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FACTORIZATION IN THE INVERTIBLE GROUP OF A C^*-ALGEBRA

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ABSTRACT. In this paper we consider the following problem: Given a unital C^*-algebra A and a collection of elements S in the identity component of the invertible group of A, denoted $\mathrm{inv}_0(A)$, characterize the group of finite products of elements of S. The particular C^*-algebras studied in this paper are either unital purely infinite simple or of the form $(A \otimes K)^+$, where A is any C^*-algebra and K is the compact operators on an infinite dimensional separable Hilbert space. The types of elements used in the factorizations are unipotents (1+ nilpotent), positive invertibles and symmetries ($s^2 = 1$). First we determine the groups of finite products for each collection of elements in $(A \otimes K)^+$. Then we give upper bounds on the number of factors needed in these cases. The main result, which uses results for $(A \otimes K)^+$, is that for A unital purely infinite and simple, $\mathrm{inv}_0(A)$ is generated by each of these collections of elements.

0. **Introduction.** In this paper we consider the following problem: Given a unital C^*-algebra A and a collection of elements S in the identity component of the invertible group of A, denoted $\mathrm{inv}_0(A)$, or in $U_0(A)$ the identity component of the unitary group, characterize the set of finite products of elements of S. The C^*-algebras considered in this paper are of the form $(A \otimes K)^+$, where K is the compact operators on an infinite dimensional separable Hilbert space and A is any C^*-algebra, and unital purely infinite simple C^*-algebras.

It is well known that for any unital Banach algebra $\mathrm{inv}_0(A)$ is equal to the set of finite products of exponentials of elements of A. For the C^*-algebras mentioned above we will characterize the groups of finite products generated by unipotents, positive invertibles, selfadjoint invertibles, symmetries and *-symmetries. A *unipotent* element has the form $1 + a$ with a nilpotent. *Symmetries* are elements that satisfy $s^2 = 1$. A **-symmetry* is a selfadjoint unitary.

The survey article [12] contains many similar factorization problems in M_n, the $n \times n$ matrices with entries in the complex numbers $\mathbf{C}$, and $L(H)$, the bounded operators on an infinite dimensional separable Hilbert space H. (In [12] symmetries are called involutions and *-symmetries are called symmetries.) For many elements it is easy to see that the set of finite products will not equal $\mathrm{inv}_0(A)$. For example, if $x \in M_n$ is a product of unipotents, it must have determinant equal to 1; the determinant condition is sufficient as well. However, in $L(H)$ many elements besides exponentials generate $\mathrm{inv}\big(L(H)\big)$, the invertible group. Included in this list are unipotents, positive invertibles, and symmetries.

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Although factorization problems have been studied extensively in M_n and $L(H)$, not much else has been done until recently. In [6] de la Harpe and Skandalis consider factorization by commutators of elements in both $\mathrm{inv}_0(A)$ and $U_0(A)$, *i.e.*, elements of the form $aba^{-1}b^{-1}$ with a and $b \in \mathrm{inv}_0(A)$ and $U_0(A)$ respectively for $(A \otimes K)^+$, unital purely infinite simple C^*-algebra and simple AF-algebras. For $A = C(X) \otimes M_n$, where X is a compact metric space, Phillips [10] characterized the group of finite products for the types of elements mentioned above as well as quasiunipotents (1+ quasinilpotent), accretives ($a+a^*$ is positive invertible), and positive stable elements $\big(\mathrm{sp}(a) \subset \{\lambda \in \mathbf{C} : \mathrm{Re}(\lambda) > 0\}\big)$. (It should be noted that this definition of accretive differs from definitions used elsewhere. See the comments before and after Lemma 4.1 of [10].) It turns out (Theorem 4.5 (1) of [10]) that for any C^*-algebra A the accretive elements generate $\mathrm{inv}_0(A)$ and the unitary accretives generate $U_0(A)$. Since accretive elements are positive stable (Corollary 4.3 of [10]), positive stable elements also generate $\mathrm{inv}_0(A)$.

This paper is organized as follows. Section 1 contains the characterization of the groups of finite products generated by unipotents, positive invertibles, and selfadjoint invertibles in $(A \otimes K)^+$. Modulo an obvious scalar factor given by the unitization, we show that each element of $\mathrm{inv}_0(A \otimes K)^+$ is a finite product of unipotents, positive invertibles, or selfadjoint invertibles. We give a partial answer for symmetries: the group of finite products is characterized in K^+. Upper bounds on the length of the factorizations in $(A \otimes K)^+$ are given in Section 2. In Section 3 we apply the results for $(A \otimes K)^+$ established in Section 1 to factorization problems in a unital purely infinite simple C^*-algebra A. We show that all the sets of elements mentioned above are generators for $\mathrm{inv}_0(A)$.

The results in this paper come from the author's doctoral dissertation at the University of Oregon under the direction of Professor N. Christopher Phillips. I would like to thank Chris for his help and encouragement throughout the project. I am also grateful to the referee for number of useful suggestions.

1. **Stable C^*-Algebras.**

A C^*-algebra A is *stable* if $A \otimes K \cong A$. In this section we will characterize the groups of finite products generated by unipotent elements, positive invertibles, selfadjoint invertibles, symmetries and *-symmetries in $(A \otimes K)^+$. Let π be the canonical projection from the unitization of a C^*-algebra onto $\mathbf{C}$.

THEOREM 1.1. *Let A be any C^*-algebra. The set of finite products of unipotents in $(A \otimes K)^+$ is equal to* $\big\{x \in \mathrm{inv}_0\big((A \otimes K)^+\big) : \pi(x) = 1\big\}$.

An important technique in many proofs in this paper will be replacing an x in $\mathrm{inv}_0(A)$ by an element that commutes with a projection. The next two lemmas allow us to do this in the proof of Theorem 1.1.

LEMMA 1.2. *Let e and f be idempotents in a unital Banach algebra A such that $\|e - f\| < 1$.*

(1) There are unipotents w and v such that $ewv = wvf$.

(2) If I is an ideal in A and $e - f \in I$, then the unipotents in (1) satisfy $w - 1,\ v - 1 \in I$.

(3) If $||e-f|| < \varepsilon$, then the unipotents in (1) satisfy

$$||w-1|| < \varepsilon(\varepsilon + ||e||)||e|| \text{ and } ||v-1|| < \frac{\varepsilon(\varepsilon + ||e||)||e||}{1-\varepsilon^2}.$$

PROOF. (1) Put $w = 1-(1-e)fe$ and $v = 1 + ef(1-e)\big(1-(e-f)^2\big)^{-1}$. Since e is idempotent, $\big((e-1)fe\big)^2 = 0$. Thus w is unipotent. To see that v is unipotent, one need only observe that both e and f commute with $1-(e-f)^2$.

To prove that $ewv = wvf$ it suffices to show that

$$ewv\left(1-(e-f)^2\right) = wvf\left(1-(e-f)^2\right).$$

The left-hand side is ef. We get ef for right-hand side since f commutes with $1-(e-f)^2$.

(2) Using the definition of w and $f^2 = f$, we have

$$w - 1 = (e-f)fe.$$

Similarly, for v we have

$$v - 1 = ef(f-e)\big(1-(e-f)^2\big)^{-1}.$$

Since $e-f \in I$ it follows that $w-1$ and $v-1$ are in I.

(3) For $w-1$, we have

$$||w-1|| \le ||e-f||\,||f||\,||e|| < \varepsilon(\varepsilon + ||e||)||e||,$$

and for $v-1$, we have

$$||v-1|| \le ||e-f||\,||f||\,||e||\,||\big(1-(e-f)^2\big)^{-1}|| < \frac{\varepsilon(\varepsilon + ||e||)||e||}{1-\varepsilon^2}.$$

This completes the proof of the lemma. ∎

LEMMA 1.3. *Let A be a unital C^*-algebra. Let $x \in A$ satisfy $||x-1|| < \frac{1}{25}$. If p is a projection in A, then there are unipotents w and v in A such that $wvxp = pwvx$ and $||wvx-1|| < 1$. Furthermore, if wvx is a product of unipotents, then so is x.*

PROOF. Suppose $||x-1|| < \frac{1}{25}$. Let p be a projection in A. Then

$$||xpx^{-1} - p|| \le ||xp - px||\,||x^{-1}|| \le 2||x-1||\,||x^{-1}|| \le \frac{2||x-1||}{1-||x-1||} < \frac{1}{12}.$$

Since xpx^{-1} and p are idempotents, by Lemma 1.2 (1) there are unipotents w and v such that $wvxpx^{-1} = pwv$, i.e. $wvxp = pwvx$.

To show $||wvx-1|| < 1$, first notice that by Lemma 1.2 (3), with $\varepsilon = \frac{1}{12}$, we have

$$||w-1|| < \frac{1}{10} \text{ and } ||v-1|| < \frac{1}{10}.$$

So

$$||wvx-1|| \le ||wvx-wv|| + ||wv-1|| \le ||w||\,||v||\,||x-1|| + ||v||\,||w-1|| + ||v-1|| < 1.$$

The last statement is obvious. This completes the proof. ∎

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LEMMA 1.4. *Let B be a C^*-algebra, and let $c \in \mathrm{inv}_0(B^+)$ with $c-1 \in B$.*

(1) Then $\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}$ is a product of 3 unipotents in $\big(M_2(B)\big)^+$ and the order of nilpotency of each factor is 2.

(2) Let $c_n \in B^+$, with $c_n - 1 \in B$ for all $n \geq 1$. Suppose $||c_n - 1|| \longrightarrow 0$ as $n \longrightarrow \infty$. If $\gamma_{n,i}$ is a factor of $\begin{pmatrix} c_n & 0 \\ 0 & c_n^{-1} \end{pmatrix}$ given by (1), then $\left\| \gamma_{n,i} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\| \longrightarrow 0$.

PROOF. (1) It is easy to check that each term in the following factorization is unipotent in $\big(M_2(B)\big)^+$ and that the order of nilpotency of each term is 2.

$$\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} = \begin{pmatrix} 1 & c-1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2-c^{-1} & c^{-1}-1 \\ 1-c^{-1} & c^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1-c & 1 \end{pmatrix}.$$

(2) If $||c_n - 1|| < \varepsilon$, then

$$||c_n^{-1} - 1|| \leq ||c_n^{-1}||\, ||c_n - 1|| \leq \frac{||c_n - 1||}{1 - ||c_n - 1||} < \frac{\varepsilon}{1+\varepsilon}.$$

The result follows by considering the factorization in the proof of (1). This completes the proof of the lemma. ∎

Now we proceed with the proof of Theorem 1.1.

PROOF (THEOREM 1.1). Suppose $x \in \mathrm{inv}_0\big((A \otimes K)^+\big)$ and $\pi(x) = 1$. We need to show that x is a product of unipotents. Choose $p_0 \in L(H)$ such that $p_0 \sim 1 - p_0 \sim 1$. Then $p = 1 \otimes p_0 \in M(A \otimes K)$ and $p \sim 1 - p \sim 1$.

Since $\mathrm{inv}_0(A \otimes K)^+$ is connected we may assume that $||x - 1|| < \frac{1}{25}$. (If not, connect x to 1 by a continuous path α. If y_1 and y_2 are on α where $||y_1 - y_2|| < \frac{1}{25}$ and y_1 is a product of unipotents, then so is y_2.) By Lemma 1.3 there are unipotents v and w in $M(A \otimes K)$ such that $pwvx = wvxp$. It follows from Lemma 1.2 (2) that w and v are in $(A \otimes K)^+$.

By Lemma 1.3, replacing x by wvx, we can assume x commutes with p and $||x-1|| < 1$. Identify $(A \otimes K)^+$ with $M_2\big(p(A \otimes K)p\big)^+$. Under this isomorphism

$$x \longmapsto \begin{pmatrix} y & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}.$$

We must factor both terms as products of unipotents. Since both p_0 and $1 - p_0$ were chosen to have infinite rank, the factorization of each term is similar. The details are given only for the first term.

As $1 - p_0$ has infinite rank we can find projections $q_2, q_3, \ldots \in L(H)$ such that $1 - p_0 = \sum_{k=2}^{\infty} q_k$, $q_k \sim p_0$ for all k, and the q_k's are pairwise orthogonal. Let $p_1 = p$ and $p_k = 1 \otimes q_k$. Let w_k be an isometry such that $w_k w_k^* = p_k$. Identify $(A \otimes K)^+$ with $(A \otimes K \otimes K)^+$ using the map $\phi\colon (A \otimes K \otimes K)^+ \longrightarrow A \otimes K$, defined by $y \otimes e_{ij} \longmapsto w_i y w_j^*$.

Combining this isomorphism with $(A \otimes K)^+ \cong \big[M_2\big(p(A \otimes K)p\big)\big]^+$ we get

$$\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \longmapsto \mathrm{diag}(y, 1, 1, \ldots).$$

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Factor as follows (see the proof of Proposition 7.1 of [5])

$$\begin{aligned}(1)\qquad \operatorname{diag}(y,1,1,\ldots) &= \operatorname{diag}(y,y^{-\frac12},y^{-\frac12},y^{\frac14},y^{\frac14},y^{\frac14},y^{\frac14},y^{-\frac18},\ldots)\\ &\quad\cdot \operatorname{diag}(1,y^{\frac12},y^{\frac12},y^{-\frac14},y^{-\frac14},y^{-\frac14},y^{-\frac14},y^{\frac18},\ldots),\end{aligned}$$

where $y^{\pm\frac{1}{2^n}}$ occurs 2^n times on the diagonal. Each term on the right-hand side of (1) can be factored further; the first term as

$$\begin{aligned}(2)\qquad &\operatorname{diag}(y^{\frac12},y^{-\frac12},1,y^{\frac18},y^{\frac18},y^{\frac18},y^{\frac18},y^{-\frac18},y^{-\frac18},y^{-\frac18},y^{-\frac18},1,1,1,1,y^{\frac{1}{32}},\ldots)\\ &\cdot \operatorname{diag}(y^{\frac12},1,y^{-\frac12},y^{\frac18},y^{\frac18},y^{\frac18},y^{\frac18},1,1,1,1,y^{-\frac18},y^{-\frac18},y^{-\frac18},y^{-\frac18},y^{\frac{1}{32}},\ldots),\end{aligned}$$

and the second as

$$(3)\qquad \operatorname{diag}(1,y^{\frac14},y^{\frac14},y^{-\frac14},y^{-\frac14},1,1,y^{\frac{1}{16}},\ldots)\cdot\operatorname{diag}(1,y^{\frac14},y^{\frac14},1,1,y^{-\frac14},y^{-\frac14},y^{\frac{1}{16}},\ldots).$$

Notice that each term in (2) and (3) is in $(A\otimes K\otimes K)^+$ since $||y^{\pm\frac{1}{2^n}}-1||\longrightarrow 0$ as $n\longrightarrow\infty$.

We factor the first term of (2), call it α. The factorization of the other terms is similar. Now α has diagonal entries of the form $\operatorname{diag}(\alpha_n,\alpha_n^{-1},1)$, where

$$\alpha_n=\operatorname{diag}(y^{\frac{1}{2\cdot 4^{n-1}}},\ldots,y^{\frac{1}{2\cdot 4^{n-1}}})\in M_{4^{n-1}}(A\otimes K)^+.$$

By Lemma 1.4 (1) $\operatorname{diag}(\alpha_n,\alpha_n^{-1},1)$ can be factored as a product of unipotents in $M_{3\cdot 4^{n-1}}(A\otimes K)^+$. Thus α can be factored as a product of three unipotent infinite block diagonal matrices. It follows from Lemma 1.4 (2) that each factor is in $(A\otimes K\otimes K)^+$. This completes the proof of the theorem. ∎

THEOREM 1.5. *Let A be any C^*-algebra. The set of finite products of positive invertibles in $(A\otimes K)^+$ is equal to* $\{x\in \operatorname{inv}_0\big((A\otimes K)^+\big) : \pi(x)\in(0,\infty)\}$.

The proof of Theorem 1.5 uses matrix factorization techniques similar to those in the proof of Theorem 1.1. The following lemmas allow us to factor certain types of matrices as products of positive invertibles. The proofs of both lemmas make extensive use of Proposition 1.4.5 of [7].

LEMMA 1.6. *Let B be a unital C^*-algebra, and let I be an ideal in B.*

(1) If $b\in B$, then $\begin{pmatrix}1&b\\0&1\end{pmatrix}$ is a product of 3 positive invertible elements of $M_2(B)$.

(2) Suppose $b_n\in B$ and $||b_n||\longrightarrow 0$. If $\gamma_{n,i}$ is one of the three positive invertible factors of $\begin{pmatrix}1&b_n\\0&1\end{pmatrix}$ given by (1), then $||\gamma_{n,i}-\begin{pmatrix}1&0\\0&1\end{pmatrix}||\longrightarrow 0$.

(3) If $b\in I$, then the three positive invertible factors of $\begin{pmatrix}1&b\\0&1\end{pmatrix}$ given by (1) are in $M_2(I)^+$.

PROOF. (1) Let $t_0=b^*b$. Then there exists $s\in B$ such that $b=s(t_0)^{\frac14}$. Put $t=(t_0)^{\frac14}$. Then

$$\begin{pmatrix}1&b\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&(1+t)^{-1}\end{pmatrix}\begin{pmatrix}1&s\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&1+t\end{pmatrix}\begin{pmatrix}1&-s\\0&1\end{pmatrix}$$

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$$= \begin{pmatrix} 1 & 0 \\ 0 & (1+t)^{-1} \end{pmatrix} \left[\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1+t \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}^* \right] \left[\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}^* \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} \right].$$

(2) Choose $s_n \in B$ as in the proof of (1) such that $b_n = s_n t_n$ and $||s_n|| \leq ||t_n||$. If $\gamma_{n,i}$ is one of the factors of $\begin{pmatrix} 1 & b_n \\ 0 & 1 \end{pmatrix}$ given by (1), then

$$\left\| \gamma_{n,i} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\| \longrightarrow 0.$$

(3) Suppose $b \in I$. Let s and t be as in the proof of (1). Then $s \in I$. Hence the three positive invertible elements in the factorization of $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ are in $M_2(I)^+$. ∎

LEMMA 1.7. *Let B be a nonunital C^*-algebra.*

(1) Suppose $y \in B^+$ with $y - 1 \in B$. Then $\operatorname{diag}(y, y^{-1}, 1)$ and $\operatorname{diag}(y, 1, y^{-1})$ are products of 24 positive invertible elements of $M_3(B)^+$.

(2) Suppose $y_n \in B^+$ with $y_n - 1 \in B$, and $||y_n - 1|| \longrightarrow 0$. If $\gamma_{n,i}$ is one of the positive invertible factors of $\operatorname{diag}(y_n, y_n^{-1}, 1)$ or $\operatorname{diag}(y_n, 1, y_n^{-1})$, then $||\gamma_{n,i} - \operatorname{diag}(1,1,1)|| \longrightarrow 0$.

PROOF. (1) Let $t = \left((y^{-1} - 1)^*(y^{-1} - 1)\right)^{\frac{1}{4}}$. Then there is an $s \in B$ such that $y^{-1} = 1 + st$. Notice that $1 + ts$ is invertible with $(1+ts)^{-1} = 1 - t(1+st)^{-1}s = 1 - tys$. Factor $\operatorname{diag}(y, y^{-1}, 1)$ as follows

$$\begin{pmatrix} y & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} y & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+ts \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & (1+ts)^{-1} \end{pmatrix}.$$

The first term is factored as

$$(*) \qquad \begin{pmatrix} 1 & 0 & ys \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -t & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -s \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ ty & 0 & 1 \end{pmatrix},$$

and the second as

$$(**) \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -ty & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -ys \\ 0 & 0 & 1 \end{pmatrix}.$$

Since s and t are in B and B is an ideal in B^+ each of the eight factors above is in $M_3(B)^+$. Use Lemma 1.6 to write each of the eight factors as products of three positive invertible elements of $M_3(B)^+$.

For $\operatorname{diag}(y, 1, y^{-1})$ the argument is similar.

$$\begin{pmatrix} y & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & y^{-1} \end{pmatrix} = \begin{pmatrix} y & 0 & 0 \\ 0 & 1+ts & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1+ts)^{-1} & 0 \\ 0 & 0 & y^{-1} \end{pmatrix}.$$

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The first term is factored as

$$\begin{pmatrix} 1 & ys & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ ty & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and the second as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -ty \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & s & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -ys & 1 \end{pmatrix}.$$

(2) We consider the case $\mathrm{diag}(y_n, y_n^{-1}, 1)$. The argument for $\mathrm{diag}(y_n, 1, y_n^{-1})$ is the same. Let $t_n = (y_n^{-1} - 1)(y_n^{-1} - 1)^*$. Choose s_n so that $y_n^{-1} - 1 = s_n t_n$ and $||s_n|| \le ||t_n||$. If $||y_n - 1|| \longrightarrow 0$, then the off diagonal terms of the eight unipotent factors of $\mathrm{diag}(y_n, y_n^{-1}, 1)$ given by $(*)$ and $(**)$ in the proof of (1) go to zero. So by Lemma 1.6 (2), if $\gamma_{n,i}$ is a positive invertible factor of $\mathrm{diag}(y_n, y_n^{-1}, 1)$, then $||\gamma_{n,i} - \mathrm{diag}(1,1,1)|| \longrightarrow 0$. This proves the lemma. ■

PROOF (THEOREM 1.5). It suffices to assume that $\pi(x) = 1$ and $||x - 1|| < 1$. Choose a projection $p \in M(A \otimes K)$ as in the proof of Theorem 1.1 such that $p \sim 1 - p \sim 1$ and $||(1-p)(x-1)(1-p)|| < 1$. Then $(1-p)x(1-p)$ is invertible in $\big[(1-p)(A \otimes K)(1-p)\big]^+$. Identify $(A \otimes K)^+$ with $\big[M_2(p(A \otimes K)p)\big]^+$. Under this isomorphism

$$x \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix},$$

where $b_1 = bd^{-1}$, $a_1 = a - bd^{-1}c$, and $c_1 = d^{-1}c$. By Lemma 2.3 of [2], a_1 is invertible, and $b_1, c_1, a_1 - 1, d - 1 \in p(A \otimes K)p$. Lemma 1.6 allows us to factor both $\begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix}$ as products of three positive invertible elements of $\big[M_2\big(p(A \otimes K)p\big)\big]^+$.

To factor $\begin{pmatrix} a_1 & 0 \\ 0 & d \end{pmatrix}$ we follow the proof of Theorem 1.1 and reduce the problem to factoring α and α_n . To factor $\alpha = \mathrm{diag}(\alpha_1, \alpha_1^{-1}, 1, \alpha_2, \alpha_2^{-1}, 1, \ldots)$ we must fit together the factors of $\mathrm{diag}(\alpha_n, \alpha_n^{-1}, 1)$ for each n, given by Lemma 1.7 (1), to form an element of $(A \otimes K \otimes K)^+$. For example, the first unipotent factor given by $(*)$ in the proof of Lemma 1.7 (1) will be

$$\mathrm{diag}\left(\begin{pmatrix} 1 & 0 & \alpha_1 s_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & \alpha_2 s_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ldots \right),$$

where $\alpha_n^{-1} - 1 = t_n s_n$ as in the proof of Lemma 1.7 (2). Call this factor β. According to the proof of Lemma 1.7 (2), if $||\alpha_n - 1|| \longrightarrow 0$, then $||t_n|| \longrightarrow 0$ and $||s_n|| \longrightarrow 0$. Therefore β is in $(A \otimes K \otimes K)^+$.

By Lemma 1.6, β is a product of positive invertibles. Let $b_n = \alpha_n s_n$. Put $w_n = (b_n b_n^*)^{\frac{1}{4}}$. As in the proof of Lemma 1.6 (1), choose z_n so that $b_n = w_n z_n$. The positive invertible

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factors of β have the following form:

$$\operatorname{diag}\left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&(1+w_1)^{-1}\end{pmatrix},\begin{pmatrix}1&0&0\\0&1&0\\0&0&(1+w_2)^{-1}\end{pmatrix},\dots\right),$$

$$\operatorname{diag}\left(\begin{pmatrix}1&0&z_1\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1+w_1\end{pmatrix}\begin{pmatrix}1&0&z_1\\0&1&0\\0&0&1\end{pmatrix}^*,\dots\right)$$

and

$$\operatorname{diag}\left(\begin{pmatrix}1&0&-z_1\\0&1&0\\0&0&1\end{pmatrix}^*\begin{pmatrix}1&0&-z_1\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&0&-z_2\\0&1&0\\0&0&1\end{pmatrix}^*\begin{pmatrix}1&0&-z_2\\0&1&0\\0&0&1\end{pmatrix},\dots\right).$$

Since $||b_n|| \longrightarrow 0$, each of these factors is in $(A\otimes K\otimes K)^+$. So β is a product of positive invertible elements of $(A\otimes K\otimes K)^+$. Similarly the other factors of $\operatorname{diag}(\alpha_n,\alpha_n^{-1},1)$ will form elements of $(A\otimes K\otimes K)^+$ which are products of positive invertibles. ∎

COROLLARY 1.8. *Let A be any C^*-algebra. The set of finite products of selfadjoint invertibles of $(A\otimes K)^+$ is equal to $\big\{x\in \operatorname{inv}_0\big((A\otimes K)^+\big) : \pi(x)\in(-\infty,\infty)\big\}$.*

When one considers factorization problems in a particular C^*-algebra one might wonder if the different factorizations have similar behavior. For example, in M_n all of the sets of finite products for elements considered in this paper are characterized by a determinant condition and are closed in $\operatorname{inv}(M_n)$. Such is not the case for $(A\otimes K)^+$.

EXAMPLE 1.9. In any C^*-algebra A a symmetry s has the form $s = 2e-1$, where e is an idempotent. If $s\in K^+$, it is easy to see that either e or $1-e$ is in K. In K idempotents are finite rank since they are the identity on their range. Let F denote the finite rank operators. Suppose $x\in K^+$ is a product of symmetries. Since F is an ideal in K, x has the form $f\pm 1$ with f finite rank. Now suppose that f is a finite rank operator. Choose a basis so that f is in M_n. By Theorem 3.6 of [13], if $\det(f+1_n) = \pm 1$ then $f+1_n$ is a finite product of symmetries in M_n. Put $f+1_n = \prod_{i=1}^k s_i$, where s_i is a symmetry in M_n. Note that $s_i\oplus 1$ is a symmetry in K^+. It follows that the set of finite products of symmetries in K^+ is not closed in $\operatorname{inv}(K^+)$.

A *-symmetry has the form $2p-1$ with p a projection. Argue as above to see that the group of finite products of *-symmetries is not closed in $U_0(K^+)$.

2. **Upper Bounds for Factorizations.** A companion problem to the factorization results is to find an upper bound on the length of the factorization. This has been done in several cases for exponentials. See the survey article [11]. For several types of elements, including unipotents, positive invertibles and selfadjoint invertibles, Phillips showed in [10] that there is no upper bound on the lengths of the factorizations in $C(X)\otimes M_2$, where X is the Hilbert Cube $[0,1]^N$. The purpose of this chapter is to establish upper bounds for the length of factorization by unipotents, positive invertibles, and selfadjoint invertibles in stable C^*-algebra. These bounds are not known to be best possible.

In [6] (Theorem 7.4) de la Harpe and Skandalis gave this upper bound result for commutators: if A is a stable C^*-algebra and $x \in \mathrm{inv}_0(A^+)$ with $\pi(x) = 1$, then x is the product of at most 6 commutators of elements of $\mathrm{inv}_0(A)$. Many ideas for this section were inspired by Lemma 7.3 and Theorem 7.4 of [6] and their proofs.

LEMMA 2.1. *Let A be a nonunital C^*-algebra, and let $x_1, x_2, \ldots, x_n$ be invertible in A^+. If $x_n x_{n-1} \cdots x_1 = 1$, then $x = \mathrm{diag}(x_1, x_2, \ldots, x_n, 1, \ldots, 1)$ is a product of three terms of the form $\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ in $\big[M_{2n}(A)\big]^+$.*

PROOF. See the proof of Lemma 7.3 of [6]. In order to prove that x is a product of three commutators they first show that x is a product of three such factors.

THEOREM 2.2. *Let A be a stable C^*-algebra. If $x \in \mathrm{inv}_0(A^+)$ with $\pi(x) = 1$, then x is the product of at most 10 unipotent elements of A^+.*

PROOF. We follow the proof of Theorem 7.4 of [6]. Let $x \in \mathrm{inv}_0(A^+)$ with $\pi(x) = 1$. There exists a projection $p \in M(A)$ such that $p \sim 1 - p \sim 1$ and

$$||(1-p)(x-1)(1-p)|| < 1.$$

Therefore $(1-p)x(1-p)$ is invertible in $\big[(1-p)A(1-p)\big]^+$. Identify A^+ with $\big[M_2(pAp)\big]^+$. Under this isomorphism

$$x \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix}, \tag{1}$$

where $a_1 = a - bd^{-1}c$, $b_1 = bd^{-1}$ and $c_1 = d^{-1}c$. Note that a_1 is invertible in $(pAp)^+$ by Lemma 2.3 of [2]. Choose $s, t \in pAp$ so that $d = 1 + st$. Then

$$\begin{pmatrix} a_1 & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} (1+ts)^{-1} & 0 \\ 0 & 1+st \end{pmatrix} \begin{pmatrix} (1+ts)a_1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2}$$

The first factor on the right-hand side of (2) is

$$\begin{pmatrix} 1 & (1+ts)^{-1}t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix} \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s(1+ts)^{-1} & 1 \end{pmatrix}.$$

Substituting in (1) we get

$$x = \begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \begin{pmatrix} (1+ts)a_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix}.$$

Now

$$\begin{pmatrix} (1+ts)a_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c_1\big((1+ts)a_1\big)^{-1} & 1 \end{pmatrix} \begin{pmatrix} (1+ts)a_1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If we put $x_1 = (1+ts)a_1$, then

$$\begin{aligned} x &= \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \begin{pmatrix} x_1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \begin{pmatrix} x_1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}. \tag{3}$$

Next we want to show that $\begin{pmatrix} x_1 & 0 \\ 0 & 1 \end{pmatrix}$ is a product of unipotents. Since x_1 is in $\mathrm{inv}_0(pAp)^+$ with $x_1 - 1 \in pAp$ and pAp is stable, Theorem 1.1 allows us to write $x_1 = \prod_{i=1}^n a_i$, where the a_i are unipotent in $(pAp)^+$ for $i = 1, \ldots, n$. Writing $1 - p$ as the sum of $2n - 1$ orthogonal projections all equivalent to p we now identify A^+ with $\big[M_{2n}(pAp)\big]^+$. Under the isomorphisms

$$\big[M_2(pAp)\big]^+ \cong A^+ \cong \big[M_{2n}(pAp)\big]^+,$$

we get

$$\begin{pmatrix} x_1 & 0 \\ 0 & 1 \end{pmatrix} \longmapsto \mathrm{diag}(x_1, 1, \ldots, 1).$$

Now

$$\mathrm{diag}(x_1, 1, \ldots, 1) = \mathrm{diag}(a_1, a_2, \ldots, a_n, 1_n)\, \mathrm{diag}\Big(\prod_{i=2}^n a_i, a_2^{-1}, \ldots, a_n^{-1}, 1_n\Big). \tag{4}$$

Since each a_i is unipotent, the first term of the right-hand side of (4) is unipotent. By Lemma 2.1 the second factor in (4) is a product of three terms of the form $\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ in $\big[M_{2n}(pAp)\big]^+$. This gives $\begin{pmatrix} x_1 & 0 \\ 0 & 1 \end{pmatrix}$ as the product of seven unipotents and hence x as the product of 11.

To get the desired number 10, let $y = \mathrm{diag}(a_1, a_2, \ldots, a_n)$ and notice that in the factorization of the second term of (4), given in the previous paragraph, the first factor has the form $\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$. Put $w = sy^{-1}$; then

$$\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}.$$

Finally notice that the fourth factor in (3) has the form $\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$ in $\big[M_2(pAp)\big]^+$. We claim that in the identification with $\big[M_{2n}(pAp)\big]^+$ this term becomes a lower triangular matrix with 1's on the diagonal, call it α. To verify the claim, consider the isomorphisms

$$\big[M_2(pAp)\big]^+ \cong A^+ \cong \big[M_{2n}(pAp)\big]^+.$$

Put $p_1 = p$ and $1 - p = \sum_{i=2}^{2n} p_i$. Let v_k be a partial isometry such that $v_k^* v_k = p_1$ and $v_k v_k^* = p_k$. Let v be the partial isometry such that $v^* v = p$ and $vv^* = 1 - p$. Using the v_k's, the isomorphisms above give

$$\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \longmapsto (1 + vz) \longmapsto \big(v_i^*(1 + vz)v_j\big)_{i,j=1}^{2n}.$$

Now $z \in pAp$, $v_k = v_k v_k^* v_k$ and the p_k's are orthogonal. So, if $i = 1$ or $j \neq 1$, then

$$v_i^* vzv_j = v_i(1 - p)vzpv_j = v_i p_i(1 - p)vzpp_k v_k = 0.$$

Therefore $\big(v_i^*(1 + vz)v_j\big)_{i,j=1}^{2n} = 1_n + (v_i^* vzv_1)_{i=1}^{2n}$, and α has the form as claimed. But then $\alpha\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}$ is also lower triangular with diagonal 1's, hence unipotent. This rearrangement gives x as the product of ten unipotents. ∎

By modifying the above proof slightly we can get upper bounds for the factorization by positive invertibles and self adjoint invertibles in a stable C^*-algebra A.

THEOREM 2.3. *Let A be a stable C^*-algebra. If $x \in \mathrm{inv}_0(A^+)$ and $\pi(x) > 0$, then x is the product of at most 31 positive invertible elements.*

PROOF. Suppose $x \in \mathrm{inv}_0(A^+)$. Since $x = a + \lambda 1 = (\lambda^{-1}a + 1)\lambda 1$, where $a \in A$ and $\lambda \in \mathbf{C}$, and $\lambda 1$ is positive and invertible we can assume that $x = a + 1$.

Arguing as in the previous proof

$$x = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}\begin{pmatrix} x_1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1}$$

in $\big[M_2(pAp)\big]^+$, where $x_1 \in \mathrm{inv}_0(pAp)^+$. Write $x_1 = \prod_{i=1}^n a_i$, where a_i is positive invertible for each i. Passing to $\big[M_{2n}(pAp)\big]^+$, and putting $z = \mathrm{diag}(x_1, 1, \ldots, 1)$, we get

$$z = \mathrm{diag}(a_1, a_2, \ldots, a_n, 1_n)\mathrm{diag}\Big(\prod_{i=2}^{n} a_i, a_2^{-1}, \ldots, a_n^{-1}, 1_n\Big). \tag{2}$$

Since each a_i is positive invertible, the first factor is positive and invertible. Put $y = \mathrm{diag}(a_1, a_2, \ldots, a_n)$. Applying Lemma 2.1 to the second term of (2) yields

$$z = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

in $\big[M_{2n}(pAp)\big]^+$. Combining (1) and (2), x is the product of 10 terms of the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$ and one positive invertible. By Lemma 1.6 (1) each term of the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$ is a product of three positive invertible elements. Therefore x is the product of 31 positive invertible elements. ∎

COROLLARY 2.4. *Let A be a stable C^*-algebra. If $x \in \mathrm{inv}_0(A^+)$ and $\pi(x) \in (-\infty, \infty)$ then x is the product of at most 31 selfadjoint invertible elements.*

3. **Purely Infinite C^*-Algebras.** In this chapter we apply the factorization results established in section 1 for stable C^*-algebras to factorization problems in the invertible group of a unital purely infinite simple C^*-algebra. A C^*-algebra A is *purely infinite* if for all $x \geq 0$ the hereditary subalgebra $\overline{xAx}$ contains an infinite projection.

It turns out that the set of finite products of any of the classes of elements considered above (commutators, unipotents, positive invertibles, selfadjoint invertibles, and symmetries), as well as others, is all of $\mathrm{inv}_0(A)$. These results are due in part to the the fact that a purely infinite simple C^*-algebra A contains a (*-isomorphic) copy of $A \otimes K$ as the following lemma (Lemma 2.3 of [5]) shows.

LEMMA 3.1. *Let p be a projection in a unital C^*-algebra A with $p \sim 1$ and $\{e_{ij}\}$ a system of matrix units for K. There is a homomorphism $\phi\colon (1-p)A(1-p) \otimes K \longrightarrow A$ such that $\phi(x \otimes e_{11}) = x$, for $x \in (1-p)A(1-p)$.*

A fact that will be used often in the proofs in this chapter is Proposition 2.2 of [3]: A unital purely infinite simple C^*-algebra A contains two orthogonal projections p and q such that $p \sim q \sim 1$ in A.

THEOREM 3.2. *Let A be a unital purely infinite simple C^*-algebra. Then* $\mathrm{inv}_0(A)$ *is the set of finite products of unipotent elements of A.*

PROOF. Suppose $x \in \mathrm{inv}_0(A)$. Choose orthogonal projections p and q in A such that $p \sim q \sim 1$. As in the proof of Theorem 1.1, we may assume that $||x-1|| < \frac{1}{25}$. Then by Lemma 1.3 we may assume that $xp = px$ and $||x-1|| < 1$. With respect to the decomposition $1 = p \oplus (1-p)$ write

$$x = \begin{pmatrix} pxp & 0 \\ 0 & 1-p \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & (1-p)x(1-p) \end{pmatrix}$$

Call the first factor on the right-hand side x_1 and the second x_2. Each must be factored as a product of unipotents.

We have $x_1 - 1 \in pAp$. Use Lemma 3.1 to choose a homomorphism $\phi\colon \big((1-q)A(1-q)\otimes K\big)^+ \longrightarrow A$. Since $p \leq 1-q$ we have $\phi\big((x-1)\otimes e_{11}\big) = x-1$. Now $\big((x-1)\otimes e_{11}\big)+1 \in \mathrm{inv}_0\big((A\otimes K)^+\big)$ and by Theorem 1.1 it is a product of unipotents in $(A\otimes K)^+$. Thus x is a product of unipotents in A.

For the second factor, since $x_2 - 1 \in (1-p)A(1-p)$ we can replace $(1-q)A(1-q)$ by $(1-p)A(1-p)$ in the above argument. ∎

THEOREM 3.3. *Let A be a unital purely infinite simple C^*-algebra. Then* $\mathrm{inv}_0(A)$ *is the set of finite products of positive invertible elements of A.*

Given a projection p in A and $x \in \mathrm{inv}_0(A)$, the next lemma will allow us to employ our usual tactic of replacing x by an element that commutes p.

LEMMA 3.4. *If p and q are projections in a unital C^*-algebra A with $||p-q|| < 1$, then there are positive invertible elements $a_1, a_2, a_3, a_4, a_5, a_6 \in A$ so that*

$$p\Big(\prod_{n=1}^{6} a_n\Big) = \Big(\prod_{n=1}^{6} a_n\Big)q.$$

PROOF. Replacing e by p and f by q, let w and v be the unipotents defined in Lemma 1.2 (1). We claim that

$$v = \big(v^{-1}(v^*)^{-1}\big)\big(v^*(p+1)v\big)(1-\frac{1}{2}p) \tag{1}$$

and

$$w = (1-\frac{1}{2}p)(ww^*)\big((w^*)^{-1}(p+1)w^{-1}\big). \tag{2}$$

Since the right hand side of both (1) and (2) is a product of three positive invertibles, one need only check the claimed equalities. Recall from the proof of Lemma 1.2 (1) that $v = 1 + pq(1-p)\big(1-(p-q)^2\big)^{-1}$ and $w = 1-(1-p)qp$, and that $pq(1-p)$ commutes with $\big(1-(p-q)^2\big)^{-1}$, which gives $vp = p$. The calculations are straightforward using these facts. ∎

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PROOF (THEOREM 3.3). Suppose $x \in \mathrm{inv}_0(A)$. Let u be the unitary part of the polar decomposition of x. It suffices to show that u is a product of positive invertibles. Choose orthogonal projections p and q in A such that $p \sim q \sim 1$. It suffices to assume that $||u-1|| < \frac{1}{2}$ and so $||upu^* - p|| < 1$. Since upu^* is a projection apply Lemma 3.4 to find positive invertible elements $a_1, a_2, a_3, a_4, a_5, a_6 \in A$ so that

$$p\Big(\prod_{n=1}^{6} a_n\Big)u = \Big(\prod_{n=1}^{6} a_n\Big)up.$$

By the proof of Lemma 3.4, $\prod_{n=1}^{6} a_n = wv$, where w and v are as in Lemma 1.3. Therefore if $||u-1|| < \frac{1}{25}$, then by Lemma 1.3

$$||\Big(\prod_{n=1}^{6} a_n\Big)u - 1|| = ||wvu - 1|| < 1.$$

Replacing u by $(\prod_{n=1}^{6} a_n)u$, we may assume $up = pu$ and $||u-1|| < 1$. (However, u might no longer be unitary.) Proceed now as in the proof of Theorem 3.2 and use Theorem 1.5 to factor u as a product of positive invertibles. ■

So far in this chapter our proofs have relied on the factorization results for $(A \otimes K)^+$. Recall that in Chapter 1 it was shown that factorization by symmetries in K^+ does not turn out as nicely as the factorization by commutators, unipotents or positive invertibles. However, in unital purely infinite simple C^*-algebras the difficulties present in the $(A \otimes K)^+$ case can be overcome.

THEOREM 3.5. *Let A be a unital purely infinite simple C^*-algebra. The set of finite products of symmetries in A is* $\mathrm{inv}_0(A)$.

PROOF. Suppose $a \in \mathrm{inv}_0(A)$. Let p be a nontrivial projection in A. Choose $v \in A$ such that $v^*v = p$ and $vv^* < p$ and put $p_1 = p - vv^*$.

First we show that without loss of generality we can assume a commutes with p_1. By Lemma 4.2 (1) of [8] there is an ε such that for any idempotent $e \in A$ if $||p_1 - e|| < \varepsilon$ then there is a symmetry $s \in A$ so that $p_1 = ses$. It suffices to assume $||a-1|| < \frac{\varepsilon}{2+\varepsilon}$. Then $||p_1 - ap_1a^{-1}|| < \varepsilon$ and, since ap_1a^{-1} is an idempotent, there exists a symmetry s such that $p_1 sa = sap_1$. Replacing a by sa, we may assume $ap_1 = p_1 a$.

Write a in matrix form with respect to the decomposition $1 = p_1 \oplus (1-p_1)$,

$$a = \begin{pmatrix} p_1 a p_1 & 0 \\ 0 & (1-p_1)a(1-p_1) \end{pmatrix}.$$

Let $x = p_1 a p_1$ and $d = (1-p_1)a(1-p_1)$.

Now we show that it suffices to assume $a = \begin{pmatrix} x & 0 \\ 0 & 1-p_1 \end{pmatrix}$ and $x \in \mathrm{inv}_0(p_1 A p_1)$. Choose a projection $q_0 \in A$ such that $1 - p_1 \sim q_0 \leq p_1$. Let h be a partial isometry such that $h^*h = 1 - p_1$ and $hh^* = q_0$. We have $(q_0 + 1 - p_1)A(q_0 + 1 - p_1) \cong M_2\big((1-p_1)A(1-p_1)\big)$. We can factor $\begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} \in M_2\big((1-p_1)A(1-p_1)\big)$ as a product of symmetries as follows

$$\begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} = \begin{pmatrix} 0 & d \\ d^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Under the isomorphism

$$\begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} \longmapsto \begin{pmatrix} d_0 & 0 \\ 0 & d^{-1} \end{pmatrix} \in (q_0 + 1 - p_1)A(q_0 + 1 - p_1),$$

where $d_0 = hdh^*$. Let $z_0 = \begin{pmatrix} d_0 & 0 \\ 0 & d^{-1} \end{pmatrix}$ and $z = p_1 - q_0 + z_0$. Then z is a product of symmetries in A.

Now

$$az = \begin{pmatrix} x & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} p_1 - q_0 + d_0 & 0 \\ 0 & d^{-1} \end{pmatrix} = \begin{pmatrix} x_0 & 0 \\ 0 & 1 \end{pmatrix},$$

and $a, z \in \mathrm{inv}_0(A)$. So $az \in \mathrm{inv}_0(A)$. Since $K_1(A) \cong \mathrm{inv}(A)/\mathrm{inv}_0(A)$ (Theorem 1.9 of [4]), x_0 is in $\mathrm{inv}_0(p_1Ap_1)$. Therefore, replacing a by az, we may assume that $a = \begin{pmatrix} x & 0 \\ 0 & 1 - p_1 \end{pmatrix}$ and $x \in \mathrm{inv}_0(p_1Ap_1)$.

Choose a projection q in A so that $p \sim q < 1 - p$ and then choose a projection r in A so that $q \sim r < 1 - p - q$. Let v_1 and v_2 be partial isometries which satisfy

$$v_1^*v_1 = p,\ v_1v_1^* = q,\ v_2^*v_2 = q, \text{ and } v_2^*v_2 = r.$$

Let $v_3 = v_1^*v_2^*$. Then $v_3^*v_3 = r$ and $v_3v_3^* = p$.

Let $w = v_1 + v_2 + vv_3$. In order to build the first of three copies of $p_1Ap_1 \otimes K$ in $(p+q+r)A(p+q+r)$ we define an infinite collection of projections using w and p_1. Let $p_k = wp_{k-1}w^*$, for $k \geq 2$, and $w_k = w^{k-1}p_1$. Then $w_kw_k^* = p_k$ and $w_k^*w_k = p_1$, and the p_k's are orthogonal equivalent projections which satisfy $p_{3n-2} < p$, $p_{3n-1} < q$ and $p_{3n} < r$, for $n \geq 1$.

Define $\chi \colon p_1Ap_1 \otimes K \longrightarrow (p+q+r)A(p+q+r)$ by $y \otimes e_{ij} \longmapsto w_iyw_j^*$. Extend χ to $(p_1Ap_1 \otimes K)^+$ by $1 \longmapsto p + q + r$.

Next we produce two other copies of $p_1Ap_1 \otimes K$ in $(p+q+r)A(p+q+r)$. For each n choose orthogonal equivalent projections $\{e^{(j)}_{3n-2} : j = 1, \ldots, 4^{n-1}\}$ such that $p_{3n-2} \sim e^{(j)}_{3n-2}$ and

$$p_{3n-2} = \sum_{j=1}^{4^{n-1}} e^{(j)}_{3n-2}.$$

Then put $e^{(j)}_{3n-1} = w\big(e^{(j)}_{3n-2}\big)w^*$ and $e^{(j)}_{3n} = w\big(e^{(j)}_{3n-1}\big)w^*$, for each n and j, and order the $e^{(j)}_i$'s as follows: $e^{(1)}_1, e^{(1)}_2, e^{(1)}_3, e^{(1)}_4, \ldots, e^{(4)}_4, e^{(1)}_5, \ldots$. Use the partial isometries which implement the equivalences $p_{3n-2} \sim e^{(j)}_{3n-2}$ and $p_{3n-2} \sim p_1$ to define partial isometries $r^{(j)}_{3n-2}$ so that $r^{(j)}_{3n-2}(r^{(j)}_{3n-2})^* = p_1$ and $(r^{(j)}_{3n-2})^*r^{(j)}_{3n-2} = e^{(j)}_{3n-2}$, and put $r^{(j)}_{3n-1} = r^{(j)}_{3n-2}w^*$ and $r^{(j)}_{3n} = r^{(j)}_{3n-1}w^*$. Then use the $r^{(j)}_i$'s to define $\phi_1 \colon (p_1Ap_1 \otimes K)^+ \longrightarrow (p+q+r)A(p+q+r)$.

Similarly choose orthogonal equivalent projections $\{f^{(j)}_i\}$ such that $p_1 = f^{(1)}_1$ and

$$p_{3n-1} = \sum_{j=1}^{2\cdot 4^{(n-1)}} f^{(j)}_{3n-1},$$

for $n \geq 1$. Then put $f_{3n}^{(j)} = w\big(f_{3n-1}^{(j)}\big)w^*$ and $f_{3n+1}^{(j)} = w\big(f_{3n}^{(j)}\big)w^*$, for any n and j. Order the $f_i^{(j)}$'s as follows:

$$f_1^{(1)}, f_2^{(1)}, f_2^{(2)}, f_3^{(1)}, f_3^{(2)}, f_4^{(1)}, f_4^{(2)}, f_5^{(1)}, \ldots, f_5^{(8)}, f_6^{(1)}, \ldots .$$

Using the partial isometries which implement $f_i^{(j)} \sim f_i^{(1)} = p_1$, define $\phi_2 \colon (p_1Ap_1 \otimes K)^+ \longrightarrow (p+q+r)A(p+q+r)$.

Now we turn to the factorization of a. Let $r_0 = 1-(p+q+r)$. By the definition of ϕ_i we have

$$\begin{aligned}\phi_i\big(\mathrm{diag}(x,1,1,\ldots)\big) &= \phi_i\big(\mathrm{diag}(x-p_1,0,0,\ldots)+1\big)\\ &= x-p_1+p+q+r\\ &= a-r_0,\end{aligned}$$

for $i=1,2$. If $a-r_0$ is a product of symmetries in $(p+q+r)A(p+q+r)$, then a is a product of symmetries in A.

Recall from the proof of Theorem 1.1 that

$$\begin{aligned}\mathrm{diag}(x,1,1,\ldots) = {}&\mathrm{diag}(x^{\frac12}, x^{-\frac12}, 1, x^{\frac18}, x^{\frac18}, x^{\frac18}, x^{\frac18}, x^{-\frac18}, \ldots)\\ &\cdot \mathrm{diag}(x^{\frac12}, 1, x^{-\frac12}, x^{\frac18}, x^{\frac18}, x^{\frac18}, x^{\frac18}, 1,1,1,1, x^{-\frac18}, \ldots)\\ &\cdot \mathrm{diag}(1, x^{\frac14}, x^{\frac14}, x^{-\frac14}, x^{-\frac14}, 1, 1, x^{\frac1{16}}, \ldots)\\ &\cdot \mathrm{diag}(1, x^{\frac14}, x^{\frac14}, 1, 1, x^{-\frac14}, x^{-\frac14}, x^{\frac1{16}}, \ldots).\end{aligned}$$

Call the factors b_1, b_2, b_3, and b_4 respectively. We must factor each b_i as a product of symmetries. For the factorizations of b_3 and b_4 replace ϕ_1 by ϕ_2 in the following argument. The factorization for b_2 is similar to that of b_1. We check the details only for b_1.

The set of bounded sequences of the product $\prod_{n=1}^{\infty} M_3\big(M_{4^{n-1}}(p_1Ap_1)\big)$ is a C^*-algebra when given the norm $||a_n|| = sup_n||a_n||$. If we let x_n be the diagonal $4^{n-1} \times 4^{n-1}$ matrix with all diagonal entries equal to $x^{\frac{1}{2\cdot 4^{n-1}}}$, then

$$b_1 = \big(\mathrm{diag}(x_1, x_1^{-1}, 1), \mathrm{diag}(x_2, x_2^{-1}, 1), \ldots, \mathrm{diag}(x_n, x_n^{-1}, 1), \ldots\big),$$

is in $\prod_{n=1}^{\infty} M_3\big(M_{4^{n-1}}(p_1Ap_1)\big)$ since $||x_n - 1|| \longrightarrow 0$, as $k \longrightarrow \infty$.

Put $q_j = w_{3n-2}^*\big(e_{3n-2}^{(j)}\big)w_{3n-2}$. Then $q_j \sim p_1$ for all j and $\sum_{j=1}^{4^{n-1}} q_j = p_1$. So we can define an isomorphism between $M_{4^{n-1}}(p_1Ap_1)$ and p_1Ap_1. Let Φ_n be the extension of this isomorphism to the 3×3 matrices over these algebras and define

$$\Phi = \prod_{n=1}^{\infty} \Phi_n : \prod_{n=1}^{\infty} M_3\big(M_{4^{n-1}}(p_1Ap_1)\big) \longrightarrow \prod_{n=1}^{\infty} M_3(p_1Ap_1).$$

Since Φ_n is an isomorphism and $||x_n - 1|| \longrightarrow 0$, as $k \longrightarrow \infty$ it follows that

$$||\Phi_n\big(\mathrm{diag}(x_n, x_n^{-1}, 1) - 1|| \longrightarrow 0.$$

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Put $\operatorname{diag}(c_n, c_n^{-1}, 1) = \Phi_n\big(\operatorname{diag}(x_n, x_n^{-1}, 1)\big)$. Then

$$c = \operatorname{diag}(c_1, c_1^{-1}, 1, c_2, c_2^{-1}, 1, c_3, \ldots)$$

is in $(p_1Ap_1 \otimes K)^+$, and $\chi(c)$ is in $(p+q+r)A(p+q+r)$.

By the definitions of χ and ϕ_1,

$$\chi(c) = \Big(\sum_{k=1}^{\infty} w_{3k-2}(c_k - p_1)w_{3k-2}^* + w_{3k-1}(c_k^{-1} - p_1)w_{3k-1}^*\Big) + (p+q+r)$$

and

$$\phi_1(b_1) = \sum_{n=1}^{\infty}\Big(\sum_{j=1}^{4^{n-1}} r_{3n-2}^{(j)}(x^{\frac{1}{2\cdot 4^{n-1}}} - p_1)r_{3n-2}^{(j)*} + r_{3n-1}^{(j)}(x^{\frac{-1}{2\cdot 4^{n-1}}} - p_1)r_{3n-1}^{(j)*}\Big) + (p+q+r).$$

Using these definitions one can check that $\chi(c) = \phi_1(b)$ and so it suffices to show that $\chi(c)$ is a product of symmetries in $(p+q+r)A(p+q+r)$.

For $k = 1, 2$ or 3, let $u_k = w^{k-1}p$. Then

$$u_k^*u_k = p(w^*)^{k-1}w^{k-1}p = p \text{ and } u_ku_k^* = w^{k-1}pp(w^*)^{k-1} = \begin{cases} p, & \text{if } k = 1 \\ q, & \text{if } k = 2 \\ r, & \text{if } k = 3 \end{cases},$$

and we have an isomorphism $(p+q+r)A(p+q+r) \longrightarrow M_3(pAp)$ defined by

$$x \longmapsto (u_i^*xu_j)_{i,j=1}^3.$$

The image of $\chi(c)$ under this isomorphism has the form $\operatorname{diag}(\alpha, \alpha^{-1}, p)$ in $M_3(pAp)$. To see this first note that if $y \in pAp$ then $y \longmapsto \operatorname{diag}(y, 0, 0)$. If $z \in qAq$ then $z \longmapsto \operatorname{diag}(0, u_2^*zu_2, 0)$. Now for every $k \geq 1$,

$$w_{3k-2}(c_k - p_1)w_{3k-2}^* \in pAp \text{ and } w_{3k-1}(c_k^{-1} - p_1)w_{3k-1}^* \in qAq.$$

Also $u_2^*w_{3k-2} = pw^*w^{3k-3}p_1 = pw^{3k-2}p_1 = w_{3k-1}$. So under the isomorphism $\chi(c)$ maps to

$$\begin{pmatrix} \Big(\sum_{k=1}^{\infty} w_{3k-2}(c_k - p_1)w_{3k-2}^*\Big) + p & 0 & 0 \\ 0 & \Big(\sum_{k=1}^{\infty} w_{3k-2}(c_k^{-1} - p_1)w_{3k-2}^*\Big) + p & 0 \\ 0 & 0 & p \end{pmatrix}.$$

Call this matrix β. Let

$$\alpha = \Big(\sum_{k=1}^{\infty} w_{3k-2}(c_k - p_1)w_{3k-2}^*\Big) + p.$$

Then

$$\alpha^{-1} = \Big(\sum_{k=1}^{\infty} w_{3k-2}(c_k^{-1} - p_1)w_{3k-2}^*\Big) + p.$$

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So

$$\beta = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & 0 & p \end{pmatrix} = \begin{pmatrix} 0 & \alpha & 0 \\ \alpha^{-1} & 0 & 0 \\ 0 & 0 & p \end{pmatrix} \begin{pmatrix} 0 & p & 0 \\ p & 0 & 0 \\ 0 & 0 & p \end{pmatrix}.$$

Each term on the right hand side is a symmetry in $M_3(p_1Ap_1)$. So $a - r_0$ is a product of symmetries in $(p + q + r)A(p + q + r)$. Thus a is a product of symmetries in A. This completes the proof of the theorem. ■

Before summarizing the unital case let us say a few words about the nonunital case. The next theorem, due to Zhang, allows us to apply the results of Section 2 to A if A is a nonunital σ-unital purely infinite simple C^*-algebra.

PROPOSITION 3.6. (THEOREM 1.2(I) OF [14]). *If A is a σ-unital purely infinite simple C^*-algebra then either A is unital or A is stable.*

We return now to the unital case. The next two theorems summarize the known results for generators of $\mathrm{inv}_0(A)$ and $U_0(A)$.

THEOREM 3.7. *Let A be a unital purely infinite simple C^*-algebra. The following sets are generators for* $\mathrm{inv}_0(A)$.

(1) Exponentials.
(2) Unipotents.
(3) Quasiunipotents.
(4) Positive invertibles.
(5) Selfadjoint invertibles.
(6) Symmetries.
(7) Commutators of elements of $\mathrm{inv}_0(A)$.
(8) Accretive elements.
(9) Positive stable elements.

PROOF. (1) This is well known and true even if A is a Banach algebra. See Proposition 3.4.3 of [1].

(2) and (3) Theorem 1.1.
(4) and (5) Theorem 1.2.
(6) and (7) Theorems 2.1 and 7.4 of [6].
(8) and (9) Theorem 4.5 (1) of [10]. ■

THEOREM 3.8. *Let A be a unital purely infinite simple C^*-algebra. The following sets are generators for $U_0(A)$.*

(1) Exponentials of skew adjoint elements.
(2) Commutators of elements of $U_0(A)$.
*(3) *-symmetries.*
(4) Accretive unitaries.

PROOF. (1) This is true for any C^*-algebra. See Proposition 3.4.5 of [1].
(2) Proposition 7.7 of [5].

(3) Use the proof of Theorem 2.1. Replace Lemma 2.3 by Lemma 1.3 (1) of [8]. If $a \in U_0(A)$ then

$$\begin{pmatrix} 0 & \alpha & 0 \\ \alpha^{-1} & 0 & 0 \\ 0 & 0 & p \end{pmatrix}$$

is a *-symmetry as α is now unitary.

(4) Theorem 4.5 (1) of [10]. ∎

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