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# ON WEAKLY AMPLE SEMIGROUPS

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**Abstract**

Let $S$ be a semigroup. Elements $a, b$ of $S$ are $\widetilde{\mathscr{R}}$-related if they have the same idempotent left identities. Then $S$ is weakly left ample if (1) idempotents of $S$ commute, (2) $\widetilde{\mathscr{R}}$ is a left congruence, (3) for any $a \in S$, $a$ is $\widetilde{\mathscr{R}}$-related to a (unique) idempotent, say $a^+$, and (4) for any element $a$ and idempotent $e$ of $S$, $ae = (ae)^+a$. Elements $a, b$ of $S$ are $\mathscr{R}^*$-related if, for any $x, y \in S^1$, $xa = ya$ if and only if $xb = yb$. Then $S$ is left ample if it satisfies (1), (3) and (4) relative to $\mathscr{R}^*$ instead of $\widetilde{\mathscr{R}}$. Further, $S$ is (weakly) ample if it is both (weakly) left and right ample. We establish several characterizations of these classes of semigroups. For weakly left ample ones we provide a construction of all such semigroups with zero all of whose nonzero idempotents are primitive. Among characterizations of weakly ample semigroups figure (strong) semilattices of unipotent monoids, and among those for ample semigroups, (strong) semilattices of cancellative monoids. This describes the structure of these two classes of semigroups in an optimal way, while, for the 'one-sided' case, the problem of structure remains open.

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## 1. Introduction and summary

The theory of semigroups represents an irresistible temptation to explore various possible generalizations. Maybe the most successful one of them is the one initiated by Fountain and continued by his collaborators. Of particular interest in the present context are his papers [1–3]. As a result there emerged a number of new classes of semigroups satisfying copious conditions. In addition. the class of unipotent semigroups came into prominence taking the place of groups, possibly satisfying a cancellation law and/or being monoids. This is a building block while one of the main classes is that of weakly left ample semigroups, see Section 2 for the definition. As in inverse semigroups, the central role is played by idempotents. This is a large class of semigroups and its structure theory necessarily depends on the choice of suitable subclasses. Besides the work alluded to above, interesting results on this subject can be found in the paper of Jackson and Stokes [7]. A most readable introduction to the subject of the paper can found in Hollings's thesis [5, Ch. 2]. Among the recent publications on the subject, it is informative to consult Gomes and Szendrei [4] for further results, Jones [8] where the varietal approach is introduced and the comprehensible historical survey by Hollings [6]. These papers contain a rich bibliography covering a large area.

We are concerned here with the structure in terms of constructions which would satisfy our intuition as to the make up of various special cases. As building blocks, we accept unipotent semigroups, or more especially cancellative monoids. As to constructions, we opted for semilattices of semigroups and one akin to Brandt semigroups.

To start with, we need a number of axioms, which are listed in Section 2 together with notation and terminology. In Section 3, we characterize the nucleus and the inverse part of a weakly left ample semigroup which represent the first inkling into the internal make up of these semigroups. The structure of weakly ample semigroups in Section 4 is the result: they are precisely (strong) semilattices of unipotent monoids. Left ample and ample semigroups form the subject of Section 5. Finally, Section 6 contains a construction of weakly left ample semigroups with zero all of whose nonzero idempotents are primitive.

## 2. Terminology and notation

Let $S$ be a semigroup; we denote by $E(S)$ the set of its idempotents and by $S^1$ the semigroup obtained from $S$ by adjoining an identity element if $S$ does not have one.

As generalizations of Green's $\mathscr{L}$- and $\mathscr{R}$-relations, we have $\mathscr{L}^*$- and $\mathscr{R}^*$-relations defined on $S$ by

$$a\mathscr{L}^*b \text{ if } (ax = ay \iff bx = by \text{ for all } x, y \in S^1),$$
$$a\mathscr{R}^*b \text{ if } (xa = ya \iff xb = yb \text{ for all } x, y \in S^1)$$

and, as generalizations of these, $\widetilde{\mathscr{L}}$ and $\widetilde{\mathscr{R}}$ defined on $S$ by

$$a\widetilde{\mathscr{L}}b \text{ if } (a = ae \iff b = be \text{ for all } e \in E(S)),$$
$$a\widetilde{\mathscr{R}}b \text{ if } (a = ea \iff b = eb \text{ for all } e \in E(S))$$

and as successive generalizations of the $\mathscr{H}$-relation,

$$\mathscr{H}^* = \mathscr{L}^* \cap \mathscr{R}^* \quad \text{and} \quad \widetilde{\mathscr{H}} = \widetilde{\mathscr{L}} \cap \widetilde{\mathscr{R}}.$$

It is well known that $\mathscr{L}^*$ is a right congruence and $\mathscr{R}^*$ is a left congruence, while this does not carry over to $\widetilde{\mathscr{L}}$ and $\widetilde{\mathscr{R}}$. Clearly, $\mathscr{L} \subseteq \mathscr{L}^* \subseteq \widetilde{\mathscr{L}}$ and $\mathscr{R} \subseteq \mathscr{R}^* \subseteq \widetilde{\mathscr{R}}$.

Definition 2.1. A semigroup $S$ is *left ample* if:

(A) its idempotents commute;
(B) every element $a \in S$ is $\mathscr{R}^*$-related to a (unique) idempotent, say $a^+$;
(C) for any $a \in S$ and $e \in E(S)$, we have $ae = (ae)^+a$.

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As a generalization of this concept, we have the following definition.

DEFINITION 2.2. A semigroup $S$ is *weakly left ample* if:

(A) its idempotents commute;
(B) every element $a \in S$ is $\widetilde{\mathscr{R}}$-related to a (unique) idempotent, say $a^+$;
(C) for any $a \in S$ and $e \in E(S)$, we have $ae = (ae)^+a$;
(D) $\widetilde{\mathscr{R}}$ is a left congruence on $S$.

Now we turn to the two-sided versions.

DEFINITION 2.3. Right ample and weakly right ample semigroups are defined by duality. A semigroup $S$ is (*weakly*) *ample* if it is both (weakly) left and right ample.

The semigroups in these definitions will be regarded as *unary* semigroups with the unary operation $a \to a^+$. Even though we will often write $S$, we will tacitly mean $(S, {}^+)$.

DEFINITION 2.4. On a weakly left ample semigroup $S$, define

$$N(S) = \{a \in S \mid a\mathscr{R}^* a^+\}, \text{ the } \textit{nucleus} \text{ of } S;$$
$$I(S) = \{a \in S \mid a\mathscr{R} a^+\}, \text{ the } \textit{inverse part} \text{ of } S.$$

Note that $I(S)$ coincides with $I(S)$ in [7].

Let $S$ be a semigroup. If $\mathcal{A}$ is a set of conditions, then we denote by $S \vDash \mathcal{A}$ that $S$ satisfies $\mathcal{A}$. If $S$ is a semilattice $Y$ of semigroups $S_\alpha$, then we write $S = (Y; S_\alpha)$. The definition of a strong semilattice $Y$ of semigroups $S_\alpha$, $S = [Y; S_\alpha, \chi_{\alpha,\beta}]$, can be found in [9, Definition III.7.8]. If $a \in S$ and $\rho$ is an equivalence relation on $S$, then $a\rho$ denotes the $\rho$-class of $a$.

We will freely use the following notation.

AXIOMS 2.5. The following are axioms a unary semigroup may satisfy.

(A1) $a = a^2 \implies a = a^+$.
(A2) $a^+ = a^{++}$.
(A3) $a = a^+a$. (A3′) $a = aa^+$.
(A4) $a^+b^+ = b^+a^+$.
(A5) $ab^+ = (ab)^+a$.
(A6) $(ab)^+ = a^+(ab)^+$. (A6′) $(ab)^+ = (ab)^+b^+$.
(A7) $(ab)^+ = (ab^+)^+$. (A7′) $(ab)^+ = (a^+b)^+$.
(A8) $a^+ = a^+a^+$.
(A9) $(ab)^+ = a^+b^+$.
(A10) $ab^+ = b^+a$.
(A11) $xa = ya \implies xa^+ = ya^+$. (A11′) $ax = ay \implies a^+x = a^+y$.

For notation and terminology, we generally follow the book [9].

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## 3. Weakly left ample semigroups

We start with axiomatization of this type of semigroups.

FACT 3.1. *A unary semigroup* $(S, {}^+)$ *is weakly left ample if and only if axioms* (A1)–(A6) *hold in* $S$.

PROOF. This is the dual of [5, Corollary 2.6.6]; see [7, Proposition 3.2]. □

FACT 3.2. *Let* $S$ *be a semigroup satisfying condition* (B) *in Definition 2.2. Then* $\widetilde{\mathscr{R}}$ *is a left congruence if and only if* $S$ *satisfies axiom* (A7).

PROOF. This is [5, Lemma 2.3.4] and the dual of [7, Proposition 1.4]. □

Some alternatives will come in handy.

LEMMA 3.3. *The following statements hold for a weakly left ample semigroup* $S$.

(i) *In Fact 3.1, axiom* (A2) *can be replaced by axiom* (A8).
(ii) *The pair* (A1) *and* (A2) *is equivalent to* $E(S) = \{a^+ \mid a \in S\}$.
(iii) $a\widetilde{\mathscr{R}}b \iff a^+ = b^+$ *for all* $a, b \in S$.

PROOF. (i) For $a \in S$,

$$a^+a^+ \overset{(A2)}{=} a^{++}a^+ \overset{(A3)}{=} a^+,$$
$$a^{++} = (a^+)^+ \overset{(A8)}{=} (a^+a^+)^+ \overset{(A7)}{=} (a^+a)^+ \overset{(A3)}{=} a^+.$$

(ii) This is trivial.

(iii) It follows from part (ii) that $E(S)$ is a transversal of $\widetilde{\mathscr{R}}$-classes and the assertion follows. □

When speaking of weakly left ample semigroups, we will freely use axioms (A1)–(A8).

Characterizations of $N(S)$ and $I(S)$ follow.

THEOREM 3.4. *Let* $S$ *be a weakly left ample semigroup.*

(i)

$$N(S) = \{a \in S \mid x, y \in Sa^+,\ xa = ya \implies x = y\}$$
$$= \{a \in S \mid xa = ya \implies xa^+ = ya^+\}$$

*and is the greatest left ample unary subsemigroup of* $S$.

(ii)

$$I(S) = \{a \in S \mid a \text{ is regular}\}$$

*and is the greatest inverse unary subsemigroup of* $S$.

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Proof. (i) Let $a \in N(S)$, $x, y \in Sa^+$ and $xa = ya$. Then $a\mathscr{R}^*a^+$ implies that $xa^+ = ya^+$, which by $x, y \in Sa^+$ yields $x = xa^+ = ya^+ = y$.

Next, let $a$ be in the second set in part (i) and let $x, y \in S$ be such that $xa = ya$. Then $xa^+a = ya^+a$ by axiom (A3), where $xa^+, ya^+ \in Sa^+$. The hypothesis implies that $xa^+ = ya^+$.

Now let $a$ be in the third set in part (i). By hypothesis, we have that $xa = ya$ implies $xa^+ = ya^+$. If $xa = a$, then $xa = a^+a$ by axiom (A3) and the hypothesis yields $xa^+ = a^+a^+$, which, by Lemma 3.3(i), yields $xa^+ = a^+$. It follows that $a\mathscr{R}^*a^+$ and $a \in N(S)$.

Let $a, b \in N(S)$ and assume that $xab = yab$. The hypothesis implies that $xab^+ = yab^+$, so axiom (A5) implies that $x(ab)^+a = y(ab)^+a$. Again by hypothesis, we get $x(ab)^+a^+ = y(ab)^+a^+$, which, by axioms (A4) and (A6), yields $x(ab)^+ = y(ab)^+$. Therefore, $ab \in N(S)$. Since the unary operation on $S$ restricts to $N(S)$, we conclude that $N(S)$ is a unary subsemigroup of $S$. Now Definition 2.1 shows that $N(S)$ is left ample. By the very definition, $N(S)$ is then the greatest left ample unary subsemigroup of $S$.

(ii) Let $a \in I(S)$. Then $a\mathscr{R}a^+$, so that $a^+ = ax$ for some $x \in S^1$. But then axiom (A3) implies that $a = a^+a = axa$ and $a$ is regular. Conversely, let $a = aba$. Then $a\mathscr{R}ab$ and hence $a\widetilde{\mathscr{R}}ab$. Since $ab \in E(S)$, we have $a^+ = ab$. Thus, $a\mathscr{R}a^+$, so that $a \in I(S)$. We have proved the first assertion.

If $a = axa$ and $b = byb$, then

$$ab = a(xa)(by)b \overset{(A1),(A4)}{=} a(by)(xa)b = (ab)yx(ab)$$

and $I(S)$ is closed under multiplication. The unary operation of $S$ restricts to $I(S)$. Therefore, $I(S)$ is a unary subsemigroup of $S$. Since $I(S)$ contains all regular elements of $S$, it is the greatest inverse unary subsemigroup of $S$. □

Corollary 3.5. *Let $S$ be a weakly left ample semigroup.*

(i) *The following statements are equivalent.*

(a) $N(S) = S$, (b) *$S$ is left ample*, (c) $\widetilde{\mathscr{R}} = \mathscr{R}^*$ *on $S$.*

(ii) *The following statements are equivalent.*

(a) $I(S) = S$, (b) *$S$ is an inverse semigroup*, (c) $\widetilde{\mathscr{R}} = \mathscr{R}$ *on $S$.*

Proof. (i) If $N(S) = S$, then $a\mathscr{R}^*a^+$ for all $a \in S$ and $S$ is left ample. Let $S$ be left ample. Since $\mathscr{R}^* \subseteq \widetilde{\mathscr{R}}$, every $\widetilde{\mathscr{R}}$-class contains an idempotent and by uniqueness we must have $\widetilde{\mathscr{R}} = \mathscr{R}^*$. If $\widetilde{\mathscr{R}} = \mathscr{R}^*$ on $S$, then $N(S) = S$ by the definition of $N(S)$.

(ii) If $I(S) = S$, then all elements of $S$ are regular and, by axiom (A4), $S$ is an inverse semigroup. If $S$ is an inverse semigroup, then $\widetilde{\mathscr{R}} = \mathscr{R}$, as is well known. If $\widetilde{\mathscr{R}} = \mathscr{R}$, then every element of $S$ is $\mathscr{R}$-related to an idempotent, so $S$ is regular and $I(S) = S$. □

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There is another way of arriving at $N(S)$ and $I(S)$. For it, we need some preparation. Let $X$ be a nonempty set. Denote by $\mathscr{PT}_X$ the semigroup of all partial transformations on $X$ written on the right of the argument and composed as such. For any $\alpha \in \mathscr{PT}_X$, let $\alpha^+$ be the identity mapping on the domain $\operatorname{dom}\alpha$ of $\alpha$. It is straightforward to verify that with the unary operation $\alpha \to \alpha^+$, $\mathscr{PT}_X$ is a weakly left ample semigroup.

The following is a formulation due to V. Gould, quoted in [5, Theorem 2.3.2], of a part of [7, Theorem 3.9] for a somewhat more general situation.

Fact 3.6. *Let $S$ be a weakly left ample semigroup, regarded as an algebra of type* $(2, 1)$. *Then the mapping $\phi : S \to \mathscr{PT}_S$ given by $s\phi = \rho_s$, where*

$$\operatorname{dom}\rho_s = S s^+ \quad \textit{and} \quad x\rho_s = xs \quad \forall x \in \operatorname{dom}\rho_s,$$

*is a representation of $S$ as a* $(2, 1)$*-subalgebra of $\mathscr{PT}_S$.*

In addition, the cited reference in [7] asserts that '$I(S)$ consists of exactly those elements corresponding to maps with inverses contained in this embedding'.

This represents another characterization of $I(S)$ and from the above results one gets immediately that

$$N(S) = \{s \in S \mid s\phi \text{ is injective}\},$$

which is yet another characterization of $N(S)$.

We first record two simple cases; their proofs are straightforward.

Proposition 3.7. *The following conditions on a weakly left ample semigroup $S$ are equivalent.*

(i) *$S$ is unipotent.*
(ii) *$N(S)$ is a unipotent left ample semigroup.*
(iii) *$I(S)$ is a unipotent inverse semigroup* (*that is, a group*).
(iv) *$\widetilde{\mathscr{R}} = \omega$, the universal relation.*
(v) $S \models a^+ = b^+$.

Corollary 3.8. *The following conditions on a weakly left ample semigroup $S$ are equivalent.*

(i) *$S$ is a unipotent monoid.*
(ii) *$N(S)$ is a right cancellative monoid.*
(iii) *$I(S)$ is the group of units of $S$.*
(iv) $S \models a^+ = b^+$, (A3′).
(v) *$S$ is either a group or an ideal extension of an idempotent-free semigroup by a group with zero whose identity is the identity of $S$.*

Semigroups in the above corollary which are not groups admit a simple construction. Recall the concepts in [9, Definition III.1.3].

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Proposition 3.9. *Let A be an idempotent-free semigroup and G be a group disjoint from A. Further, let $\varphi$ be a homomorphism mapping G onto a set of permutable invertible bitranslations of A, in notation $g\varphi = (\lambda^g, \rho^g)$ for every $g \in G$. On the set $S = A \cup G$, define a multiplication by*

$$a \circ g = a\rho^g, \quad g \circ a = \lambda^g a \quad (a \in A, g \in G),$$

*the products in A and G remaining unchanged. Then S is a unipotent monoid. Conversely, every unipotent monoid which is not a group is isomorphic to one so constructed.*

Proof. This follows easily from [9, Theorem III.2.2]. □

For semilattices of unipotent semigroups, we have the following result.

Theorem 3.10. *The following conditions on a weakly left ample semigroup S are equivalent.*

(i) *$S$ is a semilattice of unipotent semigroups.*

(ii) *The mapping*

$$a \to a^+ \quad (a \in S) \tag{3.1}$$

*is a retraction of $S$ onto $E(S)$.*

(iii) *$\widetilde{\mathscr{R}}$ is a right congruence on $S$.*

(iv) $S \vDash (A7')$.

(v) $S \vDash (A9)$.

*These conditions imply that $\widetilde{\mathscr{R}}$ is a semilattice congruence with unipotent classes and $I(S)$ is a Clifford semigroup.*

Proof. (i) implies (ii). We assume that $S = (Y; S_\alpha)$, where $S_\alpha$ is unipotent and has identity $e_\alpha$ for every $\alpha \in Y$. Let $a \in S_\alpha$. Then $a^+ \in S_\beta$ for some $\beta \in Y$ and axiom (A3) implies that $\alpha \leq \beta$. Next, $a \in S_\alpha$ implies that $a = e_\alpha a$, which, together with $a\widetilde{\mathscr{R}}a^+$, yields $a^+ = e_\alpha a^+$, whence $\beta \leq \alpha$ and equality prevails. Hence, $a^+ \in S_\alpha$.

If also $b \in S_\beta$, then $a, a^+ \in S_\alpha$ and $b, b^+ \in S_\beta$ imply that $ab, a^+b^+ \in S_{\alpha\beta}$, whence $(ab)^+ = (a^+b^+)^+$. But $a^+b^+ \in E(S)$, so that $(ab)^+ = a^+b^+$ and the mapping (3.1) is a homomorphism of $S$ into $E(S)$. By axiom (A8), it fixes elements of $E(S)$.

(ii) implies (iii). Since $E(S)$ is a transversal of the congruence induced by the mapping (3.1) and of $\widetilde{\mathscr{R}}$, it follows that this mapping induces $\widetilde{\mathscr{R}}$ and hence $\widetilde{\mathscr{R}}$ is a (right) congruence.

(iii) implies (i). The congruence $\widetilde{\mathscr{R}}$ is a left congruence by definition and a right congruence by hypothesis. For any $a, b \in S$,

$$ab\widetilde{\mathscr{R}}a^+b^+ \overset{(A4)}{=} b^+a^+\widetilde{\mathscr{R}}ba$$

and $S/\widetilde{\mathscr{R}}$ is commutative. If $e\widetilde{\mathscr{R}}a$ and $e\widetilde{\mathscr{R}}b$, then $e\widetilde{\mathscr{R}}ab$, which proves that $e\widetilde{\mathscr{R}}$ is a subsemigroup of $S$. Therefore, $\widetilde{\mathscr{R}}$ is a semilattice congruence with unipotent classes.

(iii) implies (iv). Since $a\widetilde{\mathscr{R}}a^+$, the hypothesis implies that $ab\widetilde{\mathscr{R}}a^+b$, which yields $(ab)^+ = (a^+b)^+$ and axiom (A7′) holds.

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(iv) implies (v). Indeed,

$$(ab)^+ \overset{(A7)}{=} (ab^+)^+ \overset{h}{=} (a^+b^+)^+ \overset{(A4)}{=} a^+b^+$$

and axiom (A9) holds.

(v) implies (iii). Let $a\widetilde{\mathscr{R}}b$ and $c \in S$. Then

$$(ac)^+ \overset{(A9)}{=} a^+c^+ \overset{(A4)}{=} b^+c^+ \overset{(A9)}{=} (bc)^+$$

and $\widetilde{\mathscr{R}}$ is a congruence.

The first additional statement was proved in '(iii) implies (i)'. It is checked readily that in an inverse semigroup $a\widetilde{\mathscr{R}}aa^{-1}$, so that $a^+ = aa^{-1}$. Now axiom (A9) yields $ab(ab)^{-1} = aa^{-1}bb^{-1}$, whence $abb^{-1}a^{-1} = bb^{-1}aa^{-1}$, so that $abb^{-1}a^{-1}a = bb^{-1}a$ and thus $abb^{-1} = bb^{-1}a$. By Theorem 3.4(ii), $I(S)$ is an inverse semigroup and thus it is a Clifford semigroup. □

## 4. Weakly ample semigroups

Recall that a weakly ample semigroup $S$ is both weakly left and right ample where $S$ means a unary semigroup, so both left and right regard the same multiplication and the same unary operation. This is a strong condition, as we shall see. But we need some preparation.

Lemma 4.1. *Let $S$ be a weakly left ample semigroup. Then $S$ satisfies axiom* (A3′) *if and only if each $\widetilde{\mathscr{R}}$-class of $S$ is a subsemigroup of $S$ and is a monoid.*

Proof. *Necessity.* First, $(aa)^+ \overset{(A7)}{=} (aa^+)^+ \overset{h}{=} a^+$, so that $aa\widetilde{\mathscr{R}}a$. If now $a\widetilde{\mathscr{R}}b$, then $aa\widetilde{\mathscr{R}}ab$ since $\widetilde{\mathscr{R}}$ is a left congruence and thus $a\widetilde{\mathscr{R}}ab$, so that $a\widetilde{\mathscr{R}}$ is a subsemigroup of $S$. If $a\widetilde{\mathscr{R}}b$, then $a^+ = b^+$, whence $b = bb^+ = ba^+$ and similarly $b = b^+b = a^+b$. Therefore, $a^+$ is the identity element of $S$.

*Sufficiency.* Since $a^+$ is the unique idempotent of $a\widetilde{\mathscr{R}}$ and $a\widetilde{\mathscr{R}}$ is a monoid, $a^+$ must be the identity of $a\widetilde{\mathscr{R}}$ and thus $a = aa^+$. □

Lemma 4.2. *Let $S = (Y; S_\alpha)$, where $S_\alpha$ is a unipotent monoid for every $\alpha \in Y$. Then $S$ is a strong semilattice $Y$ of semigroups $S_\alpha$.*

Proof. Let $e_\alpha$ be the identity element of $S_\alpha$ for every $\alpha \in Y$. For $\alpha \geq \beta$, define a mapping

$$\chi_{\alpha,\beta} : a \longrightarrow ae_\beta \quad (a \in S_\alpha).$$

Then $\chi_{\alpha,\beta} : S_\alpha \to S_\beta$. For $\alpha \geq \beta$ and $a, b \in S_\alpha$,

$$(a\chi_{\alpha,\beta})(b\chi_{\alpha,\beta}) = (ae_\beta)(be_\beta) = ae_\beta(be_\beta) = abe_\beta = (ab)\chi_{\alpha,\beta}.$$

For $\alpha \geq \beta \geq \gamma$,

$$e_\beta e_\gamma = e_\gamma e_\beta e_\gamma = e_\gamma e_\beta = (e_\gamma e_\beta)^2 = e_\gamma$$

since $S_\gamma$ is unipotent and thus, for any $a \in S_\alpha$,

$$a\chi_{\alpha,\beta}\chi_{\beta,\gamma} = ae_\beta e_\gamma = ae_\gamma = a\chi_{\alpha,\gamma}.$$

Trivially, $\chi_{\alpha,\alpha}$ is the identity mapping on $S_\alpha$. Therefore, $S = [Y; S_\alpha, \chi_{\alpha,\beta}]$. □

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We are now ready for the first main result of the paper.

THEOREM 4.3. *The following conditions on a unary semigroup $S$ are equivalent.*

(i) *$S$ is weakly ample.*
(ii) *$S$ is weakly left ample and satisfies axioms* (A3′) *and* (A6′).
(iii) *$S$ is weakly left ample and satisfies axiom* (A10).
(iv) *$S$ is a semilattice of unipotent monoids.*
(v) *$S$ is a strong semilattice of unipotent monoids.*
(vi) $S \models$ (A1), (A2), (A3), (A10), (A5), (A6).

*Moreover, in part* (vi), *axiom* (A5) *can be replaced by axiom* (A7).

PROOF. (i) implies (ii). This is trivial.

(ii) implies (iii). Indeed,

$$\begin{aligned} ab^+ &\overset{(A3)}{=} (ab^+)^+ab^+ \overset{(A6')}{=} (ab^+)^+b^{++}ab^+ \overset{(A2)}{=} (ab^+)^+b^+ab^+ \\ &\overset{(A4)}{=} b^+(ab^+)^+ab^+ \overset{(A3)}{=} b^+ab^+ \end{aligned}$$

and similarly $b^+a = b^+ab^+$, so that $ab^+ = b^+a$; see [7, Proposition 1.1].

(iii) implies (iv). First,

$$(ab)^+ \overset{(A7)}{=} (ab^+)^+ \overset{(A10)}{=} (b^+a)^+ \overset{(A7)}{=} (b^+a^+)^+ \overset{(A4)}{=} a^+b^+$$

and axiom (A9) holds. By Theorem 3.10, $S$ is a semilattice of unipotent semigroups. These unipotent semigroups are $\widetilde{\mathscr{R}}$-classes. Now Lemma 4.1 implies that $\widetilde{\mathscr{R}}$-classes are monoids.

(iv) implies (v). This follows directly from Lemma 4.2.

(v) implies (i). Let $S = [Y; S_\alpha, \chi_{\alpha,\beta}]$, where $S_\alpha$ is a unipotent monoid with identity element $e_\alpha$ for every $\alpha \in Y$. We set $a^+ = e_\alpha$ for every $a \in S_\alpha$ and $\alpha \in Y$. Simple verification will show that $S$ satisfies axioms (A1)–(A6) and their duals. Therefore, $S$ is weakly ample.

(iii) implies (vi). This follows from Fact 3.1.

(vi) implies (ii). It remains to verify axioms (A3′) and (A6′). Indeed, by axiom (A10), we get axiom (A3′). Next,

$$\begin{aligned} (ab)^+ &\overset{(A7)}{=} (ab^+)^+ \overset{(A10)}{=} (b^+a)^+ \overset{(A7)}{=} (b^+a^+)^+ \\ &\overset{(A10)}{=} a^+b^+ \overset{(A8)}{=} (a^+b^+)b^+ = (ab)^+b^+, \end{aligned}$$

giving axiom (A6′).

The axioms in part (vi) yield axiom (A7) by Fact 3.2. Conversely,

$$\begin{aligned} (ab)^+a &\overset{(A7)}{=} (ab^+)^+a \overset{(A10)}{=} (b^+a)^+a \overset{(A7)}{=} (b^+a^+)^+ \\ &\overset{(A10)}{=} b^+a^+a \overset{(A3)}{=} b^+a \overset{(A10)}{=} ab^+, \end{aligned}$$

giving axiom (A5). □

Of course, in the above theorem, parts (ii) and (iii) admit duals.

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Next, we list some properties of a semilattice of unipotent monoids.

Theorem 4.4. *Let $S$ be a weakly ample semigroup and $S = (Y; S_\alpha)$, where $S_\alpha$ is a unipotent monoid with identity element $e_\alpha$ for every $\alpha \in Y$.*

(i) *For every $\alpha \in Y$ and $a \in S_\alpha$, we have $a^+ = e_\alpha$ and $S_\alpha = a\widetilde{\mathscr{L}} = a\widetilde{\mathscr{R}}$.*
(ii) *Let $\rho$ be the congruence on $S$ induced by $(Y; S_\alpha)$. Then $\rho = \widetilde{\mathscr{L}} = \widetilde{\mathscr{R}}$ is the unique semilattice congruence on $S$ with classes of unipotent monoids.*
(iii) *$N(S) \subseteq \bigcup_{\alpha\in Y} N(S_\alpha)$ and $N(S)$ is a semilattice $Y$ of cancellative monoids.*
(iv) *$I(S) = \bigcup_{\alpha\in Y} I(S_\alpha)$ is a Clifford semigroup.*
(v) *Let $S = [Y; S_\alpha, \chi_{\alpha,\beta}]$. Then*

$$N(S) = [Y; N(S) \cap S_\alpha, \chi_{\alpha,\beta}|_{N(S)\cap S_\alpha}],$$
$$I(S) = [Y; I(S_\alpha), \chi_{\alpha,\beta}|_{I(S_\alpha)}].$$

Proof. (i) Let $a \in S_\alpha$. We saw in '(i) implies (ii)' in the proof of Theorem 3.10 that $a^+ = e_\alpha$. It follows that $S_\alpha \subseteq e_\alpha\widetilde{\mathscr{R}}$. Conversely, if $a \in e_\alpha\widetilde{\mathscr{R}}$, then $a^+ = e_\alpha$, which shows that $e_\alpha\widetilde{\mathscr{R}} \subseteq S_\alpha$. Hence, $S_\alpha = a\widetilde{\mathscr{R}}$ for all $a \in S_\alpha$. Dually, one obtains that $S_\alpha = a\widetilde{\mathscr{L}}$ for all $a \in S_\alpha$.

(ii) That $\rho = \widetilde{\mathscr{L}} = \widetilde{\mathscr{R}}$ follows directly from part (i). If $\lambda$ is any semilattice congruence on $S$ with classes of unipotent monoids, then we can apply part (i) to conclude that $\lambda = \widetilde{\mathscr{L}} = \widetilde{\mathscr{R}}$, establishing uniqueness.

(iii) Let $a \in N(S) \cap S_\alpha$. The implication $xa = ya$ implies that $xa^+ = ya^+$ in $S$ remains valid in $S_\alpha$ and thus $a \in N(S_\alpha)$. Hence, $N(S) \cap S_\alpha \subseteq N(S_\alpha)$, which implies that $N(S) \subseteq \bigcup_{\alpha\in Y} N(S_\alpha)$. By Corollary 3.8, $N(S_\alpha)$ is a right cancellative monoid and, by Theorem 3.4, $E(S) \subseteq N(S)$, which implies that $N(S) \cap S_\alpha$ is a right cancellative monoid. By duality, $S$ is a semilattice of cancellative monoids.

(iv) This follows from Theorem 3.4(ii) and Corollary 3.8.

(v) This follows from parts (iii) and (iv). □

## 5. Left ample and ample semigroups

As in the case of weakly left ample semigroups in Section 3, we are only able to characterize left ample semigroups by general statements and axiomatization.

Theorem 5.1. *The following conditions on a unary semigroup $S$ are equivalent.*

(i) *$S$ is left ample.*
(ii) *$S$ is weakly left ample and $\widetilde{\mathscr{R}} = \mathscr{R}^*$.*
(iii) *$S \models$ (A1)–(A6), (A11).*

Proof. (i) implies (ii). Recall that $\mathscr{R}^* \subseteq \widetilde{\mathscr{R}}$ and let $a\widetilde{\mathscr{R}}b$. Since $S$ is left ample, we have $a\mathscr{R}^*a^+$, so that $a\widetilde{\mathscr{R}}a^+$. Similarly, $b\widetilde{\mathscr{R}}b^+$. But then $a^+\widetilde{\mathscr{R}}b^+$, whence $a^+ = b^+$, since $a\widetilde{\mathscr{R}}$ contains at most one idempotent. It follows that $a\mathscr{R}^*b$. Therefore, $\widetilde{\mathscr{R}} \subseteq \mathscr{R}^*$ and equality prevails.

(ii) implies (iii). This follows from Corollary 3.5(i).

(iii) implies (i). This follows from Theorem 3.4(i) and Corollary 3.5(i). □

The case of ample semigroups can be characterized concretely, since it represents a strengthening of Theorem 4.3.

THEOREM 5.2. *The following conditions on a unary semigroup $S$ are equivalent.*

(i) $S$ *is ample.*

(ii) $S$ *is left ample and satisfies* (A3′), (A6′), (A11′).

(iii) $S$ *is weakly ample and satisfies* $\widetilde{\mathscr{L}} = \mathscr{L}^*$, $\widetilde{\mathscr{R}} = \mathscr{R}^*$.

(iv) $S$ *is a semilattice of cancellative monoids.*

(v) $S$ *is a strong semilattice of cancellative monoids.*

(vi) $S \models$ (A1)–(A6), (A11), (A3′), (A6′), (A11′).

PROOF. This follows from Corollary 3.8 and its dual, Theorem 4.3, and Theorem 5.1 and its dual. We omit the details. □

Of course, parts (ii) and (vi) admit dual statements. From the above references one can find other characterizations. However, the main characterization is that ample semigroups are precisely those which are isomorphic to some $[Y; S_\alpha, \chi_{\alpha,\beta}]$, where $S_\alpha$ is a cancellative monoid for all $\alpha \in Y$.

For Figure 1, it will be convenient to have the following definition.

DEFINITION 5.3. Let $\rho$ be an equivalence relation on a semigroup $S$. We say that $S$ is *$\rho$-abundant* if every $\rho$-class contains an idempotent.

One could also define $\rho$-adequate analogously. In Figure 1, we abbreviate $\rho$-abundant to $\rho$-ab, weakly left ample to the acronym wla, ample to a, etc. The words in parentheses denote the nomenclature customary in the literature.

With this we terminate our global discussion of the classes: weakly left ample, weakly ample, left ample and ample semigroups. Weakly ample and ample are virtually taken care of with strong semilattices. As a sample, in the next section, we provide a construction of a class of weakly left ample semigroups.

## 6. Weakly left ample semigroups with zero and primitive idempotents

For the structure of these semigroups, we have the following theorem.

THEOREM 6.1. *Let $S$ be a nontrivial semigroup with zero* 0 *and $\{R_\alpha\}_{\alpha\in A}$ be a family of nonzero right ideals of $S$ satisfying the following conditions.*

(i) $S = \bigcup_{\alpha\in A} R_\alpha$.

(ii) $R_\alpha \cap R_\beta = \{0\}$ *if* $\alpha \neq \beta$.

(iii) $E(R_\alpha) = \{e_\alpha, 0\}$, *where $e_\alpha$ is a left identity of $R_\alpha$.*

(iv) $e_\alpha e_\beta = 0$ *if* $\alpha \neq \beta$.

(v) *For any $a \in S$ and $b \in R_\beta$, $ae_\beta \neq 0$ if and only if $ab \neq 0$.*

(vi) *For any $a \in S$ and $b \in R_\beta$, $ae_\beta \neq 0$ implies that $ae_\beta = a$.*

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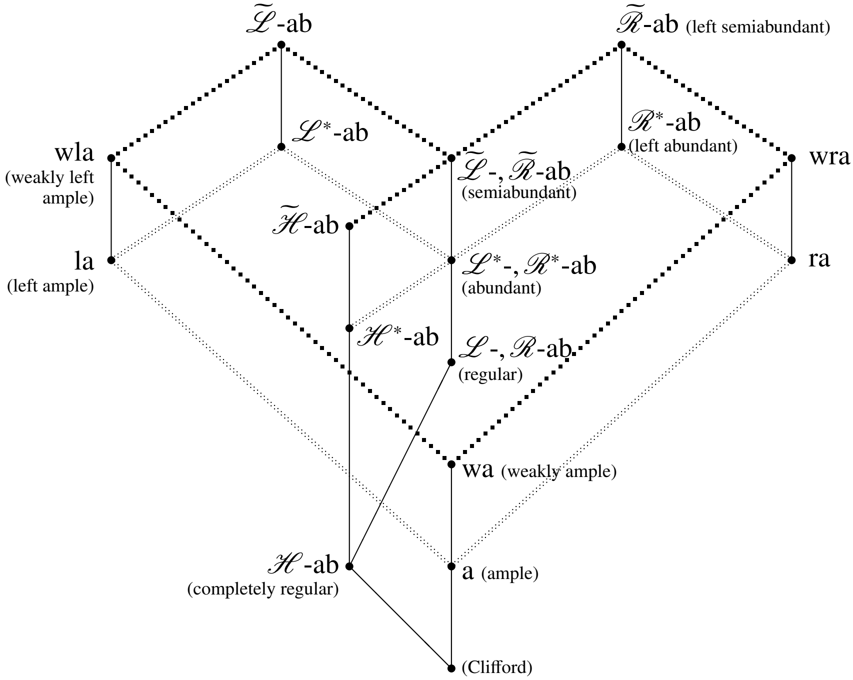



FIGURE 1. The positions of the classes studied in the paper.

*Let $a^+ = e_\alpha$ if $a \in R_\alpha \backslash \{0\}$ and $0^+ = 0$. Then $S$ is a nontrivial weakly left ample semigroup with zero $0$ all of whose nonzero idempotents are primitive. The sets $R_\alpha \backslash \{0\}$ for $\alpha \in A$ and $\{0\}$ form its complete set of $\widetilde{\mathscr{R}}$-classes.*

*Conversely, every nontrivial weakly left ample semigroup with zero all of whose nonzero idempotents are primitive is isomorphic to one so constructed.*

PROOF. *Necessity*. We will verify axioms (A1)–(A6). Axioms (A1), (A2) and (A3) follow directly from condition (iii), while axiom (A4) is a consequence of conditions (i) and (iv).

For the case that either $a = 0$ or $b = 0$, both axioms (A5) and (A6) hold trivially. Hence, let $a \in R_\alpha \backslash \{0\}$ and $b \in R_\beta \backslash \{0\}$. Then $a^+ = e_\alpha$ and $b^+ = e_\beta$. If $ab^+ = 0$, then, by condition (v), we have $ab = 0$, which verifies axiom (A5) in this case. Let $ab^+ \neq 0$. By condition (v), we have $ab \neq 0$, which, together with the hypothesis that $R_\alpha$ is a right ideal, yields $ab \in R_\alpha \backslash \{0\}$, so that $(ab)^+ = e_\alpha$. Now condition (iii) implies that $(ab)^+ a = a$. On the other hand, $ab^+ \neq 0$ by condition (vi) gives $ab^+ = a$. Therefore, axiom (A5) holds. If $ab = 0$, then axiom (A6) holds. If $ab \neq 0$, then $ab \in R_\alpha \backslash \{0\}$, whence $(ab)^+ = e_\alpha$; on the other hand, $a^+ = e_\alpha$, which shows that axiom (A6) holds as well.

Therefore, $S$ is a nontrivial weakly left ample semigroup. Condition (iii) implies that $E(S) = \{e_\alpha \mid \alpha \in A\} \cup \{0\}$ and, thus, by condition (iv), all nonzero idempotents of $S$ are primitive.

If $a\widetilde{\mathscr{R}}0$, then $0 = 00$ implies that $a = 0a = 0$. Hence, $\{0\}$ is an $\widetilde{\mathscr{R}}$-class. Let $a \in R_\alpha \backslash \{0\}$ and $b \in R_\beta \backslash \{0\}$. If $a\widetilde{\mathscr{R}}b$, then $a = e_\alpha a$ implies that $b = e_\alpha b \in R_\alpha$, since $R_\alpha$ is a right ideal of $S$, so that $\alpha = \beta$. Conversely, if $\alpha = \beta$, then $a = e_\gamma a$ if and only if $b = e_\gamma b$ for all $\gamma \in A$ and thus $a\widetilde{\mathscr{R}}b$. Therefore, the sets $R_\alpha \backslash \{0\}$, $\alpha \in A$ and $\{0\}$ form the complete collection of $\widetilde{\mathscr{R}}$-classes of $S$.

*Sufficiency*. For every $e \in E(S) \backslash \{0\}$, let

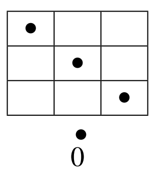
$$R_e = \{a \in S \mid a^+ = e\} \cup \{0\}.$$

Since $e \in R_e$, we have $R_e \neq \{0\}$. Let $a \in R_e$ and $b \in R_f$. If $ab = 0$, then $ab \in R_e$. Assume that $ab \neq 0$. By axiom (A2), we have $(ab)^+ab = ab$, whence $(ab)^+a \neq 0$ and again, by axiom (A2), we get $(ab)^+a^+a \neq 0$, so also $(ab)^+a^+ \neq 0$. By axiom (A4), we obtain $0 \neq (ab)^+a^+ \leq a^+$ and the hypothesis implies that $(ab)^+a^+ = a^+$. It follows that $ab \in R_e$, so $R_e$ is a right ideal.

Since $a^+$ is defined for all $a \in S$, condition (i) holds. If $R_e \cap R_f \neq \{0\}$, then, for all $a \in (R_e \cap R_f) \backslash \{0\}$, we have $a^+ = e = f$, so that $R_e = R_f$, which proves condition (ii). It follows from axiom (A8) and the definition of $R_e$ that $E(R_e) = \{e, 0\}$ and, by axiom (A3), we have $a = ea$ for any $a \in R_e$. Therefore, axiom (A3) holds as well. Let $e, f \in E(S) \backslash \{0\}$ and assume that $ef \neq 0$. Then $ef \leq e$ and $ef \leq f$, so that $e = f$. Condition (iv) follows.

Let $a \in R_e$ and $b \in R_f$. If $ab^+ \neq 0$, then, by axiom (A5), we have $ab \neq 0$. Conversely, if $ab \neq 0$, then, by axiom (A3), we get $ab^+ \neq 0$. Hence, condition (v) holds. If $ae_\beta \neq 0$, then axiom (A5) implies that $ae_\beta = (ae_\beta)^+a$ since $ae_\beta \in R_e \backslash \{0\}$. Hence, condition (vi) holds as well. □

We now denote a weakly left ample semigroup relative to the unary operation $a \to a^+$ by $(S, {}^+)$. Dually, let $(S, {}^-)$ denote the same (multiplicative) semigroup $S$ with a unary operation $a \to a^-$, which makes it a weakly right ample semigroup. Assume that $S$ is a nontrivial semigroup with zero all of whose idempotents are primitive. By Theorem 6.1 and its dual, $S$ is a 0-disjoint union of both right and of left ideals. We can visualize the situation as follows:



where horizontal classes are $\widetilde{\mathscr{R}}$-classes and vertical classes are $\widetilde{\mathscr{L}}$-classes, so that cells represent nonzero $\widetilde{\mathscr{H}}$-classes and dots indicate idempotents. No wonder this figure evokes a Brandt semigroup. We have not discussed such a semigroup but it seems worth considering. We limit ourselves to the following example.

Example 6.2. Let $M$ be a monoid with identity $e$ and let $I$ be a nonempty set. Let $S = B(M, I)$ be defined in the same way as when $M$ is a group and set

$$(i, a, j)^+ = (i, e, i), \quad (i, a, j)^- = (j, e, j), \quad 0^+ = 0^- = 0.$$

Straightforward verification will show that $(S, ^+)$ is a weakly left ample semigroup and $(S, ^-)$ is a weakly right ample semigroup.

Call the pair $((S, ^+), (S, ^-))$ *linked* if $(S, ^+)$ is a weakly left ample semigroup and $(S, ^-)$ is a weakly right ample semigroup.

PROBLEM 6.3. Find the structure (in terms of a construction) of a semigroup $S$, where $((S, ^+), (S, ^-))$ is a linked pair.

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