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DEFINING IDEALS OF BUCHSBAUM SEMIGROUP RINGS

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Introduction

Let H be a *simplicial* semigroup. We consider the semigroup ring $k[H]$ and its defining ideal I_H. For definition see the first paragraph of Section 1.

When $\dim(k[H]) = 1$, the defining ideal I_H of $k[H]$ has been studied by many authors (e.g. [1], [2], [8], [11], [3]). In this paper, we study the ideal I_H using the notion of *Gröbner bases* for arbitrary dimension.

In [9], we gave a condition for $k[H]$ to be *Cohen-Macaulay* in terms of a Gröbner bases of I_H. Our aim of this paper is to extend this characterization to the case of *Buchsbaum* semigroup rings. We show that the Buchsbaum property of $k[H]$ is determined by the form of a Gröbner bases of I_H in Theorem 2.6. As a corollary, we recover a result of [9] in Corollary 2.9. Also we see that if $k[H]$ is a Buchsbaum ring and not Cohen-Macaulay, then $k[H] \leq \mathrm{ht}\, I_H$ in Corollary 2.10.

We apply these results to determine Buchsbaum semigroup rings of codimension two. We can show the Gröbner bases of I_H explicitly in Theorem 3.1.

1. Preliminaries

In this section, we give notations and terminologies which we shall use in this paper.

Let $\mathbf{N}$ be the set of nonnegative integers and H be a finitely generated additive subsemigroup of $\mathbf{N}^r (r > 0)$ with generators $h_1, \ldots, h_{r+n} \in H$ which satisfies the following conditions:

(H-1) $h_1, \ldots, h_r$ are $\mathbf{Q}$-linearly independent
(H-2) there exists an integer $d > 0$ such that $dH \subset \sum_{i=1}^{r} \mathbf{N}h_i$.

Let k be a field. We define a homomorphism φ of polynomial rings over k as:

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$$\begin{array}{rclll} \varphi : S = k[X_1,\ldots,X_r,\ Y_1,\ldots,Y_n] & \to & k[t_1,\ldots,t_r] & \\ X_i & \mapsto & t^{h_i} & (1 \leq i \leq r) \\ Y_j & \mapsto & t^{h_{r+j}} & (1 \leq j \leq n) \end{array}$$

where we denote $t^h := t_1^{a_1} \cdots t_r^{a_r}$ for $h = (a_1, \ldots, a_r) \in \mathbf{N}^r$.

We put $k[H] = \mathrm{Im}(\varphi)$ and $I_H = \ker(\varphi)$. We denote

$$\begin{aligned} x_i &= t^{h_i}\ (1 \leq i \leq r) \\ y_j &= t^{h_{r+j}} (1 \leq j \leq n) \\ \mathbf{m} &= (x_1, \ldots, x_r, y_1, \ldots, y_n) \subset k[H]. \end{aligned}$$

Note that $\{x_1, \ldots, x_r\}$ is a *homogeneous system of parameters* of $k[H]$ by (H-1) and (H-2). Hence we have $r = \dim k[H]$ and $n = \mathrm{ht}\, I_H$.

Definition 1.1. For $\alpha, \beta \in \mathbf{N}^m$, we define

(1) $\alpha_{(i)} =$ the i-th coordinate of α
(2) $\alpha \leq \beta \Leftrightarrow \alpha_{(i)} \leq \beta_{(i)}$ for $1 \leq i \leq n$
(3) $\alpha < \beta \Leftrightarrow \alpha \leq \beta$ for $\alpha \neq \beta$
(4) $\alpha \pm \beta = (\alpha_{(1)} \pm \beta_{(1)}, \ldots, \alpha_{(m)} \pm \beta_{(m)})$.

We denote a monomial of S by $X^\alpha Y^\beta = X_1^{\alpha(1)} \cdots X_1^{\alpha(r)} Y_1^{\beta(1)} \cdots Y_n^{\beta(n)}$ for $\alpha \in \mathbf{N}^r$, $\beta \in \mathbf{N}^n$ and the set of all monomials of S by M_H.

Definition 1.2. A total order $>_S$ on M_H is called a monomial order on S if it satisfies the following conditions, for every $u, v, w \in M_H$,

$$\begin{aligned} &\text{if } u <_S v, \text{ then } uw <_S vw \\ &\text{if } 1 \neq u, \text{ then } 1 <_S u. \end{aligned}$$

Remark 1.3. It is well known that a monomial order $>_S$ on S satisfies the following.

(1) If $(\alpha, \beta) < (\gamma, \delta)$ (in $\mathbf{N}^{r+n}$), then $X^\alpha Y^\beta <_S X^\gamma Y^\delta$.
(2) Every descending sequence of monomials (w.r.t. $>_S$) is stationary. In particular, any nonempty subset of M_H has the smallest element.

For $0 \neq f \in S$, we denote the maximal term of f w.r.t. $<_S$ by $\mathrm{in}(f)$ and call it the *initial term* of f. For a subset $F \subset S$, we set

$$\mathrm{in}(F) = \{\mathrm{in}(f) \mid 0 \neq f \in F\}.$$

DEFINITION 1.4. Let I be an ideal of S and F be a finite subset of $I \setminus \{0\}$.

We call F a Gröbner bases of I, if $(\mathrm{in}(I)) = (\mathrm{in}(F))$. A Gröbner bases F of I is called *minimal,* if $in(F)$ is a minimal basis of $(\mathrm{in}(I))$.

In this case, I is generated by F (cf. [4], [10]).

Throughout this paper, we fix a monomial order $<_S$ on S defined as follows.

DEFINITION 1.5. For $X^\alpha Y^\beta \in M_H$, we denote the total degree of $\varphi(X^\alpha Y^\beta)$ by $\mathrm{wd}(X^\alpha Y^\beta)$. We define

$$X^\alpha Y^\beta >_S X^\gamma Y^\delta \Leftrightarrow \begin{cases} \mathrm{wd}(X^\alpha Y^\beta) > \mathrm{wd}(X^\gamma Y^\delta) \\ \text{or} \\ \mathrm{wd}(X^\alpha Y^\beta) = \mathrm{wd}(X^\gamma Y^\delta) \text{ and the first non zero coordinate} \\ \text{of } (\alpha, \beta) - (\gamma, \delta) \ (\in \mathbf{Z}^{r+n}) \text{ is a negative.} \end{cases}$$

In this case, the monomial order $>_S$ has the following property. If $X^\alpha Y^\beta - X^\gamma Y^\delta \in I_H$ and $X^\alpha Y^\beta >_S X^\gamma Y^\delta$, then $\alpha > 0$ implies $\gamma > 0$ since $\varphi(X^\alpha Y^\beta) = \varphi(X^\gamma Y^\delta)$. We shall use this fact freely this paper.

Next we define some notation.

NOTATION 1.6. (1) For a subset J of $k[H]$, we put

$$M(J) = \{X^\alpha Y^\beta \in M_H \mid \varphi(X^\alpha Y^\beta) \in J\}.$$

(2) For $X^\alpha Y^\beta \in M_H$, we put

$$\Sigma(X^\alpha Y^\beta) = \{X^\gamma Y^\delta \in M_H \mid \varphi(X^\alpha Y^\beta) = \varphi(X^\gamma Y^\delta),\ X^\alpha Y^\beta >_S X^\gamma Y^\delta\}.$$

Remark 1.7. By definition, we have the following.

(1) For $X^\alpha Y^\beta, X^\gamma Y^\delta \in M_H$ with $X^\alpha Y^\beta \neq X^\gamma Y^\delta$, $X^\alpha Y^\beta - X^\gamma Y^\delta \in I_H$ if and only if $X^\alpha Y^\beta \in \Sigma(X^\gamma Y^\delta)$ or $X^\gamma Y^\delta \in \Sigma(X^\alpha Y^\beta)$.

(2) For $X^\alpha Y^\beta \in M_H$, $\Sigma(X^\alpha Y^\beta) \neq \phi$ if and only if $X^\alpha Y^\beta \in (\mathrm{in}(I_H))$.

(3) If $X^\alpha Y^\beta$ is the smallest element of $\Sigma(X^\gamma Y^\delta)$ (w.r.t. $>_S$), then $\Sigma(X^\alpha Y^\beta) = \phi$.

(4) If $(\alpha, \beta) \leq (\gamma, \delta)$ (in $\mathbf{N}^{r+n}$), then $\Sigma(X^\alpha Y^\beta) \neq \phi$ implies $\Sigma(X^\gamma Y^\delta) \neq \phi$.

(5) If $X^\alpha Y^\beta - X^\gamma Y^\delta \in I_H$ and $J \subset k[H]$, then $X^\alpha Y^\beta \in M(J)$ if and only if $X^\gamma Y^\delta \in M(J)$.

(6) For $1 \leq i \leq r$, $X^\alpha Y^\beta \in M((x_i))$ if and only if there exists $X^\alpha Y^\beta - X_i X^\gamma Y^\delta \in I_H$.

(7) For $1 \leq i \leq r$, if $Y^{\beta} \in M((x_i))$, then $\sum(Y^{\beta}) \neq \phi$.

We put

$$\mathcal{R} = \{X^{\alpha}Y^{\beta} - X^{\gamma}Y^{\delta} \in S \mid X^{\gamma}Y^{\delta} \in \sum(X^{\alpha}Y^{\beta}) \text{ and } \mathrm{Gcd}(X^{\alpha}Y^{\beta}, X^{\gamma}Y^{\delta}) = 1\}.$$

Then, by Remark 1.7 (1), we have $\mathcal{R} \subset I_H$. Furthermore, the following result is standard (cf. Proposition 1.4 and Proposition 1.5 of [8]).

PROPOSITION 1.8. *We have* $I_H = (\mathcal{R})$ *and* $(\mathrm{in}(I_H)) = (\mathrm{in}(\mathcal{R}))$. *Thus we can choose a Gröbner bases of* I_H *from* $\mathcal{R}$.

2. Buchsbaum property of semigroup rings

In this section, we give a condition for $k[H]$ to be Buchsbaum in terms of a Gröbner bases of I_H.

We recall that a Noetherian local ring $(A, \mathbf{n})$ is called a Buchsbaum ring, if $l_A(A/\mathrm{q}) - e_{\mathbf{q}}(A)$ is a constant for every parameter ideal $\mathbf{q}$ of A.

$k[H]$ is called a Buchsbaum ring, if the local ring $k[H]_{\mathbf{m}}$ of $k[H]$ at $\mathbf{m}$ is a Buchsbaum ring. In this case, $k[H]$ satisfies the following conditions: for every $1 \leq i < j \leq r$,

$$[(x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}}) : x_i^{n_i}] = [(x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}}) : \mathbf{m}]$$

$$[(x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}}) : x_i^{n_i} x_j^{n_j}] = [(x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}}) : x_i^{n_i}]$$

where $n_1, \ldots, n_r \in \mathbf{N} \setminus \{0\}$ (cf. Proposition 1.10 of ch. 1 in [12]).

In [6], S. Goto proved the following criterion for $k[H]$ to be Buchsbaum.

THEOREM 2.1 (Theorem 3.1 in [6]). *The following conditions are equivalent.*

(1) $k[H]$ *is a Buchsbaum ring.*

(2) *There exists a simplicial semigroup* $H' \subset \mathbf{N}^r$ *such that* $k[H']$ *is a Cohen-Macaulay and* $\mathbf{m}k[H'] \subset k[H]$.

(3) *For* $1 \leq i \leq r$, $[(x_1^2, \ldots, x_{i-1}^2) : x_i^2] = [(x_1^2, \ldots, x_{i-1}^2) : \mathbf{m}]$ (*i.e.* $k[H]$ *is a quasi-Buchsbaum ring*).

LEMMA 2.2. *For* $t^v, t^{u_1}, \ldots, t^{u_p} \in k[H]$, *we have*

$$[(t^{u_1}, \ldots, t^{u_p}) : t^v] = \sum_{i=1}^{p} [(t^{u_i}) : t^v].$$

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Proof. It is clear that $[(t^{u_1},\dots,t^{u_p}):t^v]\supset\sum_{i=1}^{p}[(t^{u_i}):t^v]$. We show the converse inclusion.

For $f\in[(t^{u_1},\dots,t^{u_p}):t^v]$, we write $f=\sum_{i=1}^{m}c_it^{w_i}$, $c_i\neq 0$ and $w_i\in H\ (1\leq i\leq m)$. Then $t^vf=\sum_{i=1}^{m}c_it^{v+w_i}\in(t^{u_1},\dots,t^{u_p})$. Since $(t^{u_1},\dots,t^{u_p})$ is a $\mathbf{N}^r$-graded ideal, $t^{v+w_j}\in(t^{u_1},\dots,t^{u_p})$ for every $1\leq j\leq m$. Then we have $t^{v+w_j}=t^{h+u_i}$ for some $1\leq i\leq p$ and for some $h\in H$. Thus $t^{w_j}\in[(t^{u_i}):t^v]$ and $f\in\sum_{i=1}^{p}[(t^{u_i}):t^v]$. □

Hence we have the following by Theorem 2.1 and Lemma 2.2.

Proposition 2.3. *The following conditions are equivalent.*

(1) $k[H]$ *is a Buchsbaum ring.*

(2) *For every* $1\leq i<j\leq r$, $[(x_i^2):x_j^2]=[(x_i^2):\mathbf{m}]$.

(2′) *For* $1\leq i<j\leq r$ *and* $u,v\in H$, *if* $2h_j+v=2h_i+u$, *when* $(H\setminus\{0\})+v\subset H+2h_i$. □

Proposition 2.4. (Theorem 2.6 in [7]). *The following conditions are equivalent.*

(1) $k[H]$ *is a Cohen-Macaulay ring.*

(2) $x_1,\dots,x_r$ *are regular sequence of* $k[H]$.

(3) $[(x_i):x_j]=(x_i)$ *for every* $1\leq i<j\leq r$.

(3′) *For* $1\leq i<j\leq r$ *and* $u,v\in H$, *if* $h_j+v=h_i+u$, *then* $v\in H+h_i$. □

We define the subsets $\mathcal{R}_H$, $\mathcal{F}_H$ and $\mathcal{F}'$ of $\mathcal{R}$ as:

$$\mathcal{R}_H=\{X^\alpha Y^\beta-X^\gamma Y^\delta\in\mathcal{R}\mid\Sigma(Y^\delta)=\phi\}$$
$$\mathcal{F}_H=\{f\in\mathcal{R}_H\mid\mathrm{in}(f)=Y^\beta,\ \beta\in\mathbf{N}^n\}$$
$$\mathcal{F}'=\{f\in\mathcal{R}_H\mid\mathrm{in}(f)\in(\mathrm{in}(\mathcal{F}_H))\}.$$

By Proposition 1.8, Remark 1.7 (3) and (4), it is easy to see that $(\mathrm{in}(I_H))=(\mathrm{in}(\mathcal{R}_H))$.

Definition 2.5. A sequence of monomials $(Y^{\beta_1},\dots,Y^{\beta_r})$ is called a B-sequence, if it satisfies the following conditions: for every $1\leq i\neq j\leq r$,

(B-1) $\Sigma(Y^{\beta_i})=\phi$

(B-2) $Y^{\beta_i}\in M([(x_i):\mathbf{m}])$

(B-3) $Y^{\beta_i}\notin M([(x_j):\mathbf{m}])$

(B-4) $\mathrm{Gcd}(Y^{\beta_i},Y^{\beta_j})=1$

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$$\text{(B-5)}\ X_jY^{\beta_i} - X_iY^{\beta_j} \in I_H.$$

We denote by Δ_H the set of all B-sequences and put

$$\mathscr{G}_H = \{X_jY^{\beta_i} - X_iY^{\beta_j} \mid (Y^{\beta_1},\ldots,Y^{\beta_r}) \in \Delta_H,\ 1 \leq i < j \leq r\}.$$

The main purpose of this section is to prove the following result.

THEOREM 2.6. *The following conditions are equivalent.*
(1) $k[H]$ *is a Buchsbaum ring.*
(2) *We can choose a Gröbner bases of* I_H *from* $\mathscr{F}_H \cup \mathscr{G}_H$ (*or* $(in(I_H)) = (in(\mathscr{F}_H \cup \mathscr{G}_H)))$.

To prove our result, we need some lemmas.

LEMMA 2.7. *Suppose that* $k[H]$ *is a Buchsbaum ring.*
(1) *If* $X^\alpha Y^\beta \in M([(x_i):x_j])$ *and* $\alpha_{(i)} = 0$, *then* $Y^\beta \in M([(x_i):x_j])$.
(2) *If* $Y^\beta \in M([(x_i):x_j])$ *and* $\sum(Y^\beta) = \phi$, *then there exists* $(Y^{\beta_1},\ldots,Y^{\beta_r}) \in \Delta_H$ *such that* $\beta = \beta_i$.

Proof. (1) This is proved by induction on the degree of X^α. If $\alpha = 0$, then there is nothing to prove. If $\alpha > 0$, then we can find $1 \leq l \leq r$, $l \neq i$ such that $X^\alpha = X_lX^{\alpha'}$. Then we have $X^{\alpha'}Y^\beta \in M([(x_i):x_jx_l])$. Since $k[H]$ is Buchsbaum, $[(x_i):x_jx_l] = [(x_i):x_j]$. Hence, by the induction hypothesis, we have $Y^\beta \in M([(x_i):x_j])$.

(2) Since $k[H]$ is Buchsbaum, $[(x_i):x_j] = [(x_i):\mathbf{m}]$. Then, by Remark 1.7 (6), we have $X_kY^\beta - X_iX^{\alpha_k}Y^{\beta_k} \in I_H$ for $1 \leq k \neq i \leq r$. If $\sum(X^{\alpha_k}Y^{\beta_k}) \neq \phi$, then we can replace $X^{\alpha_k}Y^{\beta_k}$ by the smallest element of $\sum(X^{\alpha_k}Y^{\beta_k})$. Therefore we may assume $\sum(X^{\alpha_k}Y^{\beta_k}) = \phi$. If $\alpha_{k(k)} > 0$, then $Y^\beta \in M((x_i))$ and, by Remark 1.7 (7), $\sum(Y^\beta) \neq \phi$. This contradicts our assumption. Thus $\alpha_{k(k)} = 0$. On the other hand, $X^{\alpha_k}Y^{\beta_k} \in M([(x_k):x_i])$. By (1), this implies $Y^{\beta_k} \in ([(x_k):x_i])$. Then, by Remark 1.7 (6), there exists $X_iY^{\beta_k} - X_kX^\gamma Y^\delta \in I_H$. Hence we have

$$(X_kY^\beta - X_iX^{\alpha_k}Y^{\beta_k}) - X^{\alpha_k}(X_iY^{\beta_k} - X_kX^\gamma Y^\delta) = X_k(Y^\beta - X^{\alpha_k+\gamma}Y^\delta) \in I_H$$

and $Y^\beta - X^{\alpha_k+\gamma}Y^\delta \in I_H$, since I_H is a prime ideal. By Remark 1.7 (1) and $\sum(Y^\beta) = \phi$, we have $Y^\beta \leq_S X^{\alpha_k+\gamma}Y^\delta$ and, by the definition of the ordering $>_S$, $\alpha_k + \gamma = 0$. Hence we have $X_kY^\beta - X_iY^{\beta_k} \in I_H$ with $\sum(Y^{\beta_k}) = \phi$ for $1 \leq k \neq i \leq r$, We put $\beta_i = \beta$. Then the sequence $(Y^{\beta_1},\ldots,Y^{\beta_r})$ satisfies the conditions (B-1) and (B-2) of Definition 2.5. We show the other conditions are also satisfied.

(B-5): For every $1 \le k < l \le r$, we have the following relation

$$X_k(X_lY^{\beta_i} - X_iY^{\beta_l}) - X_l(X_kY^{\beta_i} - X_iY^{\beta_k}) = X_i(X_lY^{\beta_k} - X_kY^{\beta_l}) \in I_H.$$

Since I_H is a prime ideal, we have $X_lY^{\beta_k} - X_kY^{\beta_l} \in I_H$.

(B-3): For some $1 \le k, l \le r$, if $Y^{\beta_k} \in M([(x_l) : \mathbf{m}])$, then there exists $X_kY^{\beta_k} - X_lX^{\gamma}Y^{\delta} \in I_H$. Hence we have the relation

$$X_l(X_kY^{\beta_k} - X_lX^{\gamma}Y^{\delta}) - X_k(X_lY^{\beta_k} - X_kY^{\beta_l}) = X_k^2Y^{\beta_l} - X_l^2X^{\gamma}Y^{\delta} \in I_H$$

and $Y^{\beta_l} \in M([(x_l^2) : x_k^2])$. Since $k[H]$ is Buchsbaum, $[(x_l^2) : x_k^2] = [(x_l^2) : x_k]$ and $[(x_k) : x_l^2] = [(x_k) : x_l]$. Then it is easy to see that $[(x_l^2) : x_k^2] = x_l[(x_l) : x_k]$ and thus $Y^{\beta_l} \in M((x_l))$. Hence, by Remark 1.7 (7), $\sum(Y^{\beta_l}) \neq \phi$. This contradicts condition (B-1). Thus $Y^{\beta_k} \notin M([(x_l) : \mathbf{m}])$.

(B-4): For some $1 \le k, l \le r$, if $\mathrm{Gcd}(Y^{\beta_k}, Y^{\beta_l}) \neq 1$, then we can write

$$X_lY^{\beta_k} - X_kY^{\beta_l} = Y^{\delta}(X_lY^{\delta_k} - X_kY^{\delta_l}) \in I_H$$

where $Y^{\delta} = \mathrm{Gcd}(Y^{\beta_k}, Y^{\beta_l})$, $\delta_k = \beta_k - \delta$ and $\delta_l = \beta_l - \delta$. Then we have $X_lY^{\delta_k} - X_kY^{\delta_l} \in I_H$ and $Y^{\delta_k} \in M([(x_k) : x_l]) = M([(x_k) : \mathbf{m}])$, since $k[H]$ is Buchsbaum. Since $Y^{\delta} \neq 1$, we have $Y^{\beta_k} = Y^{\delta+\delta_k} \in M((x_k))$. Then, by (1.7), 7), $\sum(Y^{\beta_k}) \neq \phi$. This is a contradiction. Hence $\mathrm{Gcd}(Y^{\beta_k}, Y^{\beta_l}) = 1$. □

LEMMA 2.8. *Suppose that* $(\mathrm{in}(I_H)) = (\mathrm{in}(\mathscr{F}_H \cup \mathscr{G}_H))$.

(1) *For* $X^{\alpha}Y^{\beta} - X^{\gamma}Y^{\delta} \in I_H$ *with* $\sum(Y^{\beta}) = \phi$ *and* $X^{\alpha}Y^{\beta} >_S X^{\gamma}Y^{\delta}$, *there exists* $(Y^{\beta_1}, \dots, Y^{\beta_r}) \in \Delta_H$ *such that* $X^{\alpha}Y^{\beta} = X_jX^{\alpha'}Y^{\beta_i}$ *for some* $1 \le i \neq j \le r$.

(2) $\mathscr{R}_H = \mathscr{F}' \cup \mathscr{G}_H$.

Proof. (1) Since $(\mathrm{in}(I_H)) = (\mathrm{in}(\mathscr{F}_H \cup \mathscr{G}_H))$ and $\sum(Y^{\beta}) = \phi$, we have $X^{\alpha}Y^{\beta} \in (\mathrm{in}(\mathscr{G}_H))$ by Remark 1.7 (2) and (4). Thus there exists a B-sequence $(Y^{\beta_1}, \dots, Y^{\beta_r})$ and an element $X_jY^{\beta_i} - X_iY^{\beta_j} \in \mathscr{G}_H$ such that $X^{\alpha}Y^{\beta}$ is divisible by $X_jY^{\beta_i}$. If $\beta_t < \beta$, then $Y^{\beta} \in M((x_i))$ by the condition (B-2). By Remark 1.7 (7), this contradicts $\sum(Y^{\beta}) = \phi$. Hence we have $\beta = \beta_i$.

(2) Suppose that $\mathscr{R}_H \neq \mathscr{F}' \cup \mathscr{G}_H$. Then, by Remark 1.3 (2), there exists $f = X^{\alpha}Y^{\beta} - X^{\gamma}Y^{\delta} \in \mathscr{R}_H \backslash \mathscr{F}' \cup \mathscr{G}_H$ such that $\mathrm{in}(f)$ is smallest element of $\mathrm{in}(\mathscr{R}_H \backslash \mathscr{F}' \cup \mathscr{G}_H)$. Since $f \notin \mathscr{F}'$ and Remark 1.7 (2), we have $\sum(Y^{\beta}) = \phi$ and, by (1), there exists $g = X_jY^{\beta_i} - X_iY^{\beta_j} \in \mathscr{G}_H$ such that $X^{\alpha}Y^{\beta} = X_jX^{\alpha'}Y^{\beta_i}$. Thus we have

$$f - X^{\alpha'}g = X_iX^{\alpha'}Y^{\beta_j} - X^{\gamma}Y^{\delta} \in I_H.$$

If $X_iX^{\alpha'}Y^{\beta_j} = X^{\gamma}Y^{\delta}$, then $X^{\alpha'} = 1$, since $\mathrm{Gcd}(X^{\alpha}, X^{\gamma}) = 1$. Thus $f = g \in \mathscr{G}_H$.

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This is a contradiction. Hence we have $X_iX^{\alpha'}Y^{\beta_j} \neq X^{\gamma}Y^{\delta}$. We put

$$X^{\mu}Y^{\nu} = \mathrm{Gcd}(X_iX^{\alpha'}Y^{\beta_j},\ X^{\gamma}Y^{\delta}) \text{ and } g' = X^{\mu_1}Y^{\nu_1} - X^{\mu_2}Y^{\nu_2} \in I_H$$

where $X_iX^{\alpha'}Y^{\beta_j} = X^{\mu+\mu_1}Y^{\nu+\nu_1}$ and $X^{\gamma}Y^{\delta} = X^{\mu+\mu_2}Y^{\nu+\nu_2}$. Then either $g' \in \mathcal{R}_H$ or $-g' \in \mathcal{R}_H$. Since $\mathrm{in}(f) >_S \mathrm{in}(g')$, $g' \in \mathcal{F}' \cup \mathcal{G}_H$ or $-g' \in \mathcal{F}' \cup \mathcal{G}_H$ by the minimality of f. We note that $\sum(Y^{\beta_j}) = \phi$ and $\sum(Y^{\delta}) = \phi$. This implies that $\sum(Y^{\nu_p}) = \phi$ $(p = 1,2)$ and $\pm g' \notin \mathcal{F}'$. Hence we have either $g' \in \mathcal{G}_H$ or $-g' \in \mathcal{G}_H$ and there exists $(Y^{\delta_1},\ldots,Y^{\delta_r}) \in \Delta_H$ such that $g' = X_lY^{\delta_k} - X_kY^{\delta_l}$ for some $1 \leq k \neq l \leq r$. Since $\alpha_{(j)} > 0$, $\gamma_{(k)} > 0$ and $\mathrm{Gcd}(X^{\alpha}, X^{\gamma}) = 1$, we have $j \neq k$. Then

$$Y^{\beta_j} = Y^{\nu+\delta_k} \in M([(x_j) : \mathbf{m}]) \cap M([(x_k) : \mathbf{m}]).$$

This contradicts condition (B-3). Hence we have $\mathcal{R}_H = \mathcal{F}' \cup \mathcal{G}_H$. □

Proof of Theorem 2.6. (1)$\Rightarrow$(2) For $f = X^{\alpha}Y^{\beta} - X^{\gamma}Y^{\delta} \in \mathcal{R}_H$, we suppose that $\mathrm{in}(f) \notin (\mathrm{in}(\mathcal{F}_H))$. Then $\alpha > 0$ and, by the definition of $>_S$, we can find $1 \leq i < j \leq r$ such that $f = X_jX^{\alpha'}Y^{\beta} - X_iX^{\gamma'}Y^{\delta}$. Thus $X^{\alpha'}Y^{\beta} \in M([(x_i) : x_j])$ by Remark 1.7 (5). Since $\mathrm{Gcd}(X^{\alpha}, X^{\gamma}) = 1$, we have $Y^{\beta} \in M([(x_i) : x_j])$ by (2.7), 1). Then, by Remark 1.7 (2) and (4), $\sum(Y^{\beta}) = \phi$ and, by Lemma 2.7 (2), there exists $(Y^{\beta_1},\ldots,Y^{\beta_r}) \in \Delta_H$ such that $\beta = \beta_i$. Thus, $X_jY^{\beta_i} - X_iY^{\beta_j} \in \mathcal{G}_H$ and $\mathrm{in}(f) = X^{\alpha}Y^{\beta} \in (\mathrm{in}(\mathcal{G}_H))$.

Hence we have $(\mathrm{in}(I_H)) = (\mathrm{in}(\mathcal{F}_H \cup \mathcal{G}_H))$.

(2)$\Rightarrow$(1) By Proposition 2.3, it suffices to show that $[(x_i^2) : x_j^2] = [(x_i^2) : \mathbf{m}]$ for $1 \leq i < j \leq r$. Therefore we suppose that $[(x_i^2) : x_j^2] \neq [(x_i^2) : \mathbf{m}]$. Then there exists the smallest element $X^{\alpha}Y^{\beta}$ of $M([(x_i^2) : x_j^2] \setminus [(x_i^2) : \mathbf{m}]$. We note that $\alpha_{(i)} \leq 1$ and $\sum(Y^{\beta}) = \phi$. By Remark 1.7 (6), there exists $X_j^2X^{\alpha}Y^{\beta} - X_i^2X^{\gamma}Y^{\delta} \in I_H$. If $\sum(X^{\gamma}Y^{\delta}) \neq \phi$, then we can replace $X^{\gamma}Y^{\delta}$ by the smallest element of $\sum(X^{\gamma}Y^{\delta})$. Thus we may assume that $\sum(X^{\gamma}Y^{\delta}) = \phi$. Since $X^{\alpha}Y^{\beta} \notin M((x_i^2))$, we have $\gamma_{(j)} \leq 1$. Now we put

$$X^{\mu}Y^{\nu} = \mathrm{Gcd}(X_j^2X^{\alpha}Y^{\beta},\ X_i^2X^{\gamma}Y^{\delta}) \text{ and } g = X^{\mu_1}Y^{\nu_1} - X^{\mu_2}Y^{\nu_2} \in I_H$$

where $X_j^2X^{\alpha}Y^{\beta} = X^{\mu+\mu_1}Y^{\nu+\nu_1}$ and $X_i^2X^{\gamma}Y^{\delta} = X^{\mu+\mu_2}Y^{\nu+\nu_2}$. Then $g \in \mathcal{R}_H$ or $-g \in \mathcal{R}_H$. But, by Remark 1.7 (4), $\sum(Y^{\nu_1}) = \phi = \sum(Y^{\nu_2})$ and, by Remark 1.7 (2), $\mathrm{in}(g) \notin (\mathrm{in}(\mathcal{F}_H))$. Then, by Lemma 2.8 (2), we have $g \in \mathcal{G}_H$ or $-g \in \mathcal{G}_H$. On the other hand, $\mu_{1(j)} > 0$ and $\mu_{2(i)} > 0$, since $\gamma_{(j)} \leq 1$, $\alpha_{(i)} \leq 1$. Thus we have $X^{\mu_1} = X_j$ (resp. $X^{\mu_2} = X_i$) and $\alpha_{(i)} = 1$ (resp. $\gamma_{(j)} = 1$). Hence $X^{\alpha}Y^{\beta} \in M(x_i[(x_i) : \mathbf{m}]) \subset M([(x_i^2) : \mathbf{m}])$. This is a contradiction. □

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As a consequence of Theorem 2.6, we have the following corollaries.

COROLLARY 2.9 (Theorem 1.2 in [9]). *The followrng conditions are equivalent.*

(1) $k[H]$ *is a Cohen-Macaulay ring.*

(2) *We can take a Gröbner bases of* I_H *from* $\mathscr{F}_H$ (*or* $(\mathrm{in}(I_H)) = (\mathrm{in}(\mathscr{F}_H))$).

Proof. By Theorem 2.6, we may assume that $k[H]$ is a Buchsbaum ring.

If $k[H]$ is Cohen-Macaulay, then we have $[(x_i) : x_j] = (x_i)$ for every $1 \leq i \neq j \leq r$. Hence, if $Y^\beta \in M([(x_i) : \mathbf{m}])$, then $\Sigma(Y^\beta) \neq \phi$ by Remark 1.7 (7). Thus $\mathscr{G}_H = \phi$ and $\Delta_H = \phi$ (cf. Definition 2.5). Hence we have $(\mathrm{in}(I_H)) = (\mathrm{in}(\mathscr{F}_H))$.

Conversely, if $k[H]$ is not Cohen-Macaulay, then there exists $1 \leq i < j \leq r$ such that $[(x_i) : x_j] \neq (x_i)$ (cf. Proposition 2.4). Since $k[H]$ is a Buchsbaum, we have $[(x_i) : x_j] = [(x_i) : \mathbf{m}]$ and there exists the smallest element $Y^\beta \in M([(x_i) : x_j] \setminus (x_i))$ (cf. Lemma 2.7 (1)). Then, by Remark 1.7 (2), $Y^\beta \notin (\mathrm{in}(I_H))$ and $X_j Y^\beta \in (\mathrm{in}(I_H))$ since $i < j$. This shows that $(\mathrm{in}(I_H)) \neq (\mathrm{in}(\mathscr{F}_H))$. □

COROLLARY 2.10. *If* $k[H]$ *is a Buchsbaum ring and not Cohen-Macaulay, then we have*

$$\dim k[H] \leq \mathrm{ht}_S\, I_H \quad (i.e.\ r \leq n).$$

Proof. Since $k[H]$ is not Cohen-Macaulay, we have $\mathscr{G}_H \neq \phi$ by Theorem 2.6 and Corollary 2.9. Thus there exists $(Y^{\beta_1}, \ldots, Y^{\beta_r}) \in \Delta_H$. We put $n_i = \#\{k \in \{1, \ldots, r\} \mid \beta_{i(k)} > 0\}$. Then, by (B-2) of Definition 2.5, $\beta_i \neq 0$ and $n_i > 0$ for $1 \leq i \leq r$. Hence $r \leq \sum_{i=1}^r n_i$. On the other hand, $\sum_{i=1}^r n_i \leq n$ since $\mathrm{Gcd}(Y^{\beta_k}, Y^{\beta_l}) = 1$ (cf. (B-4) of Definition 2.5), for $1 \leq k \neq l \leq r$. □

In the following example, we see that there exists a Buchsbaum and not Cohen-Macaulay semigroup ring $k[H]$ with $k[H] = r$ and $\mathrm{ht}\, I_H = n$ for $2 \leq r \leq n \in \mathbf{N}$.

EXAMPLE 2.11. For $2 \leq r \leq n \in \mathbf{N}$, we let $a_0, \ldots, a_p \in \mathbf{N}$ $(p = n - r + 1)$ such that $a_p \notin \sum_{i=0}^{p-1} \mathbf{N} a_i$ and $a_p + a_j \notin \sum_{i=0}^{p-1} \mathbf{N} a_i$, for $1 \leq j \leq p$.

(e.g. $(a_0, \ldots, a_{p-1}) = 1$, $a_p = \max\{a \in \mathbf{N} \mid a \notin \sum_{i=0}^{p-1} \mathbf{N} a_i\}$, if $r < n$ or $a_0 = 2$, $a_1 = 1$, if $r = n$.)

We put $h_1, \ldots, h_{r+n}, g \in \mathbf{N}^r$ as follows:

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$$\begin{array}{lll} h_i & = (h_{i(i)} = a_0,\ h_i(j) = 0\ (j \neq i)) & (1 \leq i \leq r) \\ h_{2r+j} & = (a_j, \ldots, a_j) & (1 \leq j \leq p-1) \\ g & = (a_p, \ldots, a_p) & \\ h_{r+i} & = h_i + g & (1 \leq i \leq r) \\ H & = \sum_{i=1}^{r+n} \mathbf{N} h_i & \subset \mathbf{N}^r \\ H' & = \sum_{1 \leq j \leq r \text{ or } 2 \leq j \leq r+n} \mathbf{N} h_j + \mathbf{N} g & \subset \mathbf{N}^r. \end{array}$$

We define

$$\varphi : k[X_1, \ldots, X_r, Y_1, \ldots, Y_n] \to k[t_1, \ldots, t_r]$$

by $\varphi(X_i) = t^{h_i}$ $(1 \leq i \leq r)$, $\varphi(Y_j) = t^{h_{r+j}}$ $(1 \leq j \leq n)$,

$$\varphi' : k[X_1, \ldots, X_r, Z, Y_{r+1}, \ldots, Y_n] \to k[t_1, \ldots, t_r]$$

by $\varphi'(X_i) = t^{h_i}$ $(1 \leq i \leq r)$, $\varphi'(Y_j) = t^{h_{2r+j}}$ $(1 \leq j \leq p-1)$, $\varphi'(Z) = t^g$.

Then, by Example 2.6 in [9], $k[H'](= \operatorname{Im} \varphi')$ is a Cohen-Macaulay ring with $\mathcal{R}_{H'} = \mathcal{F}_{H'}$. Also we have $k[H]$ is a Buchsbaum ring. In fact, by the choice of $a_0, \ldots, a_p$, $(H \setminus \{0\}) + H' \subset H$. Hence, by Theorem 2.1, $k[H]$ is a Buchsbaum ring. In this case, it is easy to see that

(1) $\Delta_H = \{(Y_1, \ldots, Y_r)\}$

(2) $\mathcal{R}_H = \mathcal{F}_H \cup \mathcal{G}_H$. □

3. Codimension two Buchsbaum semigroup rings

In this section we determine simplicial semigroups which defines Buchsbaum semigroup rings of codimension two. Henceforce we put $(\operatorname{ht} I_H =) n = 2$.

When $k[H]$ is a Cohen-Macaulay ring, we determined a Gröbner bases of I_H explicitly in Summary 2.5 of [9]. Therefore it suffices to consider Buchsbaum and not Cohen-Macaulay semigroup rings.

Then we have following result.

THEOREM 3.1. *The following conditions are equivalent.*

(1) $k[H]$ *is a Buchsbaum ring and not Cohen-Macaulay.*

(2) $\dim k[H] = 2$ *and* I_H *has the following minimal basis*

$$\begin{array}{l} Y_i^{b_i+1} - X_1^{a_1-1} X_2^{a_2+1} Y_j^{b_j-1},\ Y_1 Y_2 - X_1^{a_1} X_2^{a_2}, \\ Y_j^{b_j+1} - X_1^{a_1+1} X_2^{a_2-1} Y_i^{b_i-1},\ X_2 {Y^{b_j}}_i - X_1 X_i^{b_i} \end{array}$$

where $a_1, a_2, b_1, b_2 \in \mathbf{N} \setminus \{0\}$, $\{i, j\} = \{1,2\}$.

(3) $k[H]$ *is not Cohen-Macaulay and* H *is isomorphic to*

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$< (b_1+b_2, 0), (0, b_1+b_2), (a_1b_2-1, a_2b_2+1), (a_1b_1+1, a_2b_1-1) >$ *as semigroup.*

To prove our result, we need some preliminaries.

Now, we divide $\mathscr{F}_H$ into the following subsets.

$$\mathscr{F}_1 = \{f \in \mathscr{F}_H \mid in(f) = Y_1^b,\ 0 < b \in \mathbf{N}\}$$
$$\mathscr{F}_2 = \{f \in \mathscr{F}_H \mid in(f) = Y_2^c,\ 0 < c \in \mathbf{N}\}$$
$$\mathscr{F}_3 = \{f \in \mathscr{F}_H \mid in(f) = Y_1^b Y_2^c,\ 0 < b,\ c \in \mathbf{N}\}$$

Since $\mathscr{F}_i \neq \phi$, there exists the minimal element of $\mathrm{in}(\mathscr{F}_i)$. We denote the minimal element of $\mathrm{in}(\mathscr{F}_1)$ (resp. $\mathscr{F}_2$, $\mathscr{F}_3$) by $Y_1^{b_1}$ (resp. $Y_2^{b_2}$, $Y_1^{b_{31}}Y_2^{b_{32}}$). We call $f \in \mathscr{F}_1$ (resp. $\mathscr{F}_2$, $\mathscr{F}_3$) a minimal element of $\mathscr{F}_1$ (resp. $\mathscr{F}_2$, $\mathscr{F}_3$), if $\mathrm{in}(f) = Y_1^{b_1}$ (resp. $Y_2^{b_2}$, $Y_1^{b_{31}}Y_2^{b_{32}}$).

LEMMA 3.2. *Suppose that $k[H]$ is a Buchsbaum ring. Then*

$$(\mathrm{in}(\mathscr{F}_H)) = (Y_1^{b_1},\ Y_2^{b_2},\ Y_1^{b_{31}}Y_2^{b_{32}}).$$

Proof. Since $(\mathrm{in}(\mathscr{F}_1), \mathrm{in}(\mathscr{F}_2)) = (Y_1^{b_1},\ Y_2^{b_2})$, it suffices to show that

$$\mathrm{in}(\mathscr{F}_3) \subset (Y_1^{b_1},\ Y_2^{b_2},\ Y_1^{b_{31}}Y_2^{b_{32}}).$$

Therefore, we assume that there exists an element $f = Y_1^b Y_2^{b'} - X^\alpha \in \mathscr{F}_3$ such that $Y_1^b Y_2^{b'} \notin (Y_1^{b_1},\ Y_2^{b_2},\ Y_1^{b_{31}}Y_2^{b_{32}})$. (Note that $b < b_1$, $b' < b_2$.)

Let $f_3 = Y_1^{b_{31}}Y_2^{b_{32}} - X^{\alpha_3} \in \mathscr{F}_3$ be a minimal element of $\mathscr{F}_3$. Then, by our assumption, we have either $b_{31} > b$, $b_{32} < b'$ or $b_{31} < b$, $b_{32} > b'$.

If $b_{31} > b$ and $b_{32} < b'$, then we have a relation

$$g := Y_1^{b_{31}-b}f - Y_2^{b'-b_{32}}f_3 = X^{\alpha_3}Y_2^{b'-b_{32}} - X^\alpha Y_1^{b_{31}-b} \in I_H.$$

Since $Y_1^{b_{31}-b}$, $Y_2^{b'-b_{32}} \notin (\mathrm{in}(\mathscr{F}_H))$ and Theorem 2.6, we have $\mathrm{in}(g) \in (\mathrm{in}(\mathscr{G}_H))$.

If $\mathrm{in}(g) = X^{\alpha_3}Y_2^{b'-b_{32}}$, then there exists $X_iY_2^d - X_jY_1^e \in \mathscr{G}_H$ such that $X^{\alpha_3}Y_2^{b'-b_{32}}$ is divided by $X_iY_2^d$.

Since $Y_2^d \in M([(x_j) : \mathbf{m}]) \subset M([(x_j) : y_2])$, we have $Y_t^{d+1} \in (\mathrm{in}(\mathscr{F}_2))$ by Remark 1.7 (2) and (7). Thus $Y_2^{b'-b_{32}+1} \in (\mathrm{in}(\mathscr{F}_2))$ and $b' - b_{32} + 1 \geq b_2$. Since $b_{32} > 0$, this contradicts that $(b' - b_{32} + 1 \leq) b' < b_2$. Similarly, if $\mathrm{in}(g) = X^\alpha Y_1^{b_{31}-b}$, then this contradicts that $b < b_1$.

When $b_{31} < b$ and $b_{32} > b'$, it is the same way as above. □

COROLLARY 3.3 (Theorem 2.3 in [9]). *The following conditions are equivalent.*

(1) *$k[H]$ is a Cohen-Macaulay ring of codimension two.*

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(2) *I_H is generated by at most three elements.*

Proof. If $k[H]$ is Cohen-Macaulay, then $(\mathrm{in}(I_H))$ is minimally generated by at most three elements by Corollary 2.9 and Lemma 3.2. Namely, a number of minimal Gröbner bases of I_H is at most three. Hence $\mu(I_H) \leq 3$.

The converse follows from Theorem 4.4 in [5]. □

Now, we denote the smallest element of $\sum(Y^{b_1})$ (resp. $\sum(Y_2^{b_2})$, $\sum(Y_1^{b_{31}} Y_2^{b_{32}})$) by $X^{\alpha_1} Y_2^{c_1}$ (resp. $X^{\alpha_2} Y_1^{c_2}$, X^{α_3}). We put

$$f_1 = Y_1^{b_1} - X^{\alpha_1} Y_2^{c_1} \in \mathscr{F}_1$$
$$f_2 = Y_2^{b_2} - X^{\alpha_2} Y_1^{c_2} \in \mathscr{F}_2$$
$$f_3 = Y_1^{b_{31}} Y_2^{b_{31}} - X^{\alpha_3} \in \mathscr{F}_3.$$

Remark 3.4. Since $\sum(X^{\alpha_1} Y_2^{c_1}) = \phi$ (resp. $\sum(X^{\alpha_2} Y_1^{c_2})$), we have $c_1 < b_2$ (resp. $c_2 < b_1$).

LEMMA 3.5. *If $k[H]$ is a Buchsbaum ring and not Cohen-Macaulay, then* $\dim(k[H]) = 2$ *and* $\mathscr{G}_H = \{X_2 Y_j^{b_j - 1} - X_1 Y_i^{b_i - 1}\}$ *where* $\{i, j\} = \{1,2\}$.

Proof. By Corollary 2.10, we have already $\dim(k[H]) = 2$. Also, by Theorem 2.6 and Definition 2.5, there exists $X_2 Y_j^{d_j} - X_1 Y_i^{d_i} \in \mathscr{G}_H$ with $d_1 < b_1$, $d_2 < b_2$. Since $Y_1^{d_1+1} \in (in(\mathscr{F}_H))$, $d_1 + 1 \geq b_1$. Thus $d_1 + 1 = b_1$. Similarly, we have $d_2 + 1 = b_2$.

But, by (B-2) and (B-3) of Definition 2.5, if $(Y_i^{b_i - 1}, Y_j^{b_j - 1}) \in \Delta_H$, then $(Y_j^{b_j - 1}, Y_i^{b_i - 1}) \notin \Delta_H$. Hence we have $\mathscr{G}_H = \{X_2 Y_j^{b_j - 1} - X_1 Y_i^{b_i - 1}\}$ where $\{i, j\} = \{1,2\}$. □

LEMMA 3.6. *For $\{i, j\} = \{1,2\}$, if there exists an element of the form*

$$X_2^{d_2} Y_i^{e_i} - X_1^{d_1} Y_j^{e_j} \in I_H$$

for some $d_1, d_2, e_1, e_2 > 0$, then there does not exist an element of the form

$$X_2^{d_2'} Y_j^{e_j'} - X_1^{d_1'} Y_i^{e_i'} \in I_H$$

for any $d_1', d_2', e_1', e_2' > 0$.

Proof. Suppose that there exist $X_2^{d_2} Y_i^{e_i} - X_1^{d_1} Y_j^{e_j} \in I_H$ and $X_2^{d_2'} Y_j^{e_j'} - X_1^{d_1'} Y_i^{e_i'} \in I_H$ for some $d_1, d_2, d_1', d_2', e_1, e_2, e_1', e_2' > 0$.

If $e_i \geq e'_i$, then we can find an element $X_2^{p} Y_i^{q} - X_1^{p'} Y_j^{q'} \in I_H$ such that

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$0 < p, p', 0 \leq q, q'$ and $q < e'_i$, $q \equiv e_i \pmod{e'_i}$ in the following manner.

We write $e_i = me'_i + r$, where $0 < m$ and $0 \leq r < e'_i$ and put $g' = X_2^{d'_2}Y_j^{e'_j} - X_1^{d'_1}Y_i^{e'_i}$. For every $0 \leq l \leq m$, we construct $g_l = X_2^{d_2+ld'_2}Y_i^{e_i-le'_i} - X_1^{d_1+ld'_1}Y_j^{e_j-e'_j} \in I_H$ as follows.

- $g_0 = X_2^{b_2}Y_i^{e_i} - X_1^{d_1}Y_j^{e_j}$.
- Assume that $g_0, \ldots, g_l$ are constructed for $0 \leq l < m$. Then we have

$$\begin{aligned}
&X_1^{d'_1}g_l + X_2^{d_2+ld'_2}Y_i^{e_i-(l+1)e'_i}g' \\
&= X_1^{d'_1}(X_2^{d_2+ld'_2}Y_i^{e_i-le'_i} - X_1^{d_1+ld'_1}Y_j^{e_j-le'_j}) + X_2^{d_2+ld'_2}Y_i^{e_i-(l+1)e'_i}(X_2^{d'_2}Y_j^{e'_j} - X_1^{d'_1}Y_i^{e'_i}) \\
&= X_2^{d_2+(l+1)d'_2}Y_i^{e_i-(l+1)e_i}Y_j^{e'_j} - X_1^{d_1+(l+1)d'_1}Y_j^{e_j-le'_j} \in I_H.
\end{aligned}$$

By our assumptions (H-1) and (H-2), we have $e'_j \leq e_j - le'_j$ and

$$g_{l+1} := X_2^{d_2+(l+1)d'_2}Y_i^{e_i-(l+1)e'_i} - X_1^{d_1+(l+1)d'_1}Y_j^{e_j-(l+1)e'_j} \in I_H.$$

In particular, we have $g_m = X_2^{d_2+md'_2}Y_i^{r} - X_1^{d_1+md'_1}Y_j^{e_j-me'_j} \in I_H$.

Similarly, if $e_i \leq e'_i$, then we can find an element $X_2^{p}Y_j^{q'} - X_1^{p'}Y_i^{q} \in I_H$ such that $0 < p, p'$, $0 \leq q, q'$ and $q' < e_i$, $q' \equiv e_i \pmod{e_i}$.

Thus, by the Euclidean algorithm, we can reduce to the case $e_i = 0$ or $e'_i = 0$. But, by (H-1) and (H-2), $X_2^{d_2} - X_1^{d_1}Y_j^{e_j} \notin I_H$ and $X_2^{d'_2}Y_j^{e'_j} - X_1^{d'_1} \notin I_H$ for d_1, d_2, d'_1, $d'_2 > 0$. This is a contradiction. □

Proof of Theorem 3.1. (1)⇒(2). By Lemma 3.5, we have $\dim(k[H]) = 2$ and

$$\mathcal{G}_H = \{X_2Y_j^{b_j-1} - X_1Y_i^{b_i-1}\}$$

where $\{i, j\} = \{1,2\}$. We put $b'_1 = b_1 - 1$, $b'_2 = b_2 - 1$. Then, by Theorem 2.6 and Lemma 3.2, we have following Gröbner bases of I_H,

$$\begin{aligned}
f_1 &= Y_1^{b'_1+1} - X_1^{a_{11}}X_2^{a_{12}}Y_2^{c_1} &&\in \mathcal{F}_1 \\
f_2 &= Y_2^{b'_2+1} - X_1^{a_{21}}X_2^{a_{22}}Y_1^{c_2} &&\in \mathcal{F}_2 \\
f_3 &= Y_1^{b_{31}}Y_2^{b_{32}} - X_1^{a_1}X_2^{a_2} &&\in \mathcal{F}_3 \\
g &= X_2Y_j^{b'_j} - X_1Y_i^{b'_i} &&\in \mathcal{G}_H.
\end{aligned}$$

Since $Y_1^{b'_1} \in M([(x_j) : \mathbf{m}]) \subset M([(x_j) : y_2])$, $Y_1^{b'_1}Y_2 \in (\mathrm{in}(\mathcal{F}_H))$. Namely, $Y_1^{b_{31}}Y_2^{b_{32}}$ divides $Y_1^{b'_1}Y_2$. Thus $b_{32} = 1$. Similarly, $b_{31} = 1$. Hence we have $f_3 = Y_1Y_2 - X_1^{a_1}X_2^{a_2}$.

We consider the following relation

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$$X_2 f_j - Y_j g = X_1 Y_i^{b_i'} Y_j - X_1^{a_{j1}} X_2^{a_{j2}+1} Y_i^{c_j} \in I_H.$$

Since $c_j < b_i' + 1$ (cf. Remark 3.4), $X_1 Y_i^{b_i'-c_j} Y_j - X_1^{a_{j1}} X_2^{a_{j2}+1} \in I_H$. If $a_{j1} = 0$, this contradicts our assumptions (H-1) and (H-2). Thus $g' = Y_i^{b_i'-c_j} Y_j - X_1^{a_{j1}-1} X_2^{a_{j2}+1} \in I_H$. By $b_j' > 0$, $Y_j \notin (\mathrm{in}(f_j)) = (Y_j^{b_j'+1})$ and $b_i' - c_j > 0$. Thus we have

$$g' - Y_i^{b_i'-c_j-1} f_3 = X_1^{a_1} X_2^{a_2} Y_i^{b_i'-c_j-1} - X_1^{a_{j1}-1} X_2^{a_{j2}+1} \in I_H.$$

Then, by (H-1) and (H-2), $Y_i^{b_i'-c_j-1} - X_1^{a_{j1}-1-a_1} X_2^{a_{j2}+1-a_2} \in I_H$. But $\sum(Y_i^{b_i'-c_j-1}) = \phi$. Hence $Y_i^{b_i'-c_j-1} - X_1^{a_{j1}-1-a_1} X_2^{a_{j2}+1-a_2} = 0$ and $a_{j1} = a_1 + 1$, $a_{j2} = a_2 - 1$, $c_j = b_i' - 1$. Then we have $f_j = Y_j^{b_j'+1} - X_1^{a_1+1} X_2^{a_2-1} Y_i^{b_i'-1}$.

Similarly, $f_i = Y_i^{b_i'+1} - X_1^{a_1-1} X_2^{a_2+1} Y_j^{b_j'-1}$.

Now I_H is generated by f_1, f_2, f_3, g. Thus, by Corollary 3.3, $\mu(I_H) = 4$ and $\{f_1, f_2, f_3, g\}$ is a minimal basis of I_H.

(2) $\Rightarrow$ (3). Suppose that $\dim(k[H]) = 2$ and, after the permutation of variables, I_H is minimally generated by

$$\begin{aligned} f_1 &= Y_1^{b+1} - X_1^{a_1-1} X_2^{a_2+1} Y_2^{c-1} \\ f_2 &= Y_2^{c+1} - X_1^{a_1+1} X_2^{a_2-1} Y_1^{b-1} \\ f_3 &= Y_1 Y_2 - X_1^{a_1} X_2^{a_2} \\ g &= X_2 Y_2^c - X_1 Y_1^b. \end{aligned}$$

Then, by Corollary 3.3, $k[H]$ is not Cohen-Macaulay.

We verify the second assertion. For $H = \sum_{i=1}^4 \mathbf{N} h_i$, we put

$$h_1 = (d_1, 0),\ h_2 = (0, d_2),\ h_3 = (d_{31}, d_{32}),\ h_4 = (d_{41}, d_{42}).$$

Since $f_3, g \in I_H$, we have $h_3 + h_4 = a_1 h_1 + a_2 h_2$ and $h_2 + c h_4 = h_1 + b h_3$. Thus $d_{4i} = a_i d_i - d_{3i}$ for $i = 1,2$ and

$$\begin{aligned} b d_{31} &= c d_{41} - d_1 \\ b d_{32} &= c d_{42} + d_2. \end{aligned}$$

Solving these equations, we have

$$d_{31} = \frac{d_1}{b+c}(a_1 c - 1)$$

$$d_{32} = \frac{d_2}{b+c}(a_2 c + 1)$$

$$d_{41} = \frac{d_1}{b+c}(a_1 b + 1)$$

$$d_{42} = \frac{d_2}{b+c}(a_2 b - 1).$$

We define $\mathbf{Q}$-isomorphism $T : \mathbf{Q}^2 \to \mathbf{Q}^2$ by $T(p, q) = (b + c)\left(\frac{p}{d_1}, \frac{q}{d_2}\right)$.

Then we have

$$H \cong T(H) = \langle (b + c, 0), (0, b + c), (a_1c - 1, a_2c + 1), (a_1b + 1, a_2b - 1)\rangle.$$

(3) $\Rightarrow$ (1). By the form of H, there exist elements

$$\begin{aligned} f_1 &= Y_1^{b+1} - X_1^{a_1-1}X_2^{a_2+1}Y_2^{c-1} \\ f_2 &= Y_2^{c+1} - X_1^{a_1+1}X_2^{a_2-1}Y_1^{b-1} \\ f_3 &= Y_1Y_2 - X_1^{a_1}X_2^{a_2} \\ g &= X_2Y_2^c - X_1Y_1^b \end{aligned}$$

of I_H.

Claim. $\mathcal{R}_H \setminus \mathcal{F}_H = \{g\}$.

Since $k[H]$ is not Cohen-Macaulay, $\mathcal{R}_H \setminus \mathcal{F}_H \neq \phi$ (cf. Corollary 2.9). Thus $L := \mathcal{R}_H \setminus \mathcal{F}_H \cup \{g\}$ is not empty, if $\mathcal{R}_H \setminus \mathcal{F}_H \neq \{g\}$.

If there exists $g' \in L$, then, by Lemma 3.6, we can write $g' = X_2^{d_2}Y_2^{e_2} - X_1^{d_1}Y_1^{e_1}$ where $0 < d_1, d_2, e_1, e_2$. Since $Y_1^{b+1}, Y_2^{c+1} \in (\mathrm{in}(I_H))$ and $Y_1^{e_1}, Y_2^{e_2} \notin (\mathrm{in}(I_H))$, $e_1 \leq b$ and $e_2 \leq c$. Then we have

$$Y_2^{c-e_2}g' - X_2^{d_2-1}g = X_1Y_1^{e_1}(X_2^{d_2-1}Y_1^{b-e_1} - X_1^{d_1-1}Y_2^{c-e_2}) \in I_H.$$

Hence we have $g_1 := X_2^{d_2-1}Y_1^{b-e_1} - X_1^{d_1-1}Y_2^{c-e_2} \in I_H$. Note that $g_1 \neq 0$ (since $g' \neq g$). Then, by (H-1) and (H-2), either $b - e_1 > 0$ or $c - e_2 > 0$.

If $b - e_1 > 0$, then we have

$$X_1^{d_1}g_1 - X_2^{d_2-1}Y_1^{b-2e_1}g' = X_2^{2d_2-1}Y_1^{b-2e_1}Y_2^{e_2} - X_1^{2d_1-1}Y_2^{c-e_2} \in I_H.$$

Since $d_1, d_2 > 0$, we have $2d_1 - 1$ and $2d_2 - 1 > 0$. Also, by (H-1) and (H-2), $e_2 \leq c - e_2$. Thus $0 \neq X_2^{2d_2-1}Y_1^{b-2e_1} - X_1^{2d_1-1}Y_2^{c-2e_2} \in I_H$. But this contradicts Lemma 3.6.

If $c - e_2 > 0$, we have a contradiction in the same way as above. Hence $\mathcal{R}_H = \mathcal{F}_H \cup \{g\}$.

The proof of Claim is completed.

By Theorem 2.6 and Claim, it suffices to show that $(Y_1^b, Y_2^c) \in \Delta_H$. Since we have relations $f_1, f_2, f_3 \in I_H$ (Y_1^b, Y_2^c) satisfies condition (B-2). Also it is clear that (Y_1^b, Y_2^c) satisfies conditions (B-1), (B-4) and (B-5). Then we have only to prove that (Y_1^b, Y_2^c) satisfies condition (B-3).

If $Y_1^b \in M([(x_1) : \mathbf{m}]) \subset M([(x_1) : x_2])$, then we have $X_2Y_1^b - X_1^{d_1}X_2^{d_2}Y_1^{e_1}Y_2^{e_2} \in I_H$ with $\sum(Y_1^{e_1}Y_2^{e_2}) = \phi$ and $d_1 > 0$. Since $\sum(Y_1^b) = \phi$, $d_2 = 0$ and, by (H-1)

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and (H-2), $e_1 < b$. Then we have $X_2Y_1^{b-e_1} - X^{d_1}Y_2^{e_2} \in I_H$. But, by Lemma 3.6, this is a contradiction. Hence we have $Y_1^b \notin M([(x_1):\mathbf{m}])$. Similarly, $Y_2^c \notin M([(x_2):\mathbf{m}])$. □

EXAMPLE 3.7. Let $0 < a < b < c \in \mathbf{N}$. We consider the following simplicial semigroup

$$H = \langle (c, 0), (0, c), (c-b, b), (c-a, a) \rangle.$$

and the semigroup ring $k[H] = k[t_1^c, t_2^c, t_1^{c-b}t_2^b, t_1^{c-a}t_2^a]$.

In [3], H. Bresinsky, P. Schenzel and W. Vogel discussed arithmetical Buchsbaum curves in $\mathbf{P}_k^3$ and showed that $k[H]$ is a Buchsbaum ring and not Cohen-Macaulay if and only if

$$H \cong \langle (4m, 0), (0, 4m), (2m-1, 2m+1), (2m+1, 2m-1) \rangle$$

for some $m > 0$ (cf. Theorem 3 in [3]).

We can verify this fact as follows.

By Theorem 3.1, if $k[H]$ is a Buchsbaum ring and not Cohen-Macaulay, then I_H has the following minimal basis

$$\begin{aligned} f_1 &= Y_1^{b_1+1} - X_1^{a_1-1}X_2^{a_2+1}Y_2^{b_2-1} \\ f_2 &= Y_2^{b_2+1} - X_1^{a_1+1}X_2^{a_2-1}Y_1^{b_1-1} \\ f_3 &= Y_1Y_2 - X_1^{a_1}X_2^{a_2} \\ g &= X_2Y_2^{b_2} - X_1Y_1^{b_1}, \end{aligned}$$

where $a_1, a_2, b_1, b_2 \in \mathbf{N} \setminus \{0\}$. In this case, these are homogeneous polynomials with respect to the total degree. Then we have $a_1 = a_2 = 1$, $b_1 = b_2$ and

$$H \cong \langle (2b_1, 0), (0, 2b_1), (b_1-1, b_1+1), (b_1+1, b_1-1) \rangle.$$

If $b_1 = 2m+1$ ($m \geq 0$), then $Y_1^{2m+1} - X_1^mX_2^{m+1} \in I_H$. This contradicts that $\{f_1, f_2, f_3, g\}$ is a Gröbner bases of I_H (cf. proof of Theorem 3.1). Hence we have $b_1 = 2m$ ($m > 0$) and

$$H \cong \langle (4m, 0), (0, 4m), (2m-1, 2m+1), (2m+1, 2m-1) \rangle.$$

Also I_H is generated by

$$\begin{aligned} f_1 &= Y_1^{2m+1} - X_2Y_2^{2m-1} \\ f_2 &= Y_1^{2m+1} - X_1Y_2^{2m-1} \\ f_3 &= Y_1Y_2 - X_1X_2 \\ g &= X_2Y_2^{2m} - X_1Y_1^{2m}. \end{aligned}$$

Conversely, for $H = \langle (4m, 0), (0, 4m), (2m-1, 2m+1), (2m+1, 2m-1) \rangle$, it is easy to see that $k[H]$ is not Cohen-Macaulay (cf. Theorem 3.8 in [9]). Hence, by Theorem 3.1 (3), $k[H]$ is a Buchsbaum ring. □

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