GOURSAT'S THEOREM AND THE ZASSENHAUS LEMMA

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1. Introduction. In this paper we study generalized homomorphisms between two algebras, namely the binary relations whose graphs are subalgebras of the direct product of the given algebras.[1] In 1897 Goursat proved that *every subgroup of the direct product of two groups is determined by an isomorphism between factor groups of subgroups of the given groups* (**10**, §§11, 12; **25**, pp. 15, 16). A like result is here shown for a general class of algebras, including loops and quasigroups, by a method due to Riguet (**22**). This result is used to obtain general forms of the Zassenhaus lemma and the Jordan-Hölder-Schreier theorem for normal series (**26**, §9). It is also shown how Goldie's generalization (**8**) of the latter may be derived by these methods.

For easier reading, all results are first proved for groups in §2. Although the results for groups are not new, except proposition 2, the proofs given here carry over without change to the class of algebras considered in §3. It is difficult to judge from the extensive literature whether the J.H.S. theorem for normal series has previously been extended to quasigroups in the present form, because most authors on loops and quasigroups (for example, **1**, **20**) do not count division among the operations; the first to do so was apparently Evans (**7**). The extension of these results to systems with partial and infinitary operations is discussed in §4.

To introduce our notation, we briefly review some concepts from the calculus of binary relations. A *binary relation* between two sets A and B is a triple $\rho = (R, A, B)$, where R is a subset of the Cartesian product $A \times B$, called the *graph* of ρ. One usually writes $a\rho b$ to mean $(a,b) \in R$. Relations of special interest are the *identity* relation ι_A on A, the *converse* $\rho^- = (R^-,B,A)$ of ρ and the *relative product* $\rho\sigma = (RS,A,C)$ of ρ and $\sigma = (S,B,C)$. These are defined by

1.1 $$a\iota_A a' \leftrightarrow a = a' \in A,$$

1.2 $$b\rho^- a \leftrightarrow a\rho b,$$

1.3 $$a\rho\sigma c \leftrightarrow a\rho b \text{ and } b\sigma c \text{ for some } b \in B.$$

We write $\rho \leqslant \rho' = (R',A,B)$ if R is a subset of R'.

One may think of the relation $\rho = (R,A,B)$ as a many-valued mapping of part of A into B and say that

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[1]It was pointed out to me by G. D. Findlay that ordinary homomorphisms have this property.

1.4 ρ is *universally defined* $\leftrightarrow \iota_A \leqslant \rho\rho^-$,
1.5 ρ is *onto* $\leftrightarrow \iota_B \leqslant \rho^-\rho$,
1.6 ρ is *faithful* $\leftrightarrow \rho\rho^- \leqslant \iota_A$,
1.7 ρ is *single-valued* $\leftrightarrow \rho^-\rho \leqslant \iota_B$.

If $\kappa = (K,A,A)$, one says that

1.8 κ is *symmetric* $\leftrightarrow \kappa^- \leqslant \kappa$,
1.9 κ is *transitive* $\leftrightarrow \kappa\kappa \leqslant \kappa$,
1.10 κ is *reflexive* $\leftrightarrow \iota_A \leqslant \kappa$.

An equivalence relation satisfies all of these three; but relations which are merely symmetric and transitive are also of interest. Any such relation satisfies

$$\kappa^- = \kappa, \qquad \kappa\kappa = \kappa$$

and has the same graph as an equivalence relation on a subset of A. The following definition is due to Riguet **(22)**.

1.11 ρ is *difunctional* $\leftrightarrow \rho\rho^-\rho \leqslant \rho$.

This is easily seen to imply $\rho\rho^-\rho = \rho$ and means that whenever $a\rho b'$, $a'\rho b'$ and $a'\rho b$ then $a\rho b$. Riguet showed that such a ρ determines a one-to-one correspondence between equivalence classes on subsets of A and B respectively.

We shall write

$$a\rho = \{b \mid a\rho b\};$$

more generally, for any subset A' of A,

$$A'\rho = \{b | a\rho b \text{ for some } a \in A'\}.$$

In particular, $A\rho$ is the *range* of ρ, $B\rho^-$ is its *domain*. The following rules are well known and will be used freely:

$$(\rho\sigma)\tau = \rho(\sigma\tau),$$
$$\rho\iota_B = \rho = \iota_A\rho,$$
$$\iota_A^- = \iota_A, \qquad (\rho^-)^- = \rho, \qquad (\rho\sigma)^- = \sigma^-\rho^-,$$
$$A'(\rho\sigma) = (A'\rho)\sigma.$$

We often take advantage of the first and last of these to write without brackets $\rho\sigma\tau$ and $A'\rho\sigma$.

If $\kappa = (K,A,A)$ is any binary relation such that $\kappa \leqslant \kappa\kappa = \kappa^2$, then $\kappa^n \leqslant \kappa^{n+1}$ for all $n \geqslant 1$. The union $\kappa^* = (K^*,A,A)$ of all κ^n is called the *transitive closure* of κ, since $\kappa \rightarrow \kappa^*$ is a closure operation and $\kappa = \kappa^*$ if and only if κ is transitive.

One easily verifies that

1.12 $\kappa^*\kappa^* = \kappa^*\kappa = \kappa^*, \qquad \kappa^{*-} = \kappa^{-*}$.

For any binary relation $\rho = (R,A,B)$, Riguet **(23)** defines its *difunctional closure*

1.13 $\rho^+ = (\rho\rho^-)^*\rho = \rho(\rho^-\rho)^*$.

$\rho \rightarrow \rho^+$ is again a closure operation and $\rho = \rho^+$ if and only if ρ is difunctional.

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2. Homomorphic relations between groups. To generalize the notion of a homomorphism of a group A into a group B, we call the binary relation $\rho = (R,A,B)$ *homomorphic* if and only if

(i) $1\rho 1$,

(ii) if $a\rho b$ then $a^{-1}\rho b^{-1}$,

(iii) if $a\rho b$ and $a'\rho b'$ then $aa'\rho bb'$.

Clearly then, *ρ is homomorphic if and only if its graph R is a subgroup of the direct product $A \times B$.* It is easily verified that the identity relation, the converse of a homomorphic relation and the relative product of two homomorphic relations are all homomorphic. One also verifies for any homomorphic $\rho = (R,A,B)$ that *if A' is a subgroup of A then $A'\rho$ is a subgroup of B.*

A homomorphic equivalence relation is usually called a *congruence* relation. We shall call *subcongruence* any homomorphic relation which is transitive and symmetric without necessarily being reflexive. If $\kappa = (K,A,A)$ is such a subcongruence on A, it induces a congruence relation $(K,A\kappa,A\kappa)$ on its range $A\kappa$. The factor group of $A\kappa$ modulo κ is usually written $A\kappa/\kappa$, we shall call it a *subfactor* of A. We define $\bar{\kappa} = (\bar{K},A,A\kappa/\kappa)$ by

2.1 $$a\bar{\kappa}(a'\kappa) \leftrightarrow a\kappa a',$$

so that $a\bar{\kappa} = a\kappa$. A simple calculation shows that

2.2 $$\bar{\kappa}\bar{\kappa}^{-} = \kappa,\ \bar{\kappa}^{-}\bar{\kappa} = \iota_{A\kappa/\kappa},$$

whence

2.3 $$\bar{\kappa}^{-}\kappa\bar{\kappa} = \iota_{A\kappa/\kappa},$$

Note that $\bar{\kappa}$ induces the well-known natural homomorphism $(\bar{K},A\kappa,A\kappa/\kappa)$.

PROPOSITION 1 (Riguet). *If $\rho = (R,A,B)$ is a difunctional homomorphic relation (between two groups) then*

(i) *$\kappa = \rho\rho^{-}$ is a subcongruence of A with range $B\rho^{-}$,*

(ii) *$\lambda = \rho^{-}\rho$ is a subcongruence of B with range $A\rho$,*

(iii) *ρ induces an isomorphism μ between subfactors $A\kappa/\kappa$ and $B\lambda/\lambda$ such that*

$$(a\kappa)\mu(b\lambda) \text{ if and only if } a\rho b.$$

Conversely, every isomorphism between subfactors of A and B is induced in this way.

Proof. In view of 1.11 we have

$$(\rho\rho^{-})(\rho\rho^{-}) = (\rho\rho^{-}\rho)\rho^{-} = \rho\rho^{-},$$

and anyway

$$(\rho\rho^{-})^{-} = (\rho^{-})^{-}\rho^{-} = \rho\rho^{-},$$

hence $\rho\rho^{-}$ is a subcongruence by 1.9 and 1.8. Moreover

$$A(\rho\rho^{-}) = (A\rho)\rho^{-} \subseteq B\rho^{-},$$

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and $b\rho^{-}$ is empty unless $b \in A\rho$, hence

2.4 $$A\rho\rho^{-} = B\rho^{-}.$$

This establishes (i), and by symmetry (ii). Let $\bar{\kappa} = (\bar{K},A,A\kappa/\kappa)$ and $\bar{\lambda} = (\bar{L},B,B\lambda/\lambda)$ be defined by 2.1 and put

2.5 $$\mu = \bar{\kappa}^{-}\rho\bar{\lambda},$$

then

2.6 $$\mu^{-}\mu = \iota_{B\lambda/\lambda}, \quad \mu\mu^{-} = \iota_{A\kappa/\kappa};$$

for

$$\begin{aligned}\mu^{-}\mu &= \bar{\lambda}^{-}\rho^{-}\bar{\kappa}\bar{\kappa}^{-}\rho\bar{\lambda} = \bar{\lambda}^{-}\rho^{-}\kappa\rho\bar{\lambda} \\ &= \bar{\lambda}^{-}\rho^{-}\rho\rho^{-}\rho\bar{\lambda} = \bar{\lambda}^{-}\rho^{-}\rho\bar{\lambda} = \bar{\lambda}^{-}\lambda\bar{\lambda} = \iota_{B\lambda/\lambda},\end{aligned}$$

by 2.5, 2.2, (i), 1.11, (ii) and 2.3. Hence μ is an isomorphism between $A\kappa/\kappa$ and $B\lambda/\lambda$, by 1.4 to 1.7. A further calculation shows that

2.7 $$\bar{\kappa}\mu\bar{\lambda}^{-} = \rho.$$

This gives a "canonical decomposition" **(18)** of ρ and is paraphrased by (iii).

Conversely, let μ be a given isomorphism between given subfactors $A\kappa/\kappa$ and $B\lambda/\lambda$ of A and B respectively. If ρ is defined by 2.7, a computation will show that

$$\rho\rho^{-} = \kappa, \qquad \rho^{-}\rho = \lambda, \qquad \rho\rho^{-}\rho = \rho.$$

This completes the proof of Proposition 1, which may also be written more concisely thus:

PROPOSITION 1′. *Subfactors $A\kappa/\kappa$ and $B\lambda/\lambda$ of (groups) A and B respectively are isomorphic if and only if there exists a difunctional homomorphic relation $\rho = (R,A,B)$ such that $\rho\rho^{-} = \kappa$ and $\rho^{-}\rho = \lambda$.*

The importance of the above derives from the following:

PROPOSITION 2. *Any homomorphic relation $\rho = (R,A,B)$ (between two groups) is difunctional.*

Proof. Write $f_3(x,y,z) = xy^{-1}z$, then

(†) $$f_3(x,y,y) = x, \quad f_3(y,y,z) = z.$$

Now let $a, a' \in A$ and $b, b' \in B$ and assume

$$a\rho b', \; a'\rho b', \; a'\rho b$$

then

$$a = f_3(a,a',a') \; \rho \, f_3(b',b',b) = b.$$

Thus ρ is difunctional by 1.11.

Propositions 1 and 2 together give Goursat's characterization of the subgroups of the direct product of two groups, since all such subgroups are graphs of homomorphic relations between the groups. We have seen that Proposition 1 yields a characterization of all isomorphisms between subfactors of groups. Thus the isomorphism of Zassenhaus (**26**, p. 54) may be obtained as follows.

PROPOSITION 3. *If κ and λ are subcongruences of (a group) A, then $\kappa\lambda$ induces an isomorphism between the subfactors of A modulo $\kappa\lambda\kappa$ and $\lambda\kappa\lambda$.*

Proof. By proposition 2, $\kappa\lambda$ is difunctional, hence by Proposition 1 we have subcongruences

$$(\kappa\lambda)(\kappa\lambda)^{-} = \kappa\lambda\lambda^{-}\kappa^{-} = \kappa\lambda\kappa$$

and

$$(\kappa\lambda)^{-}(\kappa\lambda) = \lambda^{-}\kappa^{-}\kappa\lambda = \lambda\kappa\lambda,$$

whose associated subfactors are isomorphic.

Zassenhaus used this result in a somewhat different form to prove the J.H.S. theorem for normal series. For the purpose of generalization, it is of interest to see how this can be done using Proposition 3 in the present form.

If C is a subgroup of A and κ is a subcongruence of A such that $C \subseteq C\kappa$, we shall call κ a subcongruence of A *over* C. Following Goldie (**9**), we call *normal series from A' to C'* any m-tuple $\kappa_1, \ldots \kappa_m$ of subcongruences of A over C such that

2.8 $$A' = A\kappa_1,\ C\kappa_1 = A\kappa_2, \ldots, C\kappa_{m-1} = A\kappa_m,\ C\kappa_m = C'.$$

Ultimately we are only interested in the case $A' = A$, $C' = C$; the more comprehensive definition is useful in the following.

PROPOSITION 4. *If λ is any subcongruence of (a group) A over (a subgroup) C then any normal series of subcongruences of A over C from A' to C' gives rise to a normal series from $A'\lambda$ to $C'\lambda$.*

Proof. If $\rho = (R,A,B)$ is difunctional and A_0 and B_0 are subgroups of A and B respectively such that $B_0 \subseteq A_0\rho$ and $A_0 \subseteq B_0\rho^{-}$ then

2.9 $$A_0\rho\rho^{-} = B_0\rho^{-}, \qquad B_0\rho^{-}\rho = A_0\rho,$$

since for instance

$$A_0\rho \subseteq B_0\rho^{-}\rho \subseteq A_0\rho\rho^{-}\rho = A_0\rho$$

by 1.11. If κ and λ are both subcongruences of A over C, it follows from 2.9, by taking $B = A$, $B_0 = A_0 = C$ and $\rho = \kappa\lambda$ or $\lambda\kappa$ that

2.10 $$C\kappa\lambda\kappa = C\lambda\kappa, \quad C\lambda\kappa\lambda = C\kappa\lambda.$$

It follows similarly from 2.4 with $B = A$ that

2.11 $$A\kappa\lambda\kappa = A\lambda\kappa, \quad A\lambda\kappa\lambda = A\kappa\lambda.$$

Using 2.8, 2.10 and 2.11 we compute

$$\begin{aligned} A'\lambda &= A\kappa_1\lambda = A\lambda\kappa_1\lambda, \\ C\lambda\kappa_i\lambda &= C\kappa_i\lambda = A\kappa_{i+1}\lambda = A\lambda\kappa_{i+1}\lambda \qquad (i = 1, \ldots m-1), \\ C\lambda\kappa_m\lambda &= C\kappa_m\lambda = C'\lambda, \end{aligned}$$

thus establishing the analogue of 2.8 which completes the proof.

In view of Propositions 3 and 4, we may now state the Jordan-Hölder-Schreier-Zassenhaus theorem in the following form:

PROPOSITION 5. *If $\kappa_1, \ldots \kappa_m$ and $\lambda_1, \ldots \lambda_n$ are normal series from A to C then the rectangular array $\{\kappa_i\lambda_j\kappa_i\}_{i\leqslant m, j\leqslant n}$ may be ordered by rows to give a "refinement" of the former, the array $\{\lambda_j\kappa_i\lambda_j\}_{i\leqslant m, j\leqslant n}$ may be ordered by columns to give a refinement of the latter, and corresponding entries of the two arrays determine isomorphic subfactors.*

3. Generalization to other algebraic systems. By an n-ary *operation* f_n on a set A is understood a mapping which assigns to each n-tuple of elements of A a single element of A, n being some finite non-negative integer. In particular, a 0-ary operation is a constant. We thus consider operations which are *finitary*, *universally defined* and *single-valued.*

Let F be a set of operation symbols with prescribed subscripts. An *algebra*, in the sense of Birkhoff **(3)**, is a representation of such a set of symbols as n-ary operations on a set A, and may be denoted by ${}_FA$. If A' is a subset of A closed under all the operations in F, the induced representation ${}_FA'$ is called a *subalgebra* of A. Two algebras ${}_FA$ and ${}_GB$ are called *similar* if $F = G$; if no confusion is likely to arise, they may be denoted merely by A and B. The Cartesian product $A \times B$ of two similar algebras is turned into another algebra of the same kind, called the *direct product* of A and B, by the familiar device, here illustrated by a unary operation,

$$f_1(a, b) = (f_1a, f_1b) \qquad (a \in A, b \in B).$$

We define *homomorphic* relations between A and B as binary relations whose graphs are subalgebras of $A \times B$. In particular, a *homomorphism* satisfies 1.4 and 1.7, an *isomorphism* also 1.5 and 1.6; a *subcongruence* satisfies 1.8 and 1.9, a *congruence* also 1.10. Factors and subfactors are defined as for groups.

Compound operations on an algebra are obtained by composition from the given operations and the selection operations I_{nk}:

$$I_{nk}(x_1, \ldots x_n) = x_k \qquad (1 \leqslant k \leqslant n).$$

Mal'cev has introduced the notion of a *primitive class* of algebras **(19)**. This is a maximal class of similar algebras subject to a given set of postulates expressed as identities between compound operations. For instance, the primitive class of groups consists of all sets with three operations

$$f_0 = 1, \quad f_1x = x^{-1}, \quad f_2(x, y) = x\cdot y,$$

subject to the postulates

$$\begin{aligned} f_2(f_2(x,y),z) &= f_2(x,f_2(y,z)), \\ f_2(f_0,x) &= x = f_2(x,f_0), \\ f_2(x,f_1x) &= f_0 = f_2(f_1x,x). \end{aligned}$$

The proofs of Propositions 1 to 5 were purposely stated in such a way that they remain valid, with minor modification of terminology, when the primitive class of groups is replaced by other primitive classes of algebras, as follows.

THEOREM I. *Proposition* 1 *holds for any primitive class of algebras, propositions* 2 *to* 5 *hold for any primitive class with a compound operation* f_3 *satisfying*

$$(\dagger) \qquad f_3(x,y,y) = x, \quad f_3(y,y,z) = z.$$

This result applies to loops and generalized loops, as seen from the first two of the following examples.

Example 1. Smiley **(24)** has considered what might be called *left-loops*, with operations 1, $\cdot$,/ subject to

$$(z/y)\cdot y = z, \quad (x\cdot y)/y = x, \quad 1\cdot y = y.$$

Taking $f_3(x,y,z) = (x/y)\cdot z$, we easily verify ($\dagger$) (see Proposition 2). Every left-loop A has a one-element subalgebra $A_0 = \{1\}$. It is easily verified that every subcongruence κ of A is uniquely determined by its range $A\kappa$ and its *kernel* $A_0\kappa$. In fact, $a\kappa a'$ if and only if $a, a' \in A$ and $a/a' \in A_0\kappa$, that is $a \in (A_0\kappa)\cdot a'$. The subfactor $A\kappa/\kappa$ may then be denoted by $A\kappa/A_0\kappa$, to conform with the usual notation in group theory. If $\rho = (R,A,B)$ is any homomorphic relation between left-loops, the subcongruence $\rho\rho^-$ gives rise to the subfactor

$$\frac{A\rho\rho^-}{A_0\rho\rho^-} = \frac{B\rho^-}{B_0\rho^-},$$

in view of 2.4 and 2.9. Hence, by Theorem I, *any homomorphic relation* $\rho = (R,A,B)$ *between left-loops induces an isomorphism between*

$$\frac{B\rho^-}{B_0\rho^-}$$

and

$$\frac{A\rho}{A_0\rho}.$$

The Zassenhaus lemma is easily seen to take the familiar form: *If*

$$\frac{U}{U'}$$

and

$$\frac{V}{V'}$$

are subfactors of a left-loop then

$$\frac{U'\cdot(U\cap V)}{U'\cdot(U\cap V')}\cong\frac{V'\cdot(V\cap U)}{V'\cdot(V\cap U')}.$$

Example 2. A *quasigroup* is a system with operations $\cdot, /, \mathbf{u}$ subject to

$$(z/y)\cdot y = z,\ (x\cdot y)/y = x,\ y\cdot(y\mathbf{u}z) = z,\ y\mathbf{u}(y\cdot x) = x.$$

Taking $f_3(x, y, z) = (x\cdot(y\mathbf{u}y))/(z\mathbf{u}y)$, we easily verify (†).[2]

Example 3. *A relatively complemented lattice* (**2**, p. 105) is a lattice in which to every element $a \cup b$ between a and $a \cup b \cup c$ there corresponds a so-called relative complement $g_3(a, b, c)$ such that

$$(a \cup b) \cup g_3(a,b,c) = a \cup b \cup c,$$
$$(a \cup b) \cap g_3(a,b,c) = a.$$

If we reckon g_3 among the operations of a relatively complemented lattice, we may put $f_3(a, b, c) = g_3(a, b, c) \cap g_3(c, b, a)$ and verify (†) by computation.[3]

Birkhoff (**2**, p. 89) has proved the J.H.S.Z. theorem for principal series for algebras whose congruence relations permute, that is, where $\kappa\lambda = \lambda\kappa$ for any two congruence relations κ and λ. It is therefore of interest that the following three statements about a primitive class of algebras are equivalent:

M1. *There is a compound operation f_3 satisfying* (†).

M2. *All homomorphic relations are difunctional.*

M3. *All pairs of congruence relations on the same algebra permute.*

In fact, M1 implies M2 by Theorem I, M2 implies M3, because

$$\kappa\lambda = \iota\kappa\lambda\iota \leqslant \lambda\kappa\lambda\kappa = (\lambda\kappa)(\lambda\kappa)^{-}(\lambda\kappa) = \lambda\kappa,$$

by 1.10 and 1.11, and symmetrically $\lambda\kappa \leqslant \kappa\lambda$. Finally, Mal'cev has shown that M3 implies M1, by an ingenious argument involving the free algebra with three generators in the primitive class.

Goldie (**8**) has generalized the J.H.S.Z. theorem for normal series to a larger class of algebras satisfying a condition of "weak associability," which is equivalent to our 2.10. We briefly indicate how Goldie's result can be deduced from proposition 1.

If A is an algebra with finitary operations then the union of any increasing sequence of subalgebras is also a subalgebra, as is well known.[4] In particular, if $\kappa = (K, A, A)$ is a homomorphic relation such that $\kappa \leqslant \kappa\kappa = \kappa^2$, then its transitive closure κ^* is also homomorphic. If A and B are similar algebras with finitary operations, it follows that the difunctional closure ρ^+ of a homomorphic relation ρ, defined by 1.13, is also homomorphic. Writing ρ^+ for ρ in Proposition 1, we find that any homomorphic ρ gives rise to two subcongruences

[2]Essentially this formula appears in the paper by Mal'cev (**19**).

[3]This construction of f_3 is implicit in the proof by Dilworth (**4**) that congruence relations on a relatively complemented lattice permute.

[4]See for instance the proof by Kurosh (**15**, p. 48) for groups.

3.1 $$\rho^{+}\rho^{+-} = (\rho\rho^{-})^{*}, \quad \rho^{+-}\rho^{+} = (\rho^{-}\rho)^{*},$$

whose associated subfactors are isomorphic. (Equation 3.1 follows from 1.13 and 1.12.) In place of Proposition 3, we find that, given subcongruences κ and λ, their relative product $\kappa\lambda$ induces an isomorphism between the subfactors belonging to the subcongruences $(\kappa\lambda\kappa)^{*}$ and $(\lambda\kappa\lambda)^{*}$. Modified Propositions 4 and 5 for normal series from A to C can then be deduced from 2.10, which is now postulated, with little more trouble than for groups. We may ask what class of algebras satisfies 2.10.

PROPOSITION 6. *Let A be an algebra with a compound operation f_3 and a subalgebra C such that*

$$f_3(a,c,c) = a, \qquad f_3(a,a,c) \in C,$$

for all $a \in A$, $c \in C$. If $\kappa = (K, A, A)$ is any homomorphic relation such that $C \subseteq C\kappa$ then

3.2 $$C\kappa\kappa^{-} = C\kappa^{-}.$$

Proof. Let $a \in C\kappa\kappa^{-}$, then $c\kappa a$ and $a\kappa a'$ for some $c \in C$ and $a' \in A$. Now $c\kappa c'$ for some $c' \in C$, since $C \subseteq C\kappa$. Hence

$$a = f_3(a,c,c)\ \kappa\, f_3(a',a',c') = c'' \in C,$$

so that $C\kappa\kappa^{-} \subseteq C\kappa^{-}$. The converse follows immediately from $C \subseteq C\kappa$. Note that 2.10 can be derived from 3.2 as from 2.9.

Example 4. Let us call *left-quasigroup* any system A with two binary operations $\cdot$ and $/$ satisfying

$$(z/y)\cdot y = z, \quad (x\cdot y)/y = x.$$

Let $f_3(x, y, z) = (x/y)\cdot z$, then $f_3(x, y, y) = x$. Given $a \in A$, call a/a a *left-unit*, and let C be the subalgebra generated by the left-units, then for any $c \in C$ we have $f(a, a, c) = (a/a)\cdot c \in C$.[5]

Example 5. Consider a system A with 1 and / satisfying

$$x/x = 1, \quad x/1 = x.$$

Let $C = \{1\}$, $f_3(x, y, z) = (x/y)/z$ or merely x/y, and verify that $f_3(a, a, 1) = 1, f_3(a, 1, 1) = a$.

4. Discussion of further generalizations. In §3 we demanded that the operations of an algebra be finitary, universally defined and single-valued. We shall briefly discuss what happens when one or more of these restrictions are relaxed.

By an *infinitary* operation on A is usually understood a mapping which assigns a value in A to any sequence of elements in A, a sequence being a

[5]Murdoch **(20)** has carried out a somewhat similar construction for certain quasigroups, without counting division among the operations.

mapping of the set I of natural numbers into A. Looking back at the proofs of propositions 1 to 5, we find that no use has been made of the fact that the operations in a group are finitary, except in the definition of compound operations and therefore in the concept of primitive class. Now the only property of a primitive class used here is the possibility of making certain constructions without going outside the class. We thus have

THEOREM II. *Theorem I remains valid for a class of similar algebras with some infinitary operations, provided the class contains all subalgebras, factors and direct products of algebras in the class.*

It seems that Goldie's generalization cannot be thus extended, as long as it depends on the fact, not in general valid for algebras with infinitary operations, that the union of an increasing sequence of subalgebras is also a subalgebra.

Example 6. A *Boolean σ-algebra* (**2**, p. 167) is a Boolean algebra with an infinitary operation "sup" such that for any sequence α of elements $1\alpha, 2\alpha, \ldots$ and any element a, $\sup \alpha \leqslant a$ if and only if $i\alpha \leqslant a$ for all $i \in I$. If $+$ denotes the so-called symmetric difference, we write $f_3(x, y, z) = x + y + z$ and verify (†). This example may be generalized to relatively complemented σ-lattices, in view of Example 3.

The familiar limit operation of analysis is a *partial* operation, in the sense that it is defined only for some sequences, which are called *convergent*. We shall use it here to illustrate arbitrary partial operations, be they finitary or infinitary, availing ourselves of the ready-made terminology that goes with the limit concept.

We consider an algebra A with a partial infinitary operation "lim." No connection is assumed between the algebraic structure of A and its "topological" structure. Interest centres on subalgebras of A which are *closed*, that is, closed under "lim," and on homomorphic relations with closed graphs. But even if ρ is an ordinary homomorphism of A into B with closed graph, B' a closed subalgebra of B, it is not in general true that $B'\rho^-$ is closed, unless ρ is *continuous*. Moreover, in extending results of group theory to groups with a limit operation, one aims to make sure that all relevant isomorphisms are *bicontinuous*. Of interest are therefore classes of similar algebras with the so-called *closed-graph property: Every isomorphism between two members of the class is continuous if it has a closed graph.*

Example 7. Given any sequence α whose set of terms is $I\alpha = \{1\alpha, 2\alpha, \ldots\}$, a *subsequence* has the form $\sigma\alpha$, where σ is a mapping of I into I. Kuratowski (**14**, p. 84) imposes the following postulates on the operation "lim."

K1. *If* $\lim \alpha = a$ *then* $\lim \sigma\alpha = a$.

K2. *If* $i\alpha = a$ *for all* $i \in I$ *then* $\lim \alpha = a$.

K3. *If every subsequence* $\sigma\alpha$ *of* α *has a subsequence* $\tau\sigma\alpha$ *such that* $\lim \tau\sigma\alpha = a$ *then* $\lim \alpha = a$.

We make no use of K2, instead we demand *compactness:*

K4. *Every sequence has a convergent subsequence.*

It is now easily shown that if A and B are sets with a limit operation satisfying K1, K3 and K4, then any one-to-one correspondence between A and B is continuous if it has a closed graph.

Theorem II can be extended to such systems as loops and quasigroups with a limit operation satisfying K1, K3 and K4. We refrain from carrying out this extension here, since the study of infinite sequences should really be replaced by that of *nets* (**11**, II). It is known that the class of locally compact quasigroups with countable base enjoys the closed-graph property (**19**, theorem 12). This suggests further extension of our results.

In view of the extensive literature on multigroups and related systems (for example, **6**; **13**), we should say a word about them. A *many-valued* operation assigns non-empty subsets of A to n-tuples of elements of A. The following example may give a clue to the extension of the present results to systems with many valued operations.

Example 8. Kuntzmann (**13**) introduces a *multiform system* with three many-valued operations $\cdot$, $/$, $\mathbf{u}$ subject to

$$c \in a \cdot b \leftrightarrow a \in c/b, \; c \in a \cdot b \leftrightarrow b \in a\mathbf{u}c.$$

Writing $f_3(x, y, z) = (x \cdot (y\mathbf{u}y))/(z\mathbf{u}y)$, in the sense of operations on complexes, we easily verify that

$$x \in f_3(x, y, y), \quad z \in f_3(y, y, z).$$

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