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OPENNESS OF VECTOR MEASURES AND THEIR INTEGRAL MAPS

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Abstract

We prove that finite dimensional nonatomic vector measures and their integral maps are open maps. These results can be found in the literature, but unfortunately the proofs presented there are not complete.

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1. Introduction

The problem of openness of vector measures and their integral maps has been treated by Anantharaman and Garg [2]. Earlier Karafiat [12] considered a continuity property of the map inverse to a vector measure which gives, in particular, the openness of the measure. Samet [16] rediscovered the problem of openness, but his proof of the openness of finite dimensional nonatomic measures contains a gap (see [16, page 472], the assertion of the openness of $\mu(\Omega(S,\varepsilon))$). Recently Professor Samet informed me that he could fill the gap. Incomplete also are some proofs in [2] and [12] (see the last section of this paper).

Our aim is to prove that $\mathbf{R}^p$ valued nonatomic measures and their integral maps are open maps. Our arguments follow, in part, [2], [12] and [15], but we include some of them for the sake of completeness.

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In this section we introduce the notation and recall some preliminary results. In Section 2 we state a few characterizations of open maps. Section 3 contains some results on semiconvex measures. Section 4 contains some rather general results on points of openness of vector measures and affine maps. In Section 5 we present a decomposition lemma due to Liapunov [15] and Karafiat [12]. The main results are presented in Section 6. Some applications of openness of vector measures and their integral maps are contained in the next section. The last section contains some remarks.

Throughout this paper S is a nonempty set, $\mathscr{A}$ is a σ-algebra of subsets of S and m is a vector measure on $\mathscr{A}$ with values in a complete metrizable locally convex space Y. It is known that there exists a finite positive measure μ on $\mathscr{A}$ equivalent to m [4; 13, page 21]; μ is called a control measure of m. Let P_0 be the set of all characteristic functions of sets in $\mathscr{A}$ and P the set of all $\mathscr{A}$-measurable $[0,1]$-valued functions in S. We will identify P_0 with $\mathscr{A}$. Note that $P_0 \subset P \subset L^\infty$, where $L^\infty = L^\infty(\mu)$. Let T_m be the integral map of m from P into Y defined by $T_m(x) = \int x\,dm$ for $x \in P$, where the integral is in the sense of [13, page 26]. On P_0 we will identify T_m with m.

The sets P_0 and P are considered under the topology w^*, the weak-star topology of L^∞. On P_0 this topology is identical with the Fréchet-Nikodym topology given by the metric $d(A,B) = \mu(A \triangle B)$, where $A \triangle B$ denotes the symmetric difference of A and B. On P the topology w^* is identical with the weak topology of $L^1 = L^1(\mu)$. The integral map T_m is continuous as a map from P into (Y, σ), where σ is the weak topology of Y (see [13, page 68]). This follows easily by applying the Radon-Nikodym theorem. The measure m is continuous as a map from P_0 into Y.

Let K denote the range of the integral map T_m. It is known that K is the closed convex hull of the range of m [13, page 76].

2. Open maps

Let X and Y be topological spaces and let $T\colon X \to Y$ be surjective. We say that T is open at $x \in X$ if for each neighbourhood U of x the image $T(U)$ is a neighbourhood of $T(x)$. We say that T is open if for each open set U in X the image $T(U)$ is open in Y. Clearly, T is open if and only if T is open at each point of X.

Denote by 2^X the family of all subsets of X and let F be a multifunction from Y to X, that is, $F\colon Y \to 2^X$. We say that F is lower semicontinuous at $y_0 \in Y$ if for each open set U in X satisfying $F(y_0) \cap U \neq \varnothing$ there exists a neighbourhood V of y_0 such that $F(y) \cap U \neq \varnothing$ for all $y \in V$. Given a map

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$T: X \to Y$ we denote by T^{-1} the multifunction from Y to X defined by the formula $T^{-1}(y) = \{x \in X: T(x) = y\}$.

Let A_α be a net of subsets of X. We define $Li\ A_\alpha$ as the set of all $x \in X$ such that for each neighbourhood U of x there exists α_0 such that $A_\alpha \cap U \neq \emptyset$ for all $\alpha > \alpha_0$, and $Ls\ A_\alpha$ as the set of all $x \in X$ such that for each neighbourhood U of x and each α there exists $\beta > \alpha$ with $A_\beta \cap U \neq \emptyset$ (see [14, Section 29]).

It is easy to prove the following lemma.

LEMMA 2.1. *Let X and Y be topological spaces, $T: X \to Y$ be surjective and $y_0 \in Y$. Consider the following conditions.*

(a) *T is open at each $x \in T^{-1}(y_0)$.*
(b) *The multifunction T^{-1} is lower semicontinuous at y_0.*
(c) *$T^{-1}(y_0) \subset Li\ T^{-1}(y_\alpha)$ for each net (y_α) in Y converging to y_0.*
(d) *$T^{-1}(y_0) \subset Ls\ T^{-1}(y_\alpha)$ for each net (y_α) in Y converging to y_0.*
(e) *For each $x \in T^{-1}(y_0)$ and each net (y_α) in Y converging to y_0 there exists a net (x_α) in X converging to x such that $x_\alpha \in T^{-1}(y_\alpha)$ for all α.*
(f) *$T^{-1}(y_0) = Ls\ T^{-1}(y_\alpha)$ for each net (y_α) in Y converging to y_0.*

The conditions (a), (b), (c) *and* (d) *are equivalent. Condition* (e) *implies* (d). *If X and Y are first countable then* (d) *implies* (e) (*in this case we replace nets in these conditions by sequences*). *If T is continuous and Y is a Hausdorff space, then* (f) *is equivalent to* (d). *In case Y is metrizable we may replace in the above nets by sequences.*

Note that condition (f) in Lemma 2.1 is due to Hájek [9]. The equivalence of conditions (a), (c) and (f) generalizes some results of Sikorski [17, page 16] and Hájek [9, Proposition 1].

3. Semiconvex measures

A measure $m: \mathscr{A} \to Y$ is called semiconvex if for each $A \in \mathscr{A}$ there exists $B \in \mathscr{A}$ such that $B \subset A$ and $m(B) = m(A)/2$ (see [10]). As easily seen, a semiconvex measure is nonatomic. In view of Liapunov's theorem [15, Theorem 1] the converse holds in case $Y = \mathbf{R}^p$. We shall need the following known lemma [3].

LEMMA 3.1. *If $m: \mathscr{A} \to Y$ is semiconvex, then*
(1) $\operatorname{ext} T_m^{-1}(y) = m^{-1}(y)$ *for each* $y \in m(P_0)$,
(2) $T_m(P) = m(P_0)$.

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Indeed, by [3, Theorem 2.2] we have $\operatorname{ext} T_m^{-1}(y) \subset P_0$ for each $y \in T_m(P)$. Since $\operatorname{ext} P = P_0$, this yields (1). In view of the Krein-Milman theorem, (1) implies (2).

The next lemma implies that, in case m is semiconvex and finite dimensional, the openness of T_m and that of m are equivalent. The second part of our proof follows an argument of [2, proof of Proposition 2.6].

LEMMA 3.2. *Let $m: \mathscr{A} \to Y$ be a semiconvex measure and let K, the common range of m and T_m, be equipped with the weak topology σ. Then, for every $y_0 \in K$, the following conditions are equivalent:*

(a) *m is open at each $x \in m^{-1}(y_0)$;*

(b) *T_m is open at each $x \in T_m^{-1}(y_0)$.*

PROOF. Assume (a) and let (y_α) be a net in K converging to y_0. Then, in view of Lemma 2.1 we have $m^{-1}(y_0) \subset Li\ m^{-1}(y_\alpha)$. Hence, by Lemma 3.1(1), $\operatorname{ext} T_m^{-1}(y_0) \subset Li\ T_m^{-1}(y_\alpha)$. This yields in view of the Krein-Milman theorem that

$$T_m^{-1}(y_0) \subset \operatorname{co} Li\ T_m^{-1}(y_\alpha).$$

Since the limes inferior of a net of closed and convex sets is closed and convex, it follows that

$$T_m^{-1}(y_0) \subset Li\ T_m^{-1}(y_\alpha).$$

In view of Lemma 2.1, this shows that (b) holds.

Assume (b) and let (y_α) be a net in K converging to y_0. In view of Lemms 2.1(f), we have $T_m^{-1}(y_0) = Ls\ T_m^{-1}(y_\alpha)$. Hence, by a result of Jerison [10], Theorem 2]

$$\operatorname{ext} T_m^{-1}(y_0) \subset Ls \operatorname{ext} T_m^{-1}(y_\alpha),$$

or, equivalently, $m^{-1}(y_0) \subset Ls\ m^{-1}(y_\alpha)$ (Lemma 3.1(1)). In view of Lemma 2.1, this shows that (a) holds.

4. Points of openness

In this section we prove that m is open at $x \in P_0$ whenever $m(x)$ is an extreme point of K, and that T_m is open at $x \in P$ whenever $T_m(x)$ is an interior point of K.

LEMMA 4.1. *If $x \in P_0$ and $m(x) \in \operatorname{ext} K$, then m is open at x (even when K is considered under the weak topology σ).*

PROOF. In view of Lemma 2.1, it is enough to find for every net (y_α) in (K, σ) converging to $m(x)$, and every net (x_α) with $x_\alpha \in m^{-1}(y_\alpha)$, a subset (x_β) of

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(x_α) convergent to x. Let (x_α) and (y_α) be such as above. Then (P, w^*) is a compact space. Thus there exists a subnet (x_β) of (x_α) w^*-convergent to some $x_0 \in P$. Since T_m is continuous $(T_m(x_\beta))$ converges to $T_m(x_0) = m(x)$. Hence, by [13, Theorem VI.1.1], $x_0 = x$.

PROPOSITION 4.2. *Let X and Y be topological linear spaces, let C be a bounded convex subset of X and let $T\colon C \to Y$ be an affine map. Then T is open at each $x \in T^{-1}(y_0)$ whenever y_0 is an interior point of $T(C)$.*

PROOF. Fix a neighbourhood V of zero in X. We first show that there exists a neighbourhood W of zero in Y such that

$$T^{-1}(y_0) \subset T^{-1}(y) + V,$$

whenever $y \in T(C)$ and $y - y_0 \in W$. By assumption, there exists $\lambda > 0$ with $\lambda(C - C) \subset V$. We assume that $\lambda \leq 1$. Choose a neighbourhood W of zero in Y with

$$y_0 + \frac{1}{\lambda} W \subset T(C).$$

Fix $y_1 \in T(C)$ and $y = (1-\lambda)y_0 + \lambda y_1$. Let $x_1 \in T^{-1}(y_1)$ and $x_0 \in T^{-1}(y_0)$ be arbitrary. Then

$$(1-\lambda)x_0 + \lambda x_1 \in T^{-1}(y),$$

whence $x_0 \in T^{-1}(y) + V$. In view of Lemma 2.1 it is enough to show that the multifunction T^{-1} is lower semicontinuous at y_0. Let U be a relatively open subset of C with $T^{-1}(y_0) \cap U \neq \varnothing$. Fix $\bar{x} \in T^{-1}(y_0) \cap U$ and let V be a neighbourhood of zero in X such that for every $x \in C$ with $x - \bar{x} \in V$ we have $x \in U$. Taking W as above, we get $T^{-1}(y) \cap U \neq \varnothing$, whenever $y \in C$ and $y - y_0 \in W$.

COROLLARY 4.3. *Let $m\colon \mathscr{A} \to Y$ be a semiconvex measure and let y_0 be an interior point of (K, σ). Then m is open at each point $x \in m^{-1}(y_0)$.*

Indeed, in view of Proposition 4.2, T_m is open at each point $x \in T_m^{-1}(y_0)$. The assertion now follows from Lemma 3.2.

5. A decomposition lemma

Properties (i)–(iv) and (v) and (vi) of the lemma below are due to Liapunov [15, page 475] and Karafiat [12, page 42], respectively.

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LEMMA 5.1. *Let $m: \mathscr{A} \to \mathbf{R}^p$ be a nonatomic measure, K its range and H a supporting hyperplane of K. Then there exists a decomposition $m = m_1 + m_2$ such that*

(i) *m_1 and m_2 are measures on $\mathscr{A}$ concentrated on disjoint sets S_1 and $S_2 = S \backslash S_1$, respectively,*

(ii) *$K = K_1 + K_2$, where K_i is the range of m_i, $i = 1, 2$,*

(iii) *K_1 lies in the linear subspace of $\mathbf{R}^p$ parallel to H and $\dim K_1 < p$,*

(iv) *K_2 has only one point in H, $K_2 \cap H = \{b\}$ say, and b is an extreme point of K_2,*

(v) *$K \cap H = K_1 + b$,*

(vi) *if (y_n) is a sequence in K convergent to $y_0 \in K \cap H$ and $y_n = y_n^1 + y_n^2$, where $y_n^1 \in K_1$ and $y_n^2 \in K_2$ for all n, then (y_n^1) converges to $y_0 - b$ and (y_n^2) converges to b.*

6. Finite dimensional measures

Now we are ready to prove that finite dimensional nonatomic measures and their integral maps are open maps.

THEOREM 6.1. *Every nonatomic measure $m: P_0 \to K$, where $K \subset \mathbf{R}^p$ is the range of m, is an open map.*

PROOF. We argue by induction on p. If $p = 1$, then K is a compact interval, and so the theorem clearly follows from Lemma 4.1 and Corollary 4.3 (or from [2, Lemma 2.1]).

Let $p > 1$ and suppose that K spans $\mathbf{R}^p$. We shall prove that m is open at each point $x \in P_0$. In view of Corollary 4.3, it is enough to consider the case where $m(x)$ is a boundary point of K. Then there exists a hyperplane H supporting K at $m(x)$. We may assume that $H \cap K \neq \{m(x)\}$ for in the contrary case $m(x) \in \operatorname{ext} K$ and the theorem follows again from Lemma 4.1. We establish the openness of m at x with the help of condition (e) of Lemma 2.1. Accordingly, let (y_n) be a sequence in K convergent to $y = m(x)$. Applying Lemma 5.1 and adopting its notation we see that $m_1(x) = y - b$ and $m_2(x) = b$. Moreover, K_1 spans a proper subspace of $\mathbf{R}^p$ and $b \in \operatorname{ext} K_2$. Therefore, by the induction hypothesis and Lemma 4.1, respectively, m_1 and m_2 are open at x. Hence there exist sequences (x_n^1) and (x_n^2) in P_0 convergent to x such that $x_n^1 \in m_1^{-1}(y_n^1)$ and $x_n^2 \in m_2^{-1}(y_n^2)$ for all n. Let $x_n(t) = x_n^1(t)$ if $t \in S_1$ and $x_n(t) = x_n^2(t)$ if $t \in S_2$, $n = 1, 2, \ldots$. Then (x_n) is a sequence in P_0 convergent to x and $x_n \in m^{-1}(y_n)$ for all n. This completes the proof.

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Now Theorem 6.1 and Lemma 3.2 yield the following theorem.

THEOREM 6.2. *If m is a finite dimensional nonatomic measure, then the integral map $T_m: P \to K$ is an open map.*

7. Applications of openness

Anantharaman and Garg [2, Proposition 2.3] proved that the set of extreme points of the closed convex hull of the range of a vector measure with values in Y is closed provided the integral map is open. However, in the case of a semiconvex measure this result can be easily obtained by applying the following lemma (see [5, Lemma 3]): let T be an open continuous map from a Hausdorff space X onto a topological space Y; for every positive integer n the set

$$F_n = \{y \in Y : T^{-1}(y) \text{ has at most } n \text{ points}\}$$

is closed.

THEOREM 7.1. *Let m be a semiconvex measure and T_m (equivalently m) be open. Then the set of extreme points of the range of m is closed.*

PROOF. It is known that $y \in \operatorname{ext} K$ if and only if $T_m^{-1}(y) = \{\chi_A\}$ for some $A \in \mathscr{A}$, where K is the range of T_m (see [1, Proposition 2]). However, m is semiconvex and so K is equal to the range of m. By the lemma recalled above we conclude that the set $\operatorname{ext} K$ is closed.

COROLLARY 7.2. *If m is a nonatomic finite dimensional valued measure then the set of extreme points of the range of m is closed.*

8. Remarks

Let P_1 be the set of all $x \in L^\infty$ with values in $[-1, 1]$ μ-almost everywhere, endowed as P with the topology w^*. Denote by T_m^1 the map on P_1 defined by the formula

$$T_m^1(x) = T_m((1+x)/2) \quad \text{for } x \in P_1,$$

or equivalently, by the equation $T_m(x) = T_m^1(2x - 1)$, where $x \in P$. Clearly, T_m and T_m^1 have the same range (see [8]). Hence in view of Lemma 2.1, T_m is open if and only if T_m^1 is open (see a problem posed in [16]).

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In [12] Karafiat applied Bolker's characterization of faces of the range of $\mathbf{R}^p$-valued nonatomic measure [5, Corollary 3.3]. But this characterization concerns only exposed faces. Consequently, the proof of Theorem 2 in [12] is complete only for $p \leq 2$.

Anantharaman and Garg [2, proof of Theorem 2.2] use the following assertion: the operation $(A, B) \mapsto A \cap B$ is continuous relative to the Hausdorff metric (see also [13, page 70]). However, this assertion is easily seen to be false.

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