-2}$; again we have $\beta = t^m/4$. So once more we see that Ψ is a smooth curve having contact of order m with H at the origin. $\square$

Let us now prove Lemma 2.10 and Proposition 2.7 in this special case. First, in the case $m = 2\ell$ even, the restriction $\mathcal{S}_\Psi \to \Psi$ of the family $\mathcal{S} \to \Delta$ to Ψ has equation

$$y^2 + y(x^m + tc_{m-2}x^{m-2} + t^2 c_{m-4}x^{m-4} + \cdots + t^{m/2}c_0) + (t^m/4) = 0.$$

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We can think of the total space $\mathcal{S}_\Psi$ of this family as a double cover of the (x,t)-plane, with branch divisor the zero locus of the discriminant

$$\delta = (x^m + tc_{m-2}x^{m-2} + t^2c_{m-4}x^{m-4} + \cdots + t^{m/2}c_0)^2 - t^m.$$

By hypothesis, for each value of t the polynomial δ is the product of the square of a polynomial $g_t(x)$ of degree $m-1$ and a quadratic polynomial $h_t(x)$. Since δ is even, g^2 and h must each be; and given the homogeneity of δ with respect to t and x we see that we can write

$$\delta = x^2(x^2-\lambda_1 t)^2(x^2-\lambda_2 t)^2\cdots(x^2-\lambda_{\ell-1}t)^2\cdot(x^2-\mu t),$$

for suitable constants $\lambda_1,\ldots,\lambda_{\ell-1}$ and μ. For example, in case $m=2$, the equation of $\mathcal{S}_\Psi$ is simply

$$y^2 + y(x^2+t) + (t^2/4) = 0$$

and the discriminant is just $\delta = x^2(x^2-2t)$. In general, the branch divisor of $\mathcal{S}_\Psi$ over the (x,t)-plane will be simply a union of the t-axis, with multiplicity 2; $\ell-1$ parabolas tangent to the x-axis at the origin, each with multiplicity 2; and one more parabola tangent to the x-axis at the origin and appearing with multiplicity 1. The double cover $\mathcal{S}_\Psi$ will thus be nodal over the double components of this branch divisor, and smooth elsewhere.

Finally, the normalization $\mathcal{S}_\Psi^\nu$ of the total space $\mathcal{S}_\Psi$ will be the double cover of the (x,t)-plane branched over the single component of multiplicity 1 in the branch divisor; that is, it will have equation

$$y^2 = x^2 - \mu t$$

and in particular, since the component $(x^2-\mu t)$ is smooth, $\mathcal{S}_\Psi^\nu$ will be smooth as well, establishing Proposition 2.7 for this particular family.

The picture in case $m=2\ell+1$ is odd is exactly the same: here $\mathcal{S}_\Psi$ has equation

$$y^2 + y(x^m + tc_{m-2}x^{m-2} + t^2c_{m-4}x^{m-4} + \cdots + t^{m/2}c_1x) - (t^m/4) = 0$$

with discriminant

$$\begin{aligned}\delta &= (x^m + tc_{m-2}x^{m-2} + t^2c_{m-4}x^{m-4} + \cdots + t^{m/2}c_1x)^2 - t^m\\ &= (x^2-\lambda_1 t)^2(x^2-\lambda_2 t)^2\cdots(x^2-\lambda_\ell t)^2\cdot(x^2-\mu t),\end{aligned}$$

for suitable constants $\lambda_1,\ldots,\lambda_\ell$ and μ. For example, in case $m=3$, the equation of $\mathcal{S}_\Psi$ will be

$$y^2 + y(x^3-3tx) - t^3 = 0,$$

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(we are scaling t here to make the coefficients nicer), and the discriminant is just

$$\begin{aligned} \delta &= (x^3 - 3tx)^2 + 4t^3 \\ &= x^6 - 6tx^4 + 9t^2x^2 + 4t^3 \\ &= (x^2 + t)^2(x^2 + 4t). \end{aligned}$$

In general, for m odd the branch divisor of $\mathcal{S}_\Psi$ over the (x, t)-plane will be simply a union of ℓ parabolas tangent to the x-axis at the origin, each with multiplicity 2; and one more parabola tangent to the x-axis at the origin and appearing with multiplicity 1. As before, the normalization $\mathcal{S}_\Psi^\nu$ of the total space $\mathcal{S}_\Psi$ will be simply the double cover of the (x, t)-plane branched over the single component $(x^2 - \mu t)$ of multiplicity 1 in the branch divisor; and as before, since this component is smooth, $\mathcal{S}_\Psi^\nu$ will be smooth as well, establishing Proposition 2.7 in this case. □

2.4.4. *The geometry of the locus* Δ_{m-1}. In order to focus on the essential aspects of the geometry of Δ_{m-1}, and in particular to remove the excess intersection of $\phi(W) \cap \Delta_{m-1}$, we will work on the blow-up $\tau : \tilde{\Delta} = \mathrm{Bl}_{\Delta_m}\Delta \to \Delta$ of Δ along Δ_m. To express our results, we have to introduce some notation. We will denote by $Z = \tau^{-1}(\Delta_m)$ the exceptional divisor of the blow up, and by $\tilde{\Delta}_{m-1}$ and $\tilde{W}$ the proper transforms of Δ_{m-1} and $\phi(W)$ in $\tilde{\Delta}$.

Our goal will be to describe the intersection $Z_{m-1} := \tilde{\Delta}_{m-1} \cap Z$. The fibers of Z over Δ_m are projective spaces $\mathbb{P}^{m-1}$ with homogeneous coordinates $\beta_0, \ldots, \beta_{m-1}$; we will denote the fiber $\tau^{-1}(0)$ of Z over the origin by Φ, by $\Phi_0 \subset \Phi$ the open set given by $\beta_0 \neq 0$, and by Q the point of Φ with coordinates $[1, 0, \ldots, 0]$ (this is the point of intersection of $\tilde{W}$ with Φ in the example above).

Note that there is a more intrinsic characterization of Φ: the tangent space to Δ_m at the origin is the subspace of $\mathcal{O}_{\mathbb{F}_n,P}/\mathcal{J}$ of polynomials divisible by y, so that Φ – the projectivization of the normal space – is just the space of polynomials in x modulo those vanishing to order m at $P = (0, 0)$ and modulo scalars. In these terms, Φ_0 is simply the subspace of polynomials not vanishing at the origin and Q the point corresponding to constants.

To study Δ_{m-1} we will take advantage of an equivariant action of the multiplicative group $\mathbb{C}^*$ on the map $\mathcal{S} \to \Delta$. Explicitly, for any $c \in \mathbb{C}^*$, we define automorphisms δ_c of Δ and σ_c of $\mathcal{S}$ simultaneously by

$$x \mapsto cx, \qquad y \mapsto c^m y,$$

$$\alpha_i \mapsto c^{m-i}\alpha_i, \qquad \beta_i \mapsto c^{2m-i}\beta_i.$$

Since δ_c and σ_c commute with the projection $\mathcal{S} \to \Delta$, δ_c preserves the loci $\Delta_k \subset \Delta$.

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In particular, since δ_c preserves Δ_m, it lifts to an automorphism $\tilde{\delta}_c$ of $\tilde{\Delta}$; and since δ_c preserves Δ_{m-1} the lifted action $\tilde{\delta}_c$ will preserve $\tilde{\Delta}_{m-1}$. We can read off from the above expression the action of $\tilde{\delta}_c$ on the fiber Φ of $\tau: \tilde{\Delta} \to \Delta$ over the origin: in terms of the homogeneous coordinates $\beta_0, \dots, \beta_{m-1}$, we have

$$\tilde{\delta}_c: [\beta_0, \dots, \beta_{m-1}] \mapsto [c^{m-1}\beta_0, c^{m-2}\beta_1, \dots, c\beta_{m-2}, \beta_{m-1}].$$

The key fact about this action, for our present purposes, follows immediately from this description:

LEMMA 2.13. *Every orbit of the action of G on $\tilde{\Delta}$ that intersects Φ_0 contains the point Q in its closure.*

We are now prepared to state and prove our main lemma on the geometry of Z_{m-1} and Δ_{m-1}.

LEMMA 2.14. (1) *The fibers of Z_{m-1} over Δ_m are unions of linear spaces.*

(2) *For any arc $\alpha(t)$ in Δ_m tending to the origin, the limiting position of the fiber $Z_{\alpha(t)}$ of Z_{m-1} over $\alpha(t)$ is contained in the complement of Φ_0.*

(3) *Φ itself is an irreducible component of Z_{m-1}.*

Proof. The proof is by induction on m, using Lemma 2.11.

First we introduce a natural stratification of the locus Δ_m. Identifying Δ_m with the space of monic polynomials of degree m in x with no x^{m-1} term, we look at the loci of polynomials with roots of given multiplicity: for any partition $m = m_1 + m_2 + \dots + m_k$ we define the locus $\Delta\{m_1, \dots, m_k\} \subset \Delta$ by

$$\Delta\{m_1, \dots, m_k\} := \{(\alpha_0, \dots, \alpha_{m-2}, 0, \dots, 0): x^m + \alpha_{m-2}x^{m-2} + \dots + \alpha_0$$
$$= (x-\lambda_1)^{m_1}(x-\lambda_2)^{m_2} \cdots (x-\lambda_k)^{m_k} \text{ for some distinct } \lambda_1, \dots, \lambda_k\}.$$

Note that the codimension of $\Delta\{m_1, \dots, m_k\}$ in Δ_m is $\sum(m_\alpha - 1)$.

Suppose α is any point of Δ_m other than the origin. Say α lies in the stratum $\Delta\{m_1, \dots, m_k\}$, and write the corresponding polynomial as

$$(x-\lambda_1)^{m_1}(x-\lambda_2)^{m_2} \cdots (x-\lambda_k)^{m_k},$$

with $\lambda_1, \dots, \lambda_k$ distinct. The fiber S_α of $\mathcal{S} \to \Delta$ over α is a reducible curve consisting of two branches, the x-axis $(y = 0)$ and the curve $y = (x-\lambda_1)^{m_1}(x-\lambda_2)^{m_2} \cdots (x-\lambda_k)^{m_k}$, which meet at the k points $r_1 = (\lambda_1, 0), \dots, r_k = (\lambda_k, 0)$ with multiplicities $m_1, \dots, m_k$.

Let $\Delta(i)$ be the versal deformation spaces $\Delta(S_\alpha, r_i)$ of the singular points $r_i \in S_\alpha$. By the openness of versality the natural map σ from a neighborhood U of α in Δ to the product $\prod \Delta(i)$ has surjective differential at α (the fibers are the equisingular deformations of S_α, in which only the locations of the points r_i on the x-axis vary). Let Δ_{m_i-1} and $\Delta_{m_i} \subset \Delta(i)$ be the loci in $\Delta(i)$ analogous to Δ_{m-1} and Δ_m in Δ, that is, the closures of the loci of deformations of the singular points

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$r_i \in S_\alpha$ with $m_i - 1$ and m_i nodes near r_i respectively. Then in the neighborhood U of α, we have

$$\Delta_m = \sigma^{-1}(\Delta_{m_1} \times \Delta_{m_2} \times \cdots \times \Delta_{m_k})$$

and

$$\Delta_{m-1} = \bigcup_{i=1}^{k} \sigma^{-1}(\Delta_{m_1} \times \cdots \times \Delta_{m_i - 1} \times \cdots \times \Delta_{m_k}).$$

In other words, the locus Δ_{m-1} will have k branches in a neighborhood of α, each containing Δ_m, along the ith of which the fibers of $\mathcal{S} \to \Delta$ will have m_j nodes tending to r_j for each $j \neq i$ and $m_i - 1$ nodes tending to r_i.

We can use this description to give a more intrinsic characterization of the fiber $Z_\alpha = \tau^{-1}(\alpha)$ of Z over the point α, analogous to the one given above for Φ. Briefly, Z_α is the projectivization of the normal space to Δ_m in Δ at α, which is the product of the normal spaces to the Δ_{m_i} in $\Delta(i)$ at the origin; this is just the space of polynomials on the x-axis modulo those vanishing to order m_i at r_i for each i.

We may now apply the induction hypothesis to describe, in these terms, the fiber of Z_{m-1} over α. By the statement of the Lemma for $m = m_i$, the proper transform of the ith branch of Δ_{m-1} will intersect Z_α in the linear subspace of Z_α corresponding to polynomials vanishing to order m_j at r_j for each $j \neq i$; the intersection with Z_α with the proper transform of Δ_{m-1} itself will be the union of these linear subspaces.

This establishes part (1) of the Lemma. Now say that $\alpha(t)$ is any arc in Δ_m tending to the origin; $\alpha(t)$ will lie in some stratum $\Delta\{m_1, \ldots, m_k\}$ for all small $t \neq 0$. As t goes to zero, the singular points $r_i(t)$ of $S_{\alpha(t)}$ approach the point P, so that the limiting position of the intersection with $Z_{\alpha(t)}$ of the proper transform of the ith branch of Δ_{m-1} will be simply the linear space of polynomials whose restriction to the x-axis vanishes to order $m - m_i$ at P; in particular, it is contained in the hyperplane $(\beta_0 = 0) \subset \Phi$ of polynomials vanishing at P. We have thus proved parts (1) and (2) of the Lemma, given part (3) for all $m_i < m$.

Finally, we need to prove for each new value of m that Φ is an irreducible component of Z_{m-1}. Now, by Lemma 2.11, the point $Q = [1, 0, \ldots, 0] \in \Phi$ lies in Z_{m-1}. But we have completely described the closure in Z_{m-1} of the inverse image $\tau^{-1}(\Delta_m \setminus \{0\})$ of the complement of the origin, and Q is not on it. Q must thus lie on an irreducible component of Z_{m-1} not meeting $\tau^{-1}(\Delta_m \setminus \{0\})$, that is to say, an irreducible component of Z_{m-1} contained in Φ; since Z_{m-1} has pure dimension $m - 1$, this irreducible component must be Φ itself. $\square$

For example, here is a picture of Z_1 in the case $m = 2$. In this case Z_1 has only two components, Φ and a component finite of degree 2 over Δ_2.

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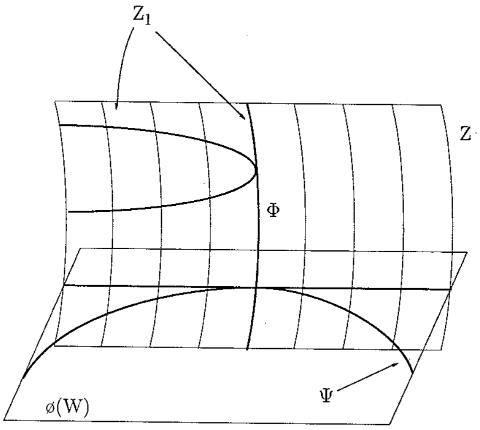

Finally, we deduce

LEMMA 2.15. (1) $\tilde{\Delta}_{m-1}$ *is smooth everywhere along* Φ_0.

(2) *The intersection multiplicity of* $\tilde{\Delta}_{m-1}$ *and* Z *along* Φ *is* m.

Proof. We use the analysis carried out in Lemma 2.11. Let $\tilde{\Lambda}_0$ be the proper transform of the linear space Λ_0 in $\tilde{\Delta}$. Since no component of Z_{m-1} other than Φ passes through Q, the only component of the intersection $\tilde{\Lambda}_0 \cap \tilde{\Delta}_{m-1}$ containing Q will be the proper transform $\tilde{\Psi}$ of the curve $\Psi \subset \Delta$ described in Lemma 2.11. Since this is smooth, and the intersection $\tilde{\Lambda}_0 \cap \tilde{\Delta}_{m-1}$ is proper in a neighborhood of Q ($\tilde{\Lambda}_0$ and $\tilde{\Delta}_{m-1}$ each have dimension m in the $(2m-1)$-dimensional $\tilde{\Delta}$, and their intersection is locally a curve) it follows that $\tilde{\Delta}_{m-1}$ must be smooth at Q. By Lemma 2.13, then, it must be smooth at every point of Φ_0.

For the second statement, notice that Lemma 2.11 asserts that this is true when restricted to the proper transform $\tilde{\Lambda}_0$, and it follows that it is true on $\tilde{\Delta}_{m-1}$ □

End of the proof of Lemma 2.10 *and Proposition* 2.7. We shall now conclude that the intersection of Δ_{m-1} with any subvariety $\Lambda \subset \Delta$ satisfying the hypotheses of Lemma 2.10 – and in particular, the image $\phi(W)$ – is the union of Δ_m and a smooth curve Ψ, such that Ψ has contact of order m with Δ_m at the origin. Notice that this will conclude the proof of Proposition 2.6 as well. To begin with, the proper transform $\tilde{\Lambda}$ of Λ in $\tilde{\Delta}$ intersects Z in a section, crossing Φ at some point R; we likewise have from Lemma 2.9 that $R \in \Phi_0$. $\tilde{\Delta}_{m-1}$ is then smooth at R. Since the tangent space to $\tilde{\Delta}_{m-1}$ at R contains the tangent space to Φ and the tangent space to $\tilde{\Lambda}$ at R is complementary to the tangent space to Φ, $\tilde{\Delta}_{m-1}$ and $\tilde{\Lambda}$ intersect transversely in a smooth curve in a neighborhood of R; since that curve is not tangent to Φ at R, its image $\Phi \subset \Delta_{m-1} \cap \Lambda$ is again a smooth curve. Finally, the intersection number of Ψ with Δ_m in Λ will be the intersection number of $\tilde{\Delta}_{m-1}$, $\tilde{\Lambda}$ and Z at R; which by Lemma 2.15 will be m. We have thus completed the proof of Lemma 2.10 and hence that of Proposition 2.7.

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3. Formulas

Before we prove our formulas, we need a simple result on the order of zeroes and poles of the cross-ratio function ϕ.

3.1. A REMARK ON THE CROSS-RATIO FUNCTION

Suppose we are given a family $f: \mathcal{X} \to B$ over a smooth one-dimensional base B, whose restriction $\tilde{f}: \tilde{\mathcal{X}} = f^{-1}(\tilde{B}) \to \tilde{B}$ to the complement $\tilde{B} = B \setminus \{b_0\}$ of a point $b_0 \in B$ is a family of smooth rational curves; and four sections $p_i: \tilde{B} \to \mathcal{X}$, disjoint over $\tilde{B}$. We get a map $\tilde{\phi}: \tilde{B} \to \bar{M}_{0,4}$, which then extends over B; and the problem is to determine the coefficient of the point b in the pullback via $\tilde{\phi}$ of the boundary components of $\bar{M}_{0,4}$. To put it another way, the cross-ratio of the four sections p_1, p_3, p_2, p_4 defines a rational function on $\tilde{B}$ and hence on B; and we ask simply for the order of zero or pole of this function at b_0.

We will answer this in terms of any completion of our family to a family of nodal rational curves. Recall first of all the set-up of Section 2.1: we have a resolution of singularities $\mathcal{Y} \to B$ of the total space of our family, such that $\mathcal{Y} \to B$ is a family of nodal curves and the extensions of the sections p_i to $\mathcal{Y}$ are disjoint. We then proceed to blow down 'extraneous' components of Y to arrive at the minimal smooth semistable model of our family: that is, a family $\mathcal{Z} \to B$ such that $\mathcal{Z}$ is smooth, the fibers Z_b are nodal, the sections p_i are disjoint and $\mathcal{Z} \to B$ is minimal with respect to these properties. Finally, we blow down the intermediate components in this chain to arrive at a family $\mathcal{W} \to B$ of 4-pointed stable curves. The special fiber W of this family will have just two components (or one, if $\ell = 0$), with a singularity of type A_ℓ at the point of their intersection.

In these terms we prove

LEMMA 3.1. *If the sections p_1 and p_2 (respectively, p_1 and p_3) meet the same component of Y, then the point b_0 is a zero (respectively, pole) of multiplicity ℓ of the function ϕ.*

Proof. We will consider the case where p_1 and p_2 meet the same component of W. Note first that if we blow down the component of W meeting p_1 and p_2, we arrive at a smooth family, that is (replacing B if necessary by a neighborhood of b_0 in B), a product $B \times \mathbb{P}^1$. (Equivalently, we could arrive at this family by blowing down the component of Z meeting p_1 and p_2, then doing the same thing on the resulting surface, and so on ℓ times.) p_3 and p_4 will remain disjoint from each other in this process, and disjoint from p_1 and p_2; but p_1 and p_2 will meet each other with contact of order ℓ: in other words, we can choose an affine coordinate z on $\mathbb{P}^1$ and a local coordinate t on B centered around b_0 so that the sections p_i are given by

$$p_1(t) = t^\ell; \qquad p_2(t) \equiv 0; \qquad p_3(t) \equiv 1; \qquad \text{and } p_4(t) \equiv \infty.$$

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The cross-ratio function is then $\phi(t) = t^\ell/(t^\ell - 1)$, which takes on the value 0 with multiplicity ℓ at $t = 0$. □

3.2. THE RECURSION FOR $\mathbb{F}_2$

Let D be any effective divisor class other than E on the ruled surface $S = \mathbb{F}_2$. We are going to find a formula for the degree $N(D)$ of the variety $V(D) \subset |D|$. To set this up, we start by choosing as usual $r_0(D) - 1$ general points on S, which we label $p_1, p_2, q_3, \ldots, q_{r_0(D)-1}$, and consider the one-parameter family $\mathcal{X} \to \Gamma$ of curves $X \in V(D) \subset |D|$ passing through $\{p_1, p_2, q_3, \ldots, q_{r_0(D)-1}\}$. As before, we let Γ^ν be the normalization of Γ and $\mathcal{X}^\nu \to \Gamma^\nu$ the normalization of the pullback family. Next, we fix general curves C_3 and $C_4 \in |C|$ in the linear series $|C|$, and adopt as usual the convention that we will choose points p_3 and p_4 on the curves X of our family lying on C_3 and C_4 respectively. Making the corresponding base change, we arrive at a family $\mathcal{X} \to B$; as before, we will denote by $\mathcal{Y}$ the minimal desingularization of $\mathcal{X}$ and by $\mathcal{Z} \to B$ the smooth semistable model.

Then we calculate the degree of the cross-ratio map $\phi\colon B \to \bar{M}_{0,4} \cong \mathbb{P}^1$ in two ways by equating the number of zeroes and poles of ϕ. We get one contribution to the degree of $\phi^*(0)$ immediately from the curves X in our family that happen to pass through either of the two points of intersection of C_3 with C_4; this gives a total contribution of $2 \cdot N(D)$ to the degree of $\phi^*(0)$.

The remaining zeroes and poles of ϕ necessarily correspond to reducible curves in the family $\{X\}$. There are two types of these: those that contain E and those that don't. Consider first a reducible curve X in our family that does not contain E. By Proposition 2.5, this must be of the form $X = X_1 + X_2$ where X_i is a general member of the family $V(D_i)$ for some pair of divisor classes D_1 and D_2 adding up to D. In particular, X_i is an irreducible rational curve with $p_a(D_i)$ nodes, and X_1 and X_2 intersect transversely in $(D_1 \cdot D_2)$ points. Moreover, by Proposition 2.6, the curve Γ will consist of $(D_1 \cdot D_2)$ smooth branches near the point $[X]$, corresponding to the points of intersection of X_1 and X_2; thus there are $(D_1 \cdot D_2)$ points in the normalization Γ^ν lying over each such point $[X] \in \Gamma$.

How does such a fiber of the family $\mathcal{X} \to B$ contribute to the degrees of either $\phi^*(0)$ or $\phi^*(\infty)$? It depends on how the points p_i are distributed. If three or four lie on one component, it does not contribute to either, but if there are two on each it may: for example, if p_1 and p_2 lie on the same component – say X_1 – of X, and p_3 and p_4 on the other, we get a zero of ϕ. Now, as we observed in the proof of Proposition 2.5, each component X_i of X must contain exactly $r_0(D_i)$ of the points $p_1, p_2, q_3, \ldots, q_{r_0(D)-1}$. If X_1 is to contain p_1 and p_2, it will contain $r_0(D_1) - 2$ of the points q_α, and X_2 will contain the remaining $r_0(D) - r_0(D_1) + 1 = r_0(D_2)$. Thus, to specify such a fiber, we have first to break the $r_0(D) - 3$ points q_α into disjoint sets of $r_0(D_1) - 2$ and $r_0(D_2)$. The curve X_1 can then be any of the $N(D_1)$ irreducible rational curves in the linear series $|D_1|$ passing through p_1, p_2 and the

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first set, while X_2 can then be any of the $N(D_2)$ irreducible rational curves in the linear series $|D_2|$ passing through the second set. Altogether, then, we see that there will be

$$N(D_1)N(D_2)\binom{r_0(D)-3}{r_0(D_1)-2},$$

points in Γ of this type, and correspondingly

$$N(D_1)N(D_2)(D_1\cdot D_2)\binom{r_0(D)-3}{r_0(D_1)-2},$$

such points in the normalization Γ^ν. Finally, if a fiber of $\mathcal{X}\to B$ lying over such a point of Γ^ν is to contribute to $\phi^*(0)$, we have to choose p_3 and p_4 to lie on X_2, that is, to be any of the $(D_2\cdot C)$ points of intersection of X_2 with C_3 and C_4 respectively. There are thus a total of $(D_2\cdot C)^2$ fibers of $\mathcal{X}\to B$ of this type lying over each such point of Γ^ν.

To complete the calculation of the contribution of fibers of this type to the degree of $\phi^*(0)$, we observe that the fiber of the normalization $\mathcal{X}^\nu$ over such a point will have two components, the normalizations of the curves X_i, meeting at one point (the point of each lying over the new node). Moreover, by Proposition 2.7, the total space $\mathcal{X}^\nu$ will be smooth at such a point; and it follows by Lemma 3.1 that the corresponding point of B will be a simple zero of ϕ. In sum, then, fibers of $\mathcal{X}\to B$ of this type contribute a total of

$$N(D_1)N(D_2)(D_1\cdot D_2)\binom{r_0(D)-3}{r_0(D_1)-2}(D_2\cdot C)^2,$$

to the degree of $\phi^*(0)$.

The contribution of such fibers to the degree of the divisor $\phi^*(\infty)$ is found analogously, the only difference being that, in order to get a pole of the cross-ratio, the points p_1 and p_3 must lie on one component – say X_1 – of X, while p_2 and p_4 will lie on the other. Thus, instead of breaking the $r_0(D)-3$ points q_α into subsets of $r_0(D_1)-2$ and $r_0(D_2)$, we divide them into subsets of $r_0(D_1)-1$ and $r_0(D_2)-1$; and instead of $N(D_1)N(D_2)\binom{r_0(D)-3}{r_0(D_1)-2}$ such points in Γ of this type we have $N(D_1)N(D_2)\binom{r_0(D)-3}{r_0(D_1)-1}$. Similarly, instead of choosing p_3 among the $(D_2\cdot C)$ points of $X_2\cap C_3$, we choose it among the $(D_1\cdot C)$ points of $X_1\cap C_3$; so that instead of $(D_2\cdot C)^2$ zeroes of the cross-ratio lying over each such point of Γ^ν there will be $(D_1\cdot C)(D_2\cdot C)$. Again, each pole of the cross-ratio corresponding to a fiber of this type will have multiplicity one; so the total contribution to the degree of $\phi^*(\infty)$ is

$$N(D_1)N(D_2)(D_1\cdot D_2)\binom{r_0(D)-3}{r_0(D_1)-1}(D_1\cdot C)(D_2\cdot C).$$

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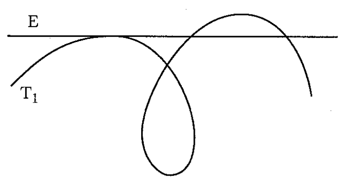
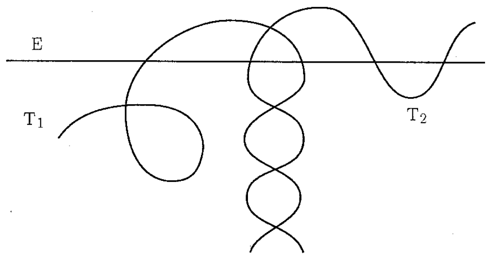
It remains to add up the number of zeroes and poles of ϕ coming from members of our family containing E. Proposition 2.5 describes all such curves, and the description is particularly simple, given that we are on the surface $\mathbb{F}_2$. There are only two types: a degenerate member X of our family must consist either of

(1) the union of E and an irreducible rational nodal curve $X_1 \in |D - E|$, simply tangent at one point (which will be a smooth point of X_1) and meeting transversely elsewhere; or



(2) the union of E and two curves $X_i \in |D_i|$, which will correspond to general points of the varieties $V(D_i)$ for some pair of divisor classes D_1 and D_2 with $D_1 + D_2 = D - E$. In particular, X_1 and X_2 will intersect each other and E transversely.



Now, we can forget about curves of the first type; in fact, since E cannot contain any of the points $p_1, \ldots, p_4$, these will be distinct points of X_1. Hence the cross-ratio function will not be zero or infinite at such a point of B. On the other hand, fibers of the second type may contribute. To see what our family looks like in a neighborhood of such a curve, recall first that by Proposition 2.6, as we approach X along any branch of Γ, all the points of intersection of X_1 and X_2, as well as all but one of the points of intersection of each curve X_i with E, will be old nodes; exactly one of the points of intersection of each X_i with E will be new. The fiber of the normalized family $\mathcal{X}^\nu \to \Gamma^\nu$ will thus consist of the normalizations of X_1 and X_2, each meeting a copy of E in one point and disjoint from each other

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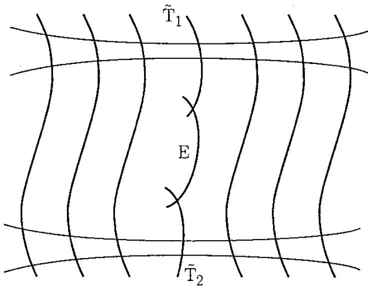

Recall also that the total space of $\mathcal{X}^\nu$ will be smooth along such a fiber.

Again, E can't contain any of the points p_i, and if three or four lie on either curve X_i the corresponding point of B will be neither a zero nor a pole of ϕ; but we may get a contribution if two are on each X_i. Specifically, if p_1 and p_2 lie on one component – say X_1 – and p_3 and p_4 on the other, we get a zero of ϕ; while if p_1 and p_3 lie on a component – again, call this one X_1 – and p_2 and p_4 on the other, we get a pole of ϕ. That said, we can count the number of such fibers exactly as in the preceding case.

We do the zeroes first. We begin by specifying a point $[X]$ in Γ – that is, we break the points q_α into subsets of size $r_0(D_1) - 2$ and $r_0(D_2)$ respectively, and choose X_1 among the $N(D_1)$ irreducible rational curves in $|D_1|$ through p_1, p_2 and the first set and X_2 among the $N(D_2)$ irreducible rational curves in $|D_2|$ through the second set. Next, a point in Γ^ν: we can take any of the $(D_1 \cdot E)(D_2 \cdot E)$ points of Γ^ν lying over $[X] \in \Gamma$. Lastly, we have to choose p_3 and p_4 among the $(D_2 \cdot C)$ points of intersection of X_2 with C_3 and C_4 respectively. We have, in sum,

$$N(D_1)N(D_2)\binom{r_0(D) - 3}{r_0(D_1) - 2}(D_1 \cdot E)(D_2 \cdot E)(D_2 \cdot C)^2,$$

zeroes of ϕ of this type.

The poles of the cross-ratio coming from such curves are counted in the same way; the differences being exactly as in the preceding case: in specifying the point $[X] \in \Gamma$ we have to choose a subset of $r_0(D_1) - 1$ rather than $r_0(D_1) - 2$ of the points q_α; and p_3 must be chosen among the $(D_1 \cdot C)$ points of $X_1 \cap C_3$. There are thus a total of

$$N(D_1)N(D_2)\binom{r_0(D) - 3}{r_0(D_1) - 1}(D_1 \cdot E)(D_2 \cdot E)(D_1 \cdot C)(D_2 \cdot C),$$

poles of this type.

There is one important difference between this case and the previous, however: here, the fiber of the normalization $\mathcal{X}^\nu \to \Gamma^\nu$ has three components, with the components X_1 and X_2 containing the points p_i separated by the component E. Since by Proposition 2.7 the total space $\mathcal{X}^\nu$ is smooth, we see by Lemma 3.1 that

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such points will be double zeroes and poles of ϕ. The contribution to the degrees of these divisors coming from fibers of this type is thus twice the number of such fibers.

We can now calculate the degree of the divisors $\phi^*(0)$ and $\phi^*(\infty)$. We have

$$\begin{aligned}\deg(\phi^*(0)) &= 2\cdot N(D)\\ &\quad+\sum_{\substack{D_1+D_2=D\\ D_1,D_2\neq E}} N(D_1)N(D_2)\binom{r_0(D)-3}{r_0(D_1)-2}(D_1\cdot D_2)(D_2\cdot C)^2\\ &\quad+2\cdot\sum_{\substack{D_1+D_2=D-E\\ D_1,D_2\neq E}} N(D_1)N(D_2)\binom{r_0(D)-3}{r_0(D_1)-2}\\ &\quad\times(D_1\cdot E)(D_2\cdot E)(D_2\cdot C)^2.\end{aligned}$$

Similarly,

$$\begin{aligned}&\deg(\phi^*(\infty))\\ &=\sum_{\substack{D_1+D_2=D\\ D_1,D_2\neq E}} N(D_1)N(D_2)\binom{r_0(D)-3}{r_0(D_1)-1}(D_1\cdot D_2)(D_1\cdot C)(D_2\cdot C)\\ &\quad+2\cdot\sum_{\substack{D_1+D_2=D-E\\ D_1,D_2\neq E}} N(D_1)N(D_2)\binom{r_0(D)-3}{r_0(D_1)-1}\\ &\quad\times(D_1\cdot E)(D_2\cdot E)(D_1\cdot C)(D_2\cdot C).\end{aligned}$$

To express the final result we introduce the notation

$$\begin{aligned}\gamma(D_1,D_2) &:= N(D_1)N(D_2)\left[\binom{r_0(D)-3}{r_0(D_1)-1}(D_1\cdot C)(D_2\cdot C)\right.\\ &\qquad\left.-\binom{r_0(D)-3}{r_0(D_1)-2}(D_2\cdot C)^2\right].\end{aligned}$$

We now write $\deg(\phi^*(0)) = \deg(\phi^*(\infty)))$ and solve the resulting equation for $N(D)$ to arrive at the recursion formula for $N(D)$ on $\mathbb{F}_2$.

THEOREM 3.2 *Let* $D \in \mathrm{Pic}(\mathbb{F}_2)$ *and let* $N(D)$ *be the number of irreducible rational curves in the linear series* $|D|$ *that pass through* $r_0(D)$ *general points of* $\mathbb{F}_2$*; then we have*

$$N(D) = \frac{1}{2}\sum_{\substack{D_1+D_2=D\\ D_1,D_2\neq E}} \gamma(D_1,D_2)(D_1\cdot D_2)$$

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$$+ \sum_{\substack{D_1+D_2=D-E \\ D_1,D_2\neq E}} \gamma(D_1,D_2)(D_1\cdot E)(D_2\cdot E).$$

3.3. THE CLASS $2C$ ON F_n

We now analyze the linear series $|2C|$ on the ruled surface $\mathbb{F}_n$ for any n. We arrive at a closed-form expression for $N(2C)$ rather than a recursion. This is clear: since every linear series $|D|$ on $\mathbb{F}_n$ with $D < 2C$ that actually contains irreducible curves has arithmetic genus 0, we can say immediately how many degenerate fibers of each type there are in our one-parameter family of curves in $|2C|$.

The dimension of $|2C|$ is $3n+2$. The arithmetic genus of the curves in the series is $n-1$, so that the expected dimension of the Severi variety is $r_0(2C) = 2n+3$. This is in fact the actual dimension: any irreducible nodal curve $D \in |2C|$ is be disjoint from E (if it met E, it would contain it, having intersection number 0 with it); so that the nodes of D impose independent conditions on $|2C|$.

So, we choose as usual $2n+2$ general points on $\mathbb{F}_n$, which we label $p_1, p_2, q_3, \ldots$, q_{2n+2} and consider the one-parameter family of curves $X \in |2C|$ passing through $\{p_1, p_2, q_3, \ldots, q_{2n+2}\}$; we denote this family $\mathcal{X} \to \Gamma$. As before, we let Γ^ν be the normalization of Γ and $\mathcal{X}^\nu \to \Gamma^\nu$ the normalization of the pullback family. Next, we fix general curves C_3 and $C_4 \in |C|$ in the linear series $|C|$, and adopt the convention that we choose points p_3 and p_4 on the curves X of our family lying on C_3 and C_4 respectively. Making the corresponding base change, we arrive at a family $\mathcal{X} \to B$; as before, we denote by $\mathcal{Y}$ the minimal desingularization of $\mathcal{X}$ and by $\mathcal{Z} \to B$ the smooth semistable model.

Now we consider the cross-ratio map $\phi\colon B \to \bar{M}_{0,4} \cong \mathbb{P}^1$ as before; we shall obtain a formula for $N(D)$ from

$$\deg \phi^*(0) = \deg \phi^*(\infty).$$

Of course, we get one contribution to the degree of $\phi^*(0)$ from the curves in our family that pass through any of the n points of intersection of C_3 with C_4; this gives a total contribution of $n \cdot N(2C)$ to the degree of $\phi^*(0)$.

The remaining zeroes and poles of ϕ correspond to reducible curves in the family $\{X\}$. As before we look first at curves that do not contain E. They can only be of the form $X = D_1 + D_2$ where D_1 and D_2 are each linearly equivalent to C.

Such a fiber of the family $\mathcal{X} \to B$ can be either a pole or a zero of ϕ, depending of course on how the points p_i are distributed. Namely, if p_1 and p_2 lie on the same component D_i and p_3 and p_4 on the other, we get a zero. To specify such a fiber, we simply have to break the $2n$ points q_α into disjoint sets of $n-1$ and $n+1$. The two components D_i of the curve X will be the (unique) curve in the series $|C|$ containing p_1, p_2 and the first subset; and the unique curve in the series $|C|$ containing the second subset.

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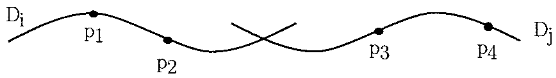
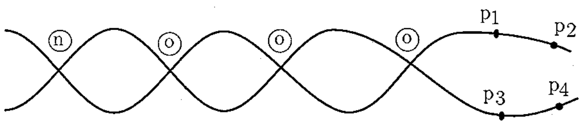


Next, we have to count the number of points of B lying over each point of Γ corresponding to such a curve. Since the general curve of our family has $n-1$ nodes, and the n nodes of $X = D_1 \cup D_2$ impose independent conditions on the series $|2C|$, the curve Γ has n (smooth) branches at the point $[X]$; thus the normalization of Γ has n points lying over $[X]$ (cf. Proposition 2.6). Moreover, for each of these points there will be a point of B for every possible choice of points p_3 and p_4; these can be any of the $(C \cdot C) = n$ points of intersection of the component D_i not containing p_1 or p_2 with C_3 and C_4 respectively. There are thus a total of $\binom{2n}{n-1} \cdot n \cdot n^2$ fibers of $\mathcal{X} \to \Gamma$ of this type.

Now, for each such fiber of $\mathcal{X} \to \Gamma$, the fiber of $\mathcal{X} \to B$ will be simply the normalization of X at the $n-1$ old nodes.



In particular, it has just two components and is stable, and as we have seen $\mathcal{X}^\nu$ already is smooth at the node of such a fiber. Each such point is thus a simple zero of ϕ; so the total contribution to the degree of $\phi^*(0)$ of such curves is

$$\binom{2n}{n-1} \cdot n^3.$$

Similarly, we could have p_1 and p_3 on the same component D_i and p_2 and p_4 on the other; in this case, we get a point of $\phi^*(\infty)$. The only difference in this case is that to specify such a fiber, we have to break the $2n$ points $\{q_\alpha\}$ into two disjoint sets of n points apiece. The two components D_i of the curve X will be the (unique) curves in the series $|C|$ containing p_1 and the first subset; and the unique curve in the series $|C|$ containing p_2 and the second subset. The rest of the analysis is exactly the same – for each such curve, we get n^3 points of B, each of which is a simple pole of the cross-ratio function ϕ – so the total contribution to the degree of $\phi^*(\infty)$ of such curves is

$$\binom{2n}{n} \cdot n^3.$$

The remainder of the calculation will be spent evaluating the contributions to the degree of the pullbacks of the boundary components of $\bar{M}_{0,4}$ coming from the curves in our original family containing E. As we have indicated, these curves are

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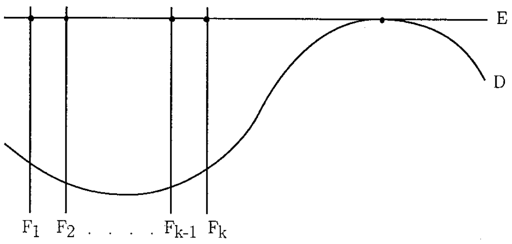
of $n-1$ types: for each $k = 1, \ldots, n-1$ we have a finite number of curves in our family consisting of the sum of E, k fibers $F_1, \ldots, F_k$ of $\mathbb{F}_n \to \mathbb{P}^1$ and an irreducible curve D linearly equivalent to $C + (n-k)F$, with D having a single point of $(n-k)$-fold intersection with E, as pictured below. For each of these types, there are a number of possibilities for the distribution of the points $p_1, \ldots, p_4$ on the various components. For each such distribution corresponding to points b in the inverse image of a boundary component Δ of $\bar{M}_{0,4}$, we consider the number of fibers X_b of that type and the coefficient with which the corresponding points $b \in B$ appear in the divisor $\tilde{\phi}^*(\Delta)$; in the end we will sum up the contributions to arrive at an expression for $N(2C)$ on $\mathbb{F}_n$.



$\bullet_1$ $p_1, p_2 \in D$; $p_3, p_4 \in F_i$. In such a curve, the fiber components F_j must each contain one of the points q_α. To specify such a curve, then, we must first choose a subset of k of the $2n$ points q_α and take $F = \cup F_i$ the unique curve in the linear series $|k \cdot F|$ containing them. Next, we have to single out one of these k points, and label the corresponding fiber F_i. At this point p_3 and p_4 will be determined, as the unique points of intersection of F_i with the curves C_3 and C_4 respectively. Finally, we choose a curve $D \in |C + (n-k)F|$ passing through the remaining $2n-k$ of the points q_α and having a point of $(n-k)$-fold tangency with E. (Note that the ordering of the k points q_α chosen to lie on fibers does not matter; all that counts is which one is chosen to lie on F_i.)

Now, the linear series $|C + (n-k)F|$ cuts on the curve $E \cong \mathbb{P}^1$ the complete linear series $|\mathcal{O}_{\mathbb{P}^1}(n-k)|$. This linear series is parametrized by the space $\mathbb{P}^{n-k}$ of polynomials of degree $n-k$ on $\mathbb{P}^1$ modulo scalars; and in that projective space the divisors consisting simply of $n-k$ times a single point – that is, $(n-k)$th powers of linear forms – form a rational normal curve. In the linear series $|C + (n-k)F|$, then, the locus of curves D having a single point of $(n-k)$-fold intersection with E is a cone over a rational normal curve in $\mathbb{P}^{n-k}$ (with vertex the subseries $E + |(2n-k)F| \subset |C + (n-k)F|$ of curves containing E); in particular, it has degree $n-k$. There are thus exactly $n-k$ curves D linearly equivalent to $|C + (n-k)F|$ passing through p_1, p_2 and the remaining $2n-k$ of the points q_α and having a point of $(n-k)$-fold tangency with E. In sum, the number of fibers X of this type in our family is $\binom{2n}{k} \cdot k \cdot (n-k)$.

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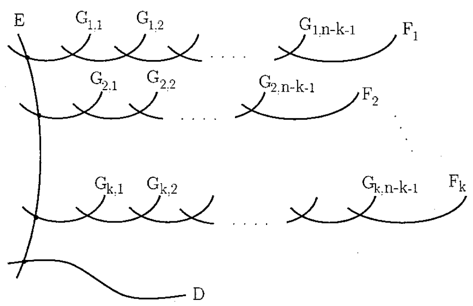
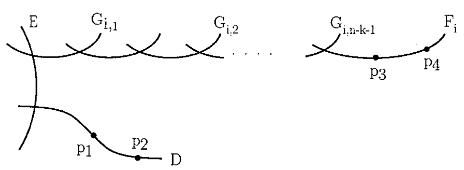
It remains to determine, for each such fiber of our family, the coefficient with which the corresponding point $b \in B$ appears in the pullback divisor $\phi^*(0)$. To do this, we need to know the local geometry of the family near $b \in B$; in particular, we need to have the picture of the corresponding fibers of the families $\mathcal{X} \to B$ and $\mathcal{Y} \to B$. For the first, the only thing we need to know is which of the singular points of the fiber X are limits of nodes of nearby fibers and (in the case of the point of intersection of D with E) how many. The answer, as provided in Proposition 2.7, is that the points of intersection of D with the fibers F_i are all limits of nodes on nearby curves; and the remaining $(n-k-1)$ nodes of the general fiber of the family tend to the point of intersection of D with E. When we normalize the total space of the family, then, the curves D and F_i are pulled apart; and the point of intersection of D with E becomes a node, so that the fiber of $\mathcal{X} \to B$ over b consists of a rational curve E with k fibers $F_1, \ldots, F_k$ and the curve D attached.

But as we also saw in Proposition 2.7, $\mathcal{X}$ will not be smooth: at the point lying over each point of intersection of E with a fiber F_i, $\mathcal{X}$ will have a singularity of type A_{n-k-1}. Resolving each of these, we arrive at this picture of the fiber of $\mathcal{Y} \to B$ over b



Finally, we can blow down the extraneous curves F_j and $G_{j,*}$ for $j \neq i$ to arrive at the picture of the fiber Z of the family $Z \to B$ of semistable 4-pointed curves with smooth total space



Inasmuch as there are $(n-k)$ rational curves in the chain connecting the components D and F_i containing the pairs $\{p_1, p_2\}$ and $\{p_3, p_4\}$, each such fiber

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represents a point of multiplicity $n-k+1$ in the divisor $\phi^*(0)$. In sum, then, the fibers of this type contribute a total of

$$\sum_{k=1}^{n-1} \binom{2n}{k} \cdot k \cdot (n-k) \cdot (n-k+1),$$

to the degree of $\phi^*(0)$.

$\bullet_2$ $p_1, p_3 \in D$; $p_2, p_4 \in F_i$ or $p_2, p_4 \in D$; $p_1, p_3 \in F_i$. In the first of these cases we are simply exchanging the locations of p_2 and p_3 to obtain a fiber X corresponding to a point b in the inverse image $\phi^*(\infty)$; this will affect the count of the number of such fibers, but not the final configuration on the semistable model with smooth total space, so the multiplicity of each such point in the divisor $\phi^*(0)$ will be as in the preceding case $n-k+1$.

The difference here is that, because the fixed point p_2 lies on one of the fiber components, we can put the remaining fiber components through only $k-1$ of the points q_α; at the same time, we can put the curve D through p_1 and the remaining $2n-k+1$. To specify such a curve, then, we must first choose a subset of $k-1$ of the $2n$ points q_α and take $F = \cup F_j$ the unique curve in the linear series $|k \cdot F|$ containing them and p_2; the component of F containing p_2 we call F_i. As in the preceding case, there will be exactly $n-k$ curves in the linear series $|C+(n-k)F|$ passing through the remaining $2n-k+1$ points q_α and the point p_1 and having a point of intersection multiplicity $n-k$ with E; the curve D can be any of these. At this point p_4 will be determined, as the unique point of intersection of F_i with the curve C_4; while p_3 can be taken to be any of the

$$(D \cdot C_3) = ((C + (n-k)F) \cdot C) = 2n - k,$$

points of intersection of D with C_3. The number of fibers X of this type in our family is thus $\binom{2n}{k-1} \cdot (n-k) \cdot (2n-k)$.

As we said, each such fiber X of our family is a pole of order $n-k+1$ of the cross-ratio function ϕ. Finally, since exchanging p_1 with p_4 (as in the second case above) yields an identical result, the total contribution of the fibers of these types to the poles of ϕ is

$$2 \cdot \sum_{k=1}^{n-1} \binom{2n}{k-1} \cdot (n-k) \cdot (2n-k) \cdot (n-k+1).$$

$\bullet_3$ $p_1, p_2 \in D$; $p_3 \in F_i$ and $p_4 \in F_j$, $i \neq j$. This case is very similar to the first; again, we have first to select a subset of k of the $2n$ points q_α and take $F = \cup F_i$ the unique curve in the linear series $|k \cdot F|$ containing them. We then have to single out two of these k points, and label the corresponding fibers F_i and F_j, which in turn determines the points $p_3 = F_i \cap C_3$ and $p_4 = F_j \cap C_4$. Finally, we take as before D to be any of the $n-k$ curves in $|C+(n-k)F|$ passing through p_1 and p_2 and the

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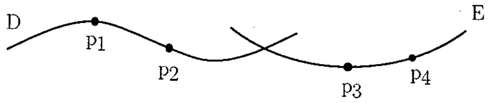
remaining $2n-k$ of the points q_α and having a point of $(n-k)$-fold tangency with E. Thus, the number of fibers X of this type in our family is $\binom{2n}{k}\cdot k\cdot(k-1)\cdot(n-k)$.

At this point, we see another difference with the preceding case: here, to arrive at the fiber of the family of semistable 4-pointed curves with smooth total space we blow down the curves F_m and $G_{m,*}$ for all m including i and j, to arrive at the simpler fiber



Since this is already semistable, each such fiber represents a simple zero of ϕ. In sum, then, the fibers of this type contribute a total of

$$\sum_{k=1}^{n-1}\binom{2n}{k}\cdot k\cdot(k-1)\cdot(n-k),$$

to the degree of $\phi^*(0)$.

For the remaining cases we indicate only the distribution of the points p_i and the resulting contribution; the reader should find no difficulty in supplying the details.

$\bullet_4$ $p_1,p_3\in D$; $p_2\in F_i$ and $p_4\in F_j$, $i\neq j$; or $p_2,p_4\in D$; $p_1\in F_i$ and $p_3\in F_j$, $i\neq j$. The fibers of this type contribute a total of

$$2\cdot\sum_{k=1}^{n-1}\binom{2n}{k-1}\cdot(k-1)\cdot(n-k)\cdot(2n-k),$$

to the degree of $\phi^*(\infty)$.

$\bullet_5$ $p_3,p_4\in D$; $p_1\in F_i$ and $p_2\in F_j$, $i\neq j$. Such fibers contribute a total of

$$\sum_{k=1}^{n-1}\binom{2n}{k-2}\cdot(n-k)\cdot(2n-k)^2,$$

to the degree of $\phi^*(0)$.

$\bullet_6$ $p_1\in D$, $p_2\in F_i$ and $p_3,p_4\in F_j$, $i\neq j$; or $p_1\in F_i$, $p_2\in D$ and $p_3,p_4\in F_j$, $i\neq j$. We get a contribution of

$$2\cdot\sum_{k=1}^{n-1}\binom{2n}{k-1}\cdot(k-1)\cdot(n-k)^2,$$

to the degree of $\phi^*(0)$.

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$\bullet_7$ $p_1 \in D, p_3 \in F_i$ and $p_2, p_4 \in F_j, i \neq j$; or $p_2 \in D, p_4 \in F_i$, and $p_1, p_3 \in F_j$, $i \neq j$. These fibers contribute a total of

$$2 \cdot \sum_{k=1}^{n-1} \binom{2n}{k-1} \cdot (k-1) \cdot (n-k)^2,$$

to the degree of $\phi^*(\infty)$.

$\bullet_8$ $p_3 \in D, p_1 \in F_i$ and $p_2, p_4 \in F_j, i \neq j$; or $p_4 \in D, p_2 \in F_i$, and $p_1, p_3 \in F_j$, $i \neq j$. Contributing a total of

$$2 \cdot \sum_{k=1}^{n-1} \binom{2n}{k-2} \cdot (2n-k) \cdot (n-k)^2,$$

to the degree of $\phi^*(\infty)$.

We come now to the last three cases, those in which none of the four points p_i lie on D. The next one is the last to contribute to the degree of $\phi^*(0)$.

$\bullet_9$ $p_1 \in F_i$, $p_2 \in F_j$ and $p_3, p_4 \in F_\ell$, $i \neq j \neq \ell \neq i$. The total contribution of such fibers is

$$\sum_{k=1}^{n-1} \binom{2n}{k-2} \cdot (k-2) \cdot (n-k)^2.$$

$\bullet_{10}$ $p_1 \in F_i$ $p_3 \in F_j$ and $p_2, p_4 \in F_\ell$; or $p_2 \in F_i$ $p_4 \in F_j$ and $p_1, p_3 \in F_\ell$, $i \neq j \neq \ell \neq i$. Counting both possible exchanges, we see that the total contribution of such fibers to the degree of $\phi^*(\infty)$ is

$$2 \cdot \sum_{k=1}^{n-1} \binom{2n}{k-2} \cdot (k-2) \cdot (n-k)^2.$$

$\bullet_{11}$ $p_1, p_3 \in F_i$ and $p_2, p_4 \in F_j$, $i \neq j$ The total contribution of such fibers to the degree of $\phi^*(\infty)$ is thus

$$\sum_{k=1}^{n-1} \binom{2n}{k-2} \cdot (n-k) \cdot (2n-2k).$$

We are now ready to add up all the contributions to the degrees of $\phi^*(0)$ and $\phi^*(\infty)$, equating the results and solving for $N(2C)$. We have

$$\deg(\phi^*(0)) \;=\; n \cdot N(2C) + n^3 \binom{2n}{n-1}$$

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$$+\sum_{k=1}^{n-1}(n-k)\left[\binom{2n}{k}(k(n-k+1)+k(k-1))\right.$$
$$+\binom{2n}{k-1}(2(k-1)(n-k))$$
$$\left.+\binom{2n}{k-2}((2n-k)^2+(k-2)(n-k))\right].$$

while on the other hand

$$\begin{aligned}&\deg(\phi^*(\infty))\\ &=n^3\binom{2n}{n}+\sum_{k=1}^{n-1}(n-k)\left[\binom{2n}{k-1}(2(2n-k)(n-k+1)\right.\\ &\qquad +2(2n-k)(k-1)+2(k-1)(n-k))\\ &\qquad +\binom{2n}{k-2}(2(2n-k)(n-k)\\ &\qquad \left.+2(k-2)(n-k)+2(n-k))\right].\end{aligned}$$

Combining these, we arrive at the expression

$$n\cdot N(2C)=n^3\left(\binom{2n}{n}-\binom{2n}{n-1}\right)+S,$$

where

$$S=\sum_{k=1}^{n-1}(n-k)\left[\binom{2n}{k}\cdot(-kn)+\binom{2n}{k-1}\cdot 2(2n-k)n\right.$$
$$\left.+\binom{2n}{k-2}\cdot(-kn)\right].$$

(Note that we can now enlarge the formal limits of summation to include $k=0$ without affecting the sum; this will be convenient in the following calculations.) To reduce this further, we separate it into two terms: we write $S=S'-S''$, where

$$S'=\sum_{k=0}^{n-1}4n^2(n-k)\binom{2n}{k-1}$$

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and

$$S'' = \sum_{k=0}^{n-1} (n-k) \cdot kn \cdot \left[\binom{2n}{k} + 2\binom{2n}{k-1} + \binom{2n}{k-2} \right].$$

The expression for S'' telescopes nicely and we have simply

$$S'' = \sum_{k=0}^{n-1} (n-k) \cdot kn \cdot \binom{2n+2}{k}.$$

As for S', we can combine that with the remaining two terms in the expression for $N(2C)$, and together they simplify. To start with, observe that

$$n^3 \left(\binom{2n}{n} - \binom{2n}{n-1} \right) = n^2 \binom{2n}{n-1}.$$

Now, combining this with the expression for S' above, we have

$$\begin{aligned} & n^2 \binom{2n}{n-1} + \sum_{k=0}^{n-1} 4n^2 (n-k) \binom{2n}{k-1} \\ & = n^2 \left(\binom{2n}{n-1} + 4\binom{2n}{n-2} + 8\binom{2n}{n-3} + \cdots + (4n-4)\binom{2n}{0} \right). \end{aligned}$$

Now we use standard binomial identities to reduce this to

$$\begin{aligned} & n^2 \binom{2n}{n-1} + \sum_{k=0}^{n-1} 4n^2 (n-k) \binom{2n}{k-1} \\ & = n^2 \left(\binom{2n+2}{n-1} + 2\binom{2n+1}{n-2} + 3\binom{2n+1}{n-3} + \cdots + n\binom{2n+1}{0} \right) \\ & = n^2 \sum_{k=0}^{n-1} (n-k) \binom{2n+2}{k}. \end{aligned}$$

Finally, we can combine this and the expression above for S'': we have

$$\begin{aligned} n \cdot N(2C) &= n^3 \left(\binom{2n}{n} - \binom{2n}{n-1} \right) + S' - S'' \\ &= n^2 \sum_{k=0}^{n-1} (n-k) \binom{2n+2}{k} - n \sum_{k=0}^{n-1} k(n-k) \binom{2n+2}{k} \\ &= n \sum_{k=0}^{n-1} (n-k)^2 \binom{2n+2}{k}. \end{aligned}$$

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We have therefore proved the

THEOREM 3.3. *Let $N(2C)$ be the number of irreducible rational curves in the linear series $|2C|$ on $\mathbb{F}_n$ passing through $2n+3$ points, then*

$$N(2C) = \sum_{k=0}^{n-1} (n-k)^2 \binom{2n+2}{k}.$$

For example, on $\mathbb{F}_2$ we have $N(2C) = 10$; on $\mathbb{F}_3$ we have $N(2C) = 69$; and on $\mathbb{F}_4$ we have $N(2C) = 406$ and so on.

We now show how to arrive at an expression of $N(2C)$ on $\mathbb{F}_n$ as a coefficient of a simple generating function. We simply write out the sum involved, and then telescope it using the standard binomial relations as before: that is, we write

$$N(2C) = \binom{2n+2}{n-1} + 4\binom{2n+2}{n-2} + 9\binom{2n+2}{n-3} + \cdots + n^2\binom{2n+2}{0}$$

and use the relations

$$\binom{2n+2}{n-1} + \binom{2n+2}{n-2} = \binom{2n+3}{n-1},$$

$$3\binom{2n+2}{n-2} + 3\binom{2n+2}{n-3} = 3\binom{2n+3}{n-2}$$

and so on to rewrite this as

$$\begin{aligned} N(2C) &= \binom{2n+3}{n-1} + 3\binom{2n+3}{n-2} + 6\binom{2n+3}{n-3} \\ &\quad + \cdots + \frac{n(n+1)}{2}\binom{2n+3}{0} \\ &= \sum_{k=0}^{n-1} \binom{n-k+1}{2}\binom{2n+3}{k}. \end{aligned}$$

We can also think of this as the coefficient of t^n in the product of the power series

$$\sum \binom{2n+3}{k} t^k = (1+t)^{2n+3}$$

and

$$\sum \binom{\ell+2}{2} t^\ell = \frac{1}{(1-t)^3},$$

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so that we can write $N(2C)$ on $\mathbb{F}_n$ as the coefficient

$$N(2C) = \left[\frac{(1+t)^{2n+3}}{(1-t)^3}\right]_{t^n}.$$

3.4. A FORMULA FOR $\mathbb{F}_n$

We conclude our paper with a formula for the general ruled surface $\mathbb{F}_n$. Here we define the function $\gamma_{i_1,\ldots,i_t}(D_{i_1},\ldots,D_{i_t})$ giving the contribution to the cross-ratio corresponding to a given decomposition $D = D_1 + D_2 + \cdots + D_t + E$ or $D = D_1 + D_2$, with $D_j \in V_{i_j}(D_j)$. Recall that the variety $V_i(D)$ is the closure in $|D|$ of the locus of irreducible rational curves that have a point of contact of order i with the exceptional curve E. We define

$$\gamma_{i_1,\ldots,i_t}(D_1,\ldots,D_t) := \prod (i_j N_{i_j}(D_j)) \cdot$$

$$\cdot \left[\binom{r_0(D)-3}{r_0^{i_1}(D_1)-1, r_0^{i_2}(D_2)-1, r_0^{i_3}(D_3), \ldots} \right.$$

$$\times \left[\sum_{j\geqslant 3} \frac{(C\cdot D_j)}{i_j} \left(\frac{(C\cdot D_1)}{i_1} + \frac{(C\cdot D_2)}{i_2} \right) - \sum_{j\geqslant 3} \frac{(C\cdot D_j)^2}{i_j} \right]$$

$$- \binom{r_0(D)-3}{r_0^{i_1}(D_1)-2, r_0^{i_2}(D_2), r_0^{i_3}(D_3), \ldots}$$

$$\left. \times \left[\sum_{j\geqslant 2} (C\cdot D_j)^2 \left(\frac{1}{i_j} + \frac{1}{i_1} \right) + \frac{1}{i_1} \sum_{2\leqslant j<k\leqslant t} (C\cdot D_j)(C\cdot D_k) \right] \right]$$

In these terms, we can state

THEOREM 3.4. *Let D be a divisor on the surface $\mathbb{F}_n$. Let $N(D)$ be the number of irreducible rational curves in $|D|$ that pass through $r_0(D)$ general points of $\mathbb{F}_n$. Then*

$$\begin{aligned} nN(D) &= \sum_{D_1+D_2=D} (D_1\cdot D_2)\gamma_{1,1}(D_1,D_2) \\ &\quad + \sum_{t=2}^{n} \sum_{D_1+D_2+\cdots+D_t=D-E} \\ &\quad \times \sum_{i_1,\ldots,i_t} \prod_{j:i_j=1} (E\cdot D_{i_j})\gamma_{i_1,\ldots,i_t}(D_1,\ldots,D_t). \end{aligned}$$

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