Canad. Math. Bull. Vol. 34 (2), 1991 pp. 220–223

GLOBAL HOLOMORPHIC APPROXIMATION ON THE PRODUCT OF CURVES IN $\mathbf{C}^n$

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ABSTRACT. We prove that every continuous function on the product of certain curves can be asymptotically approximated by entire functions.

1. **Introduction.** Let γ be the image in $\mathbf{C}^m$ of the real axis under a proper continuous imbedding. It was shown by Alexander in [1], that if γ is smooth, then every continuous function on γ can be asymptotically approximated on γ by a holomorphic function in $\mathbf{C}^m$. The term "smooth" means "piecewise C^1".

The aim of this work is to prove asymptotic approximation of continuous functions on the product of smooth curves by entire functions.

Before stating the result, we fix some notation. For X a compact subset of $\mathbf{C}^n$, let $C(X)$ be the Banach algebra of continuous complex valued functions on X with the supremum norm and $P(X)$, the closure in $C(X)$ of polynomials. The polynomially convex hull $X^\wedge$ of X is

$$\{z \in \mathbf{C}^n;\ |p(z)| \leqq \|p\|_X \text{ for every polynomial } p\}.$$

For $i = 1, \ldots, k, \Gamma^i$ will denote a smooth properly imbedded image in $\mathbf{C}^{n_i}$ of the real axis. Let $n = \sum_{i=1}^k n_i$ and $\Gamma = \prod_{i=1}^k \Gamma^i$. Let B_r^i denote the open ball in $\mathbf{C}^{n_i}$ centered at the origin and with radius r. By "arc" we mean a homeomorphic image of the closed unit interval.

In this note we establish the following.

THEOREM. *For every continuous function f on Γ and every positive continuous function ϵ on Γ, there exists a function g holomorphic in $\mathbf{C}^n$ such that $|f(p) - g(p)| < \epsilon(p)$ for each p in Γ.*

2. **Preliminaries.** For the proof of the theorem, we shall need some preliminary lemmas.

The following is a consequence of Stolzenberg's result [2]. For the proof see [1].

LEMMA 2.1. *Let X be a compact polynomially convex subset of C^n and let α and β be disjoint smooth arcs in $\mathbf{C}^n$ such that $\alpha \cap X$ and $\beta \cap X$ each contains a single point. Then,*

a) $X \cup \alpha \cup \beta$ is polynomially convex;

Research supported by NSERC of Canada and FCAR of Québec.
Received by the editors October 3, 1989.
AMS subject classification: 32E30.

b) $C(X \cup \alpha \cup \beta) \cap P(X) = P(X \cup \alpha \cup \beta)$.

In the following lemma, we establish an approximation result on the union of two products of compacts sets

LEMMA 2.2. *For* $i = 1, \ldots, k$, *let* X^i *be a compact polynomially convex subset of* $\mathbf{C}^{n_i}$ *and* γ^i *a smooth arc in* $\mathbf{C}^{n_i}$. *Suppose that* $X^i \cup \gamma^i = X^i \cup \alpha^i \cup \beta^i$, *where* α^i *and* β^i *are disjoint smooth arcs in* $\mathbf{C}^{n_i}$ *such that* $\alpha^i \cap X^i$ *and* $\beta^i \cap X^i$ *each contains a single point. If* $X := \prod_{i=1}^k X^i \cup \prod_{i=1}^k \gamma^i$, *then*

$$C(X) \cap P\left(\prod_{i=1}^{k} X^i\right) = P(X).$$

PROOF. Let μ a measure on X orthogonal to polynomials. By the Hahn-Banach Theorem, it suffices to show that μ is orthogonal to the space $C(X) \cap P(\prod_{i=1}^k X^i)$. For this it suffices to show that the support of μ is contained in $\prod_{i=1}^k X^i$. Let ϕ then be a continuous function on X such that the support of ϕ doesn't meet $\prod_{i=1}^k X^i$; we must show that $\mu(\phi) = 0$. Let $n = \sum_{i=1}^k n_i$ and for $z \in \mathbf{C}^n$, write $z = (z^1, \ldots, z^k)$, where $z^i \in \mathbf{C}^{n_i}$. For $j = 1, \ldots, k$, define θ_j on $\mathbf{C}^n$ by

$$\theta_j(z) = \inf_{w \in X^j} \|z^j - w\|.$$

Let h be the function of $\mathbf{C}^n$ which is equal to $\phi / \sum_{i=1}^k \theta_i$ on supp ϕ and which is zero elsewhere. Note that h is continuous and has the same support as ϕ. Since $\phi = \sum_{i=1}^k \theta_i h$, it suffices to show that $\theta_j \mu = 0$ for $j = 1, \ldots, k$.

Considered as a function of z^j, θ_j vanishes on X^j and is continuous everywhere. Then by Lemma (2.1) and the assumption on $X^j \cup \gamma^j$, θ_j is the uniform limit of polynomials on $X^j \cup \gamma^j$ and hence on X. If we combine this with the fact that μ is orthogonal to polynomials, it follows that $\theta_j \mu$ is orthogonal to polynomials. Note also that the support of $\theta_j \mu$ is contained in $\prod_{i=1}^k \gamma^i$ since $\theta_j \equiv 0$ on $\prod_{i=1}^k X^i$.

Let f be a continuous function on $\prod_{i=1}^k \gamma^i$. Using the Stone-Weirstrass Theorem and the polynomial approximation on smooth arcs, we can approximate f uniformly on $\prod_{i=1}^k \gamma^i$ by polynomials and then $\theta_j \mu(f) = 0$ since $\theta_j \mu$ is orthogonal to polynomials.

3. **Proof of the theorem.** We may assume that Γ contains the origin. For each i, define as in [1], γ_r^i to be the subarc of $\Gamma^i \cap B_r^i$ which contains the origin and σ_r^i the set $\Gamma^i \setminus \{$ the two unbounded components of $\Gamma^i \setminus B_r^i\}$. Then γ_r^i and σ_r^i are bounded open arcs in $\mathbf{C}^{n_i}$.

For each i, define by induction a sequence of real numbers $(r(j,i))_{j \geqq 0}$ as follows: put $r(0,i) = 1$ and $r(j,i) > r(j-1,i) + 1$ for $j > 0$ such that

$$\sigma^i_{r(j-1,i)} \subset B^i_{r(j,i)} \tag{3.1}$$

$$(\bar{B}^i_{r(j-1,i)} \cup \bar{\sigma}^i_{r(j-1,i)})^{\wedge} \cap (\bar{\sigma}^i_{r(j,i)} \setminus \gamma^i_{r(j,i)}) = \emptyset. \tag{3.2}$$

Condition (3.1) can be achieved since σ_r^i is bounded and (3.2) since the first set in compact and the second goes to infinity since Γ^i does. For simplicity, we write $\gamma_j^{\,i}, \sigma_j^i$ and B_j^i for $\gamma_{r(j,i)}^{\,i}, \sigma_{r(j,i)}^i$ and $B_{r(j,i)}^i$ respectively.

For each i, define $X_j^i = (\bar{B}_{j-2}^i \cup \bar{\gamma}_{j-1}^{\,i})^\wedge$ for $j \geqq 2$. It can be shown as in [1], that $X_j^i \cup \bar{\gamma}_{j+2}^{\,i} = X_j^i \cup (\alpha_j^i \cup \beta_j^i)$, where α_j^i and β_j^i are disjoint smooth arcs each intersecting X_j^i at a single point. For $j \geqq 2$, let $X_j = \prod_{i=1}^k X_j^i$ and $Y_j = \prod_{i=1}^k X_j^i \cup \prod_{i=1}^k \bar{\gamma}_{j+2}^{\,i}$.

We are now ready to prove the theorem in the case where $\epsilon(x) \equiv 1$. Choose a sequence $(\epsilon_i)_{i \geqq 0}$ of positive numbers such that $\sum_{i \geqq 0} \epsilon_i = 1$. Let f be a continuous function on Γ. We may suppose that f is in fact continuous on $\mathbf{C}^n$. By the Stone-Weirstrass Theorem and by the polynomial approximation on smooth arcs in each $\mathbf{C}^{n_i}$, there exists a polynomial g_0 such that

$$\|f - g_0\|_{\prod_i \bar{\gamma}_3^{\,i}} < \epsilon_0. \tag{3.3}$$

Let $\alpha_1 \in C^\infty(\mathbf{C}^n), 0 \leqq \alpha_1 \leqq 1, \alpha_1 \equiv 1$ on $\prod_{i=1}^k B_1^i$ and $\alpha_1 \equiv 0$ outside $\prod_{i=1}^k B_2^i$. Let $f_1 = \alpha_1 g_0 + (1 - \alpha_1)f$. By Lemma (2.2), there exists a polynomial g_1 such that

$$\|f_1 - g_1\|_{Y_2} < \epsilon_1. \tag{3.4}$$

This gives in particular

$$\|g_1 - g_0\|_{X_2} < \epsilon_1, \tag{3.5}$$

since each X_2^i is contained in $\bar{B}_1^i$.
Since $f - f_1 = \alpha_1(f - g_0)$, it follows from (3.3) and (3.4) that

$$\|f - g_1\|_{\prod_{i=1}^k \bar{\gamma}_3^{\,i}} < \epsilon_0 + \epsilon_1. \tag{3.6}$$

since $f_1 = f$ on the complement of $\prod_{i=1}^k B_2^i$ and this complement contains $\prod_{i=1}^k \bar{\gamma}_4^{\,i} \setminus \prod_{i=1}^k \gamma_3^{\,i}$, it follows from (3.4) and (3.6) that

$$\|f - g_1\|_{\prod_{i=1}^k \bar{\gamma}_4^{\,i}} < \epsilon_0 + \epsilon_1. \tag{3.7}$$

Suppose that there exist polynomials $g_0, g_1, \ldots, g_{m-1}$ satisfying

$$\|g_j - g_{j-1}\|_{X_{j+1}} < \epsilon_j, \quad \text{for} \quad 1 \leqq j \leqq m-1 \tag{3.8}$$

and

$$\|f - g_j\|_{\prod_{i=1}^k \bar{\gamma}_{j+3}^{\,i}} < \sum_{\nu=0}^{j} \epsilon_\nu, \quad \text{for} \quad 0 \leqq j \leqq m-1. \tag{3.9}$$

Let $\alpha_m \in C^\infty(\mathbf{C}^n), 0 \leqq \alpha_m \leqq 1, \alpha_m \equiv 1$ on $\prod_{i=1}^k B_m^i$ and $\alpha_m = 0$ outside $\prod_{i=1}^k B_{m+1}^i$. Let $f_m = \alpha_m g_{m-1} + (1 - \alpha_m)f$. In the same way as we established (3.4), there exists a polynomial g_m such that

$$\|f_m - g_m\|_{Y_{m+1}} < \epsilon_m.$$

https://doi.org/10.4153/CMB-1991-035-1 Published online by Cambridge University Press

This gives (3.8) for $j = m$ and from the induction hypotheses and an argument similar to the proof of (3.7) we also have (3.9) for $j = m$.

Since the inequality (3.8) is satisfied for all j and since X_j contains $\prod_{i=1}^{k} B_{j-2}^{i}$, it follows that the sequence $(g_j)_{j \geqq 0}$ converges uniformly on compacts subsets of $\mathbf{C}^n$. Let $g = \lim g_j$, then g is holomorphic in $\mathbf{C}^n$. We show now that g approximates f on Γ. Fix $j \geqq 0$. Then for $m > j$, we have $g_m = g_j + \sum_{\nu=j+1}^{m} (g_\nu - g_{\nu-1})$ and

$$(3.10) \qquad \|f - g_m\|_{\prod_{i=1}^{k} \bar{\gamma}_j^{\,i}} \leqq \|f - g_j\|_{\prod_{i=1}^{k} \bar{\gamma}_j^{\,i}} + \sum_{\nu=j+1}^{m} \|g_\nu - g_{\nu-1}\|_{\prod_{i=1}^{k} \bar{\gamma}_j^{\,i}}.$$

Since $\prod_{i=1}^{k} \bar{\gamma}_j^{\,i}$ is contained in $\prod_{i=1}^{k} \bar{\gamma}_{j+3}^{\,i}$ and also in X_{j+1} and since (3.8) and (3.9) are satisfied for all j and from (3.10), it follows

$$\|f - g\|_{\prod_{i=1}^{k} \bar{\gamma}_j^{\,i}} \leqq \|f - g_j\|_{\prod_{i=1}^{k} \bar{\gamma}_j^{\,i}} + \sum_{\nu=j+1}^{\infty} \epsilon_\nu < \sum_{\nu=0}^{j} \epsilon_\nu + \sum_{\nu=j+1}^{\infty} \epsilon_\nu = 1.$$

Since the set $\prod_{i=1}^{k} \bar{\gamma}_j^{\,i}$ expand to cover Γ, it follows that $|f(p) - g(p)| < 1$, for $p \in \Gamma$.

We have proved the theorem in case of $\epsilon(p) \equiv 1$. Suppose now that ϵ is a continuous positive function on Γ and f is continuous on Γ. Then there exists a function g holomorphic in $\mathbf{C}^n$ such that $|-1 + \log \epsilon - g| < 1$ on Γ. This implies Re $g < \log \epsilon$ on Γ. Also there exists a function h_0 holomorphic in $\mathbf{C}^n$ such that $|h_0 - f \exp(-g)| < 1$ on Γ. If we put $h = h_0 \exp(g)$, then h approximates f on Γ within $\epsilon(p)$.

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