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GENERALISED HELLY AND RADON NUMBERS

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At the Second Oklahoma Conference on Convexity and Related Combinatorial Geometry (1980) G.Sierksma posed several problems dealing with the generalised Helly and Radon numbers of a convexity space. The aim of this note is to give answers to and comment on some of Sierksma's questions.

1. INTRODUCTION

We begin with some preliminary definitions. A *convexity space* is a pair $(X, \mathcal{C})$ consisting of a set X and a family $\mathcal{C} \subset \mathcal{P}(X)$, the power set of X, such that ϕ, $X \in \mathcal{C}$ and $\mathcal{C}$ is closed under arbitrary intersections. $\mathcal{C}$ is called a *convexity structure* for X and the elements of $\mathcal{C}$ are called *$\mathcal{C}$-convex sets*. The *$\mathcal{C}$-hull* of any S in X, denoted by $\mathcal{C}(S)$, is the intersection of all $\mathcal{C}$-convex sets containing S. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are convexity structures on the same set X and $\mathcal{C}_1 \subset \mathcal{C}_2$, then $\mathcal{C}_1$ is said to be a *substructure* of $\mathcal{C}_2$.

A *Radon t-partition* of a set S in X is a partition

$$S = S_1 \cup \ldots \cup S_t$$

into t pairwise disjoint subsets such that

$$\cap\{\mathcal{C}(S_i) : i = 1, \ldots, t\} \neq \emptyset.$$

The *t-Radon number* of a convexity space $(X, \mathcal{C})$ is the smallest integer $r(t)$ (if such exists) such that each set S in X of cardinality at least $r(t)$ admits a Radon t-partition.

The *t-core* of any set S in X is defined by

$$\operatorname{core}_t(S) = \cap\{\mathcal{C}(S \setminus M) : M \subset S, |M| \leqslant t\}.$$

(Throughout this note $|M|$ denotes the cardinality of M). Now we can define the *t-Helly number* of a convexity space $(X, \mathcal{C})$ as the smallest integer $h(t)$ (if such exists) such that any set S with $|S| \geqslant h(t) + 1$ has non-empty t-core.

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The (generalised) Helly and Radon numbers play a very important part in the theory of axiomatic convexity and combinatorial geometry. The numbers, their properties and relationships have been investigated in many papers; see, among others, [**1**, **2**, **3**, **4**, **5**, **9**, **10**, **11**, **12**].

In Section 2 we consider relationships between the t-Helly and t-Radon numbers of two spaces $(X, \mathcal{C}_1)$ and $(X, \mathcal{C}_2)$ under the assumption that $\mathcal{C}_1$ is a substructure of $\mathcal{C}_2$. In Section 3 we comment on relationships between the number of vertices of $\mathrm{core}_t(S)$ and the number of Radon partitions of a set in the case of ordinary convexity space $(\mathbf{R}^n, \mathrm{conv})$. Finally, in Section 4, we consider some variants of Levi's inequality.

2. Convexity parameters for substructures

We start with quoting Problem 4 from [**11**]:

> Let $h_1(t)$ and $h_2(t)$ be the t-Helly numbers of $(X, \mathcal{C}_1)$ and $(X, \mathcal{C}_2)$, respectively. If $\mathcal{C}_1 \subset \mathcal{C}_2$, what is the relationship between $h_1(t)$ and $h_2(t)$? This problem is also open for the other numbers.

Below we establish the relationship for the t-Helly and t-Radon numbers and we show that, in general, any similar relationships for the Carathéodory numbers and exchange numbers do not exist.

First we observe the following simple lemma.

Lemma. *Let $(X, \mathcal{C}_1)$ and $(X, \mathcal{C}_2)$ be convexity spaces with $\mathcal{C}_1 \subset \mathcal{C}_2$. Then for each set S in X we have $\mathcal{C}_2(S) \subset \mathcal{C}_1(S)$.*

Theorem 1. *Let $(X, \mathcal{C}_2)$ have t-Helly number $h_2(t)$ and t-Radon number $r_2(t)$. Then each space $(X, \mathcal{C}_1)$, where $\mathcal{C}_1$ is a substructure of $\mathcal{C}_2$, has t-Helly and t-Radon numbers $h_1(t)$ and $r_1(t)$ and*

(a) $h_1(t) \leqslant h_2(t)$,
(b) $r_1(t) \leqslant r_2(t)$.

Proof: To show (a) we first notice the following simple consequence of the Lemma:

$$\begin{aligned} \mathcal{C}_2 - \mathrm{core}_t(A) &= \bigcap\{\mathcal{C}_2(A \setminus M) : M \subset A,\ |M| \leqslant t\} \\ &\subset \bigcap\{\mathcal{C}_1(A \setminus M) : M \subset A,\ |M| \leqslant t\} \\ &= \mathcal{C}_1 - \mathrm{core}_t(A). \end{aligned}$$

From the definition of $h_2(t)$ it follows that $\mathcal{C}_2 - \mathrm{core}_t(A)$ is non-empty for each set A in X with $|A| \geqslant h_2(t) + 1$. This and the above inclusion imply $\mathcal{C}_1 - \mathrm{core}_t(A) \neq \emptyset$ for each set consisting of at least $h_2(t) + 1$ elements. This, in turn, means that $h_1(t) \leqslant h_2(t)$.

The proof of (b) is also straightforward. ∎

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Simple examples can be given illustrating that both inequalities in the previous theorem are sharp.

We recall the following two definitions.

The *Carathéodory number* of a convexity space $(X, \mathcal{C})$ is the smallest integer c (if such exists) such that for each S in X the following holds

$$\mathcal{C}(S) = \bigcup\{\mathcal{C}(T) : T \subset S,\ |T| \leqslant c\}.$$

The *exchange number* of a convexity space $(X, \mathcal{C})$ is defined as the smallest integer e (if such exists) such that for each point p and each set A in X with $e \leqslant |A| < \infty$ the following holds

$$\mathcal{C}(A) \subset \bigcup\{\mathcal{C}(\{p\} \cup (A \setminus \{a\})) : a \in A\}.$$

It is not surprising that there are examples of convexity structures $\mathcal{C}_1$ and $\mathcal{C}_2$ with $\mathcal{C}_1 \subset \mathcal{C}_2$ for which the Carathéodory and the exchange numbers satisfy the inequalities $c_1 \leqslant c_2$ and $e_1 \leqslant e_2$. Now we are going to present an example illustrating that for these numbers, opposite inequalities are possible as well. We can even show that the existence of $c_2(e_2)$ does not imply the existence of $c_1(e_1)$.

EXAMPLE: On the plane we denote by p_k the point $(1 + 3k, 0)$, $k \geqslant 0$, and by D_k a regular $(k + 4)$-gon inscribed in the circle with the centre at p_k and radius 1. Now we consider the sequence of convexity spaces $(\mathbf{R}^2, \mathcal{C}_k)$, $k \geqslant 0$, where $\mathcal{C}_k$ consists of ϕ, $\mathbf{R}^2$ and *proper* (ordinary) convex subsets of D_j's for $j \leqslant k$. It is obvious that $\mathcal{C}_k \subset$ conv for each $k \geqslant 0$. We show that the Carathéodory number c_k and the exchange number e_k of $(\mathbf{R}^2, \mathcal{C}_k)$ are equal to $k + 4$, although $(\mathbf{R}^2, \text{conv})$ has Carathéodory number $c = 3$ and exchange number $e = 3$. To this end we denote by W_k the set of vertices of D_k. For the sets D_k we have

$$\mathcal{C}_k(W_k) = \mathbf{R}^2,$$

$$\bigcup\{\mathcal{C}_k(T) : T \subset W_k,\ |T| \leqslant k + 3\} = D_k,$$

and

$$\bigcup\{\mathcal{C}_k(\{p_k\} \cup (W_k \setminus \{w\})) : w \in W_k\} = D_k.$$

This means that both c_k and e_k are greater than $k + 3$ and simple reasoning gives $c_k = e_k = k + 4$.

3. VERTICES OF $(t-1)$-CORE AND THE NUMBER OF RADON t-PARTITIONS

In this section we discuss the connection between the number of vertices of the $(t - 1)$-core and the number of Radon partition in the case of ordinary convexity space

$(\mathbf{R}^n, \operatorname{conv})$. Let us recall some information from [8] and [11] about $\operatorname{core}_{t-1}(S)$. First, that $\operatorname{core}_{t-1}(S)$ can be also defined as the intersection of all closed halfspaces containing at least $|S| - t + 1$ points of the set S (so $\operatorname{core}_{t-1}(S)$ is a polytope in $\mathbf{R}^n$). Second, $\operatorname{conv}(D_t) \subset \operatorname{core}_{t-1}(S)$, where D_t stands for the set of all points p such that p belongs to the intersection $\bigcap\{C(S_i) : i = 1, \ldots, t\}$ for some partition $S_1, \ldots, S_t$ of the set S, and in the case of $(\mathbf{R}^2, \operatorname{conv})$ we in fact have equality $\operatorname{conv}(D_t) = \operatorname{core}_{t-1}(S)$. Thus in $(\mathbf{R}^2, \operatorname{conv})$ for D_t and $\operatorname{core}_{t-1}(S)$ we have

$$\operatorname{conv}(D_t) = \operatorname{core}_{t-1}(S) = \operatorname{conv}(V(\operatorname{core}_{t-1}(S))),$$

where $V(Y)$ denotes the set of vertices of the polytope Y. This all justifies the following question: (Problem 9 [**11**])

> Is the number of Radon t-partitions dependent on the number of vertices of $\operatorname{core}_{t-1}(S)$ for finite sets S in $\mathbf{R}^n$? That is, does the set have the minimum number of Radon t-partitions when the $(t-1)$-core is a single point?

It is well-known that 7-element sets in $(\mathbf{R}^2, \operatorname{conv})$ have 3-Radon partitions (see [**13**]). Now we give two examples of a 7-element set in the plane. As we will see, there need not be a strict connection between the number of vertices of $(t-1)$-core and the number of Radon t-partitions. Indeed, consider set S_1 given in Figure 1. In this case $\operatorname{core}_2(S_1) = \{0\}$. It is easy to check that S_1 has the following ten different Radon 3-partitions:

$$\begin{array}{ll}
(\{1, 3\}, \{2, 4\}, \{5, 6, 7\}), & (\{1, 3\}, \{5, 6\}, \{2, 4, 7\}), \\
(\{1, 3\}, \{2, 7\}, \{4, 5, 6\}), & (\{1, 6\}, \{3, 5\}, \{2, 4, 7\}), \\
(\{1, 6\}, \{2, 7\}, \{3, 4, 5\}), & (\{1, 6\}, \{2, 4\}, \{3, 5, 7\}), \\
(\{2, 7\}, \{5, 6\}, \{1, 3, 4\}), & (\{2, 7\}, \{3, 5\}, \{1, 4, 6\}), \\
(\{2, 4\}, \{5, 6\}, \{1, 3, 7\}), & (\{2, 4\}, \{3, 5\}, \{1, 6, 7\}).
\end{array}$$

Now examine the set S_2 given in Figure 2. Note that $\operatorname{core}_2(S_2)$ has 7 vertices and this is the greatest number of vertices for any 7-element set. However, as can be simply verified, S_2 has only the following seven Radon 3-partitions:

$$\begin{array}{ll}
(\{A, C\}, \{E, F\}, \{B, D, G\}), & (\{A, C\}, \{B, G\}, \{D, E, F\}), \\
(\{A, G\}, \{B, D\}, \{C, E, F\}), & (\{A, G\}, \{C, E\}, \{B, D, F\}), \\
(\{B, D\}, \{E, F\}, \{A, C, G\}), & (\{B, G\}, \{D, F\}, \{A, C, E\}). \\
(\{C, E\}, \{D, F\}, \{A, B, G\}). &
\end{array}$$

In a similar way, we may also give examples showing that there is no connection between the number of Radon partitions and the number of vertices of $\operatorname{core}_{t-1}(S)$ in higher dimensions and for greater t.

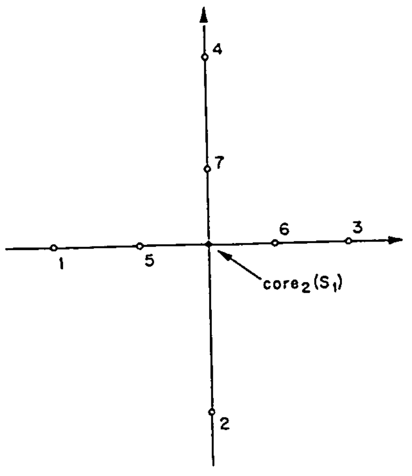

Figure 1.

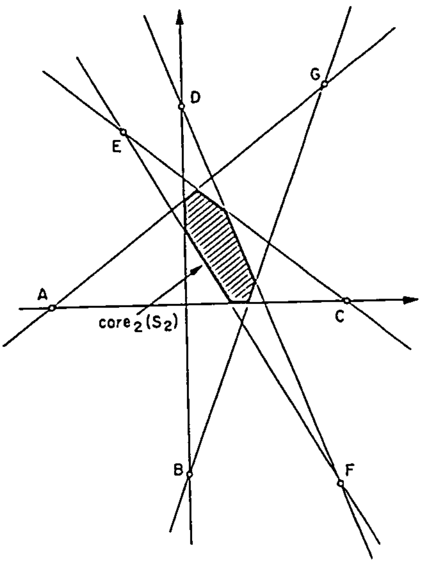

Figure 2.

4. Larman type generalisation of Helly and Radon numbers

The next two definitions are taken from [**11**]. Let $n \geqslant 0$ be an integer. The number $LR(t, n)$ is the infimum of all positive integers k such that for each set S in X with $|S| \geqslant k$ and each set T in S with $|T| \leqslant n$ there exists a t-partition

$$S = S_1 \cup \ldots \cup S_t$$

such that

$$\bigcap\{\mathcal{C}(S_i \setminus T) : i = 1, \ldots, t\} \neq \emptyset.$$

The number $LH(t, n)$ of a convexity space is defined as the infimum of all non-negative integers k such that each S in X with $|S| \geqslant k+1$ and each T in S with $|T| \leqslant n$ has the property that

$$\bigcap\{\mathcal{C}(S \setminus (M \setminus T)) : M \subset S, \quad |M| \leqslant t\} \neq \emptyset.$$

Sierksma asks if the inequality $LH(t, n) \leqslant LR(t+1, n) - 1$ holds ([**11**, Problem 13a]). Note that in the case when $t = 1$ and $n = 0$ it reduces to the well-known Levi's inequality $h(1) \leqslant r(2) - 1$ [**7**]. We show that the answer to Sierksma's question is in the affirmative.

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We begin by listing some properties of the numbers $LR(t, n)$ and $LH(t, n)$ which we will need.

(1) $LH(t, 0) = h(t)$ and $LR(t, 0) = r(t)$.
(2) $LR(t_1, n) \leqslant LR(t_2, n)$ for $t_1 \leqslant t_2$.
(3) $LR(t, n_1) \leqslant LR(t, n_2)$ for $n_1 \leqslant n_2$.
(4) $LH(t, n) \leqslant h(t)$.
(5) $LR(t, n) = r(t) + n$.

To show (4), take any set S with $|S| = h(t) + 1$. Obviously

$$\bigcap\{C(S \setminus M) : M \subset S,\, |M| \leqslant t\} \neq \emptyset.$$

For each $T \subset S$ we have

$$C(S \setminus M) \subset C(S \setminus (M \setminus T)).$$

Hence

$$\bigcap\{C(S \setminus M) : M \subset S,\, |M| \leqslant t\} \subset \bigcap\{C(S \setminus (M \setminus T)) : M \subset S,\, |M| \leqslant t\}.$$

This means that $LH(t, n) \leqslant h(t)$.

Concerning (5), we first show that $LR(t, n) \leqslant r(t) + n$. To this end take any $(r(t) + n)$-element set S in X and a subset T of S with $|T| \leqslant n$. The set $S \setminus T$ has at least $r(t)$ elements, hence there exists a Radon t-partition

$$S \setminus T = S_1 \cup \ldots \cup S_t.$$

The sets $S_i^* = S_i$, $i = 1, \ldots, t-1$, and $S_t^* = S_t \cup T$ form a partition of S for which we have

$$\bigcap\{C(S_i^* \setminus T) : i = 1, \ldots, t\} = \bigcap\{C(S_i) : i = 1, \ldots, t\} \neq \emptyset.$$

So $LR(t, n) \leqslant r(t) + n$. To establish the reverse it suffices to show that any $(LR(t, n) - n)$-element set S in X has a Radon t-partition. For each $LR(t, n)$-element set B containing S and the set $T = B \setminus S$ there exists a partition

$$B = B_1 \cup \ldots \cup B_t$$

such that

$$\bigcap\{C(B_i \setminus T) : i = 1, \ldots, t\} \neq \emptyset.$$

From this it follows that the sets $S_i = B_i \cap S$, $i = 1, \ldots, t$, form the desired Radon t-partition for S. This gives $r(t) \leqslant LR(t, n) - n$ and completes the proof.

THEOREM 2. *For any convexity space having the Radon number $r(2)$ the following inequalities*

$$LH(t, n) \leqslant LR(t+1, n) - 1$$

are true.

PROOF: From results in [1] and [2] it follows that the existence of $r(2)$ implies the existence of all Radon numbers $r(t)$. By the inequality $h(t) \leqslant r(t+1) - 1$, due to Doignon, Reay and Sierksma (D-R-S inequality) [1] we have existence of all Helly numbers $h(t)$. Now using the above mentioned properties and the D-R-S inequality we get

$$LH(t, n) \leqslant h(t) \leqslant r(t+1) - 1 = LR(t+1, 0) - 1 \leqslant LR(t+1, n) - 1,$$

which gives $LH(t, n) \leqslant LR(t+1, n) - 1$ and establishes the theorem. $\square$

Note that from the established inequaltiy we get Levi's inequality (in the case $t = 1$ and $n = 0$) and the D-R-S inequality (for $n = 0$).

We have just solved the Problem 13a in its original form; however we cannot leave it without any comment. The names of the numbers $LR(t, n)$ and $LH(t, n)$ suggest their connection with Larman's consideration [6]. However, Larman considered partitions of the type $S = S_1 \cup S_2$ which remain Radon partitions even after "stealing" any arbitrary element of S, that is,

$$C(S_1 \setminus \{x\}) \cap C(S_2 \setminus \{x\}) \neq \emptyset, \text{ for each } x \in S.$$

We are now going to change the definition of the $LR(t, n)$ numbers in the following way: The $LR^*(t, n)$ number is the infimum (if such exists) of all integers k such that each set S in X with $|S| \geqslant k$ has a partition $S = S_1 \cup \ldots \cup S_t$ such that

$$\bigcap \{C(S_i \setminus M) : i = 1, \ldots, t\} \neq \emptyset \text{ for each } M \subset S,\ |M| \leqslant n.$$

For such $LR^*(t, n)$ numbers we have the following estimates.

THEOREM 3. *If a convexity space has Radon number $r(2)$ then all $LR^*(t, n)$ numbers exist and moreover*

$$(n+1)t \leqslant LR^*(t, n) \leqslant r(nt + t).$$

PROOF: For the same reason as in Theorem 2 all numbers $r(t)$ exist. So, take a set S in X, $|S| = r(nt + t)$. The set has a Radon $(nt + t)$-partition

$$S = (S_1 \cup \ldots \cup S_t) \cup (S_{t+1} \cup \ldots \cup S_{2t}) \cup \ldots \cup (S_{nt+1} \cup \ldots \cup S_{nt+t})$$

such that

$$(*)\qquad \bigcap\{\mathcal{C}(S_i) : i = 1, \ldots, nt + t\} \neq \emptyset.$$

Now consider the sets

$$A_j = \bigcup\{S_{kt+j} : k = 0, \ldots, n\} \text{ for } j = 1, \ldots, t.$$

As we will see the sets form a Larman-type t-partition for S. Indeed, for each $M \subset S$ with $|M| \leqslant n$ and for each j there exists, $k(j)$ such that $S_{k(j)t+j} \subset A_j \setminus M$. Moreover, we have

$$\begin{aligned}\bigcap\{\mathcal{C}(S_i) : i = 1, \ldots, nt + t\} &\subset \bigcap\{\mathcal{C}(S_{k(j)t+j}) : j = 1, \ldots, t\} \\ &\subset \bigcap\{\mathcal{C}(A_j \setminus M) : j = 1, \ldots, t\}.\end{aligned}$$

The above inclusions and (*) imply that the intersection $\bigcap\{\mathcal{C}(A_j \setminus M) : j = 1, \ldots, t\}$ is non-empty for all M. This establishes the right side inequality in our theorem. The left side inequality is obvious. So the proof is complete. $\square$

It is worth adding that the bounds cannot, in general, be improved. Indeed, consider any convexity space having Radon number $r(2) = 2$ and, of course, $r(t) = t$ (such spaces exist!). It is easy to check that in such a space we also have $LR^*(t, n) = (n + 1)t$. It would be interesting to find better estimates for $LR^*(t, n)$ in special cases. In particular, does the inequality $h(t + n) \leqslant LR^*(t + 1, n) - 1$ hold for the ordinary convexity space $(\mathbf{R}^d, \mathrm{conv})$?

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