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SMOOTHABILITY, STRONG SMOOTHABILITY AND DENTABILITY IN BANACH SPACES[1]

BY

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ABSTRACT. It is shown that dentability of the unit ball of a conjugate Banach space X^* does not imply smoothability of the unit ball of X, answering a question raised by Kemp. A property called strong smoothability is introduced and is shown to be dual to dentability. The results are used to provide new proofs of the facts that X is an Asplund space whenever it has an equivalent Fréchet differentiable norm, or whenever X^* has the Radon-Nikodým Property.

Introduction. In connection with the property of dentability, Edelstein, [7], has introduced a notion called smoothability. The notion has been reformulated by Kemp, [10], and it is to this version that we henceforth refer (see Definition 3.1 below).

The definitions of dentability and smoothability suggest that the two properties are dual to each other. As it turns out, they are indeed dually related, but the duality is not complete: dentability of $B(X)$ implies smoothability of $B(X^*)$ and weak* dentability of $B(X^*)$ implies smoothability of $B(X)$, [10], but neither implication has a valid converse, [12]. (Here, X denotes a real Banach space, and $B(X)$, its closed unit ball.)

In section 2 we define a property called strong smoothability and show that there is complete duality with dentability in the sense indicated.

Part of that duality relates strong smoothability of $B(X)$ to *weak** dentability of $B(X^*)$, and we may ask if one must conclude that $B(X)$ is strongly smoothable when $B(X^*)$ is merely dentable. Kemp, [10], has asked if one may even conclude that $B(X)$ is smoothable. Section 3 answers both questions in the negative.

In Section 4, we show that strong smoothability is separably determined (Theorem 4.1). This fact is then used to obtain new proofs of previously known results: a Banach space is an Asplund space if every separable subspace is an Asplund space (see Phelps, [16]), or if X^* has the Radon-Nikodým Property (Stegall, [18]), or if the norm of X is Fréchet differentiable (Ekeland, Lebourg [8]).

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We mention that Sullivan [20] has been able to use the methods of Theorem 4.1 to obtain a very short proof of the above-mentioned result of Stegall. In [20], and earlier in [19], Sullivan used a property called malleability and the final section of this paper briefly compares the notions of malleability, smoothability, and strong smoothability.

2. **Strong smoothability and dentability.** If A is a convex subset of the real Banach space X, and $x \in X$, the closed convex cone generated by $A - x$ is denoted $k(x, A)$, that is, $k(x, A) \equiv c\ell \cup \{t(A-x) : t \geq 0\}$, where "$c\ell$" denotes closure in the norm topology. Further, if $f \in X^*$ and $\alpha > 0$, $S(f, \alpha, A) \equiv \{x \in A : f(x) \geq \sup f(A) - \alpha\}$ is called a *slice* of A; if X is a conjugate space, $X = Y^*$, and $f \in Y$ the above subset is called a *weak* slice* of A [15].

2.1 A convex subset A of X is called *strongly smoothable* if for every $\varepsilon > 0$ there is some $x \in X \backslash c\ell\ A$ and some $f \in S(X^*) \equiv \{g \in X^* : \|g\| = 1\}$, such that

$$\{y \in B(X) : f(y) \geq \varepsilon\} \subset k(x, A). \tag{2.1.1}$$

When $X = Y^*$, A is *weak* strongly smoothable* if for every $\varepsilon > 0$ there is $x \in X \backslash w^*\text{-}c\ell\ A$ and $f \in Y$ satisfying (2.1.1). (Here, of course, "$w^*\text{-}c\ell$" denotes closure in the weak* topology.)

2.2 $A \subset X$ is said to be *dentable* (resp., *weak* dentable*) if it has slices (resp., weak* slices) of arbitrarily small diameter, that is, if for each $\varepsilon > 0$ there is a slice of A such that diam $S(f, \alpha, A) < \varepsilon$.

The main result of this section is

2.3. THEOREM. *Let A be a closed and bounded convex subset of the real Banach space X, and suppose that the origin is interior to A. Then*

(i) *A is dentable if and only if A^0 is strongly smoothable.*

(ii) *A is strongly smoothable if and only if A_0 is weak* dentable.*

We recall that the *polar* of A is the subset $A^0 = \{f : \sup f(A) \leq 1\}$ of X^* or, if $X = Y^*$, $A^0 = \{y \in Y : \sup y(A) \leq 1\}$.

In order to prove the theorem, we will need three lemmas. The proof of the first lemma uses the easily verified fact that for $x \in X \backslash A$, the polar of $k(x, A)$ is the subset $\{f : \sup f(A) \leq f(x)\}$ of the dual or predual according to what polar we want.

2.4. LEMMA. *Let A be a closed convex subset of X with $0 \in A$. Then, for each $x \in X \backslash A$, the convex cones $k(x, A)$ and $k(0, S(x, \sup x(A^0) - 1, A^0))$ are polar to each other. If X is a conjugate space and A is weak* closed, the same conclusions holds with A^0 replaced by A_0.*

Proof. Let $K = k(x, A)$ and $L = k(0, S)$ where $S = S(x, \sup x(A^0) - 1, A^0) = \{f \in A^0 : f(x) \geq 1\}$. Now, if $f \in S$ then $\sup f(A) \leq 1 \leq f(x)$ so $S \subset K^0$ and, hence, $L \subset K^0$. Since $\{x\}$ is compact and not in the closed convex set A, it follows that

K^0 is the closure of the set $G=\{g\in X^*:\sup g(A)<g(x)\}$. Since $0\in A$, if $g\in G$, then $g(x)>0$ and $g(x)^{-1}\cdot g\in S$; so $K^0\subset L$ and the first part of the lemma is proved. The proof of the second part is similar and is omitted.

For the special case where A is the closed unit ball, Lemma 2.4 can be stated as follows:

2.5 COROLLARY. *Let $x\in S(X)$ [resp., let $x\in S(X^*)$], and let $0<\varepsilon<1$. The closed convex cones generated by $x+\varepsilon B(X)$ and by $S(-x, 1-\varepsilon, B(X^*))$ [resp., by $x+\varepsilon B(X^*)$ and by $S(-x, 1-\varepsilon, B(X))$] are polar to each other.*

The following lemma relates the diameter of a bounded slice to the size of a ball generating a cone that contains the slice.

2.6. LEMMA. *Suppose that $K\equiv k(-x, \varepsilon B(X))$ contains the slice $S\equiv S(f, \alpha, A)$ where $\|x\|=1$, $0<\varepsilon<1$, $\alpha>0$, $1\le \sup f(A)\le 1+\alpha$ and $\sup\{\|y\|:y\in S\}\le M$. Then, $\operatorname{diam} S\le 8\varepsilon\|f\|M^2+2\alpha M$.*

Proof. Let $y_1, y_2\in S$. *Then*

$$\begin{aligned}\|y_1-y_2\|&\le\|y_1-f(y_2)\cdot y_1\|+\|f(y_2)y_1-f(y_1)y_2\|+\|f(y_1)\cdot y_2-y_2\|\\&\le 2\alpha M+\|f(y_2)y_1-f(y_1)y_2\|\end{aligned}$$

Now, setting $u_i=y_i/\|y_i\|$, $i=1,2$, we obtain $\|u_i-x\|\le 2\varepsilon$. Thus

$$\begin{aligned}\|f(y_2)y_1-f(y_1)y_2\|=\|y_1\|\cdot\|y_2\|(\|f(u_2)u_1-f(u_1)u_2\|)\le M^2(\|f(u_2-u_1)\cdot u_1\|\\+\|f(u_1)(u_1-u_2)\|)\le 2M^2\|f\|\cdot\|u_1-u_2\|\le 8\varepsilon\|f\|M^2\end{aligned}$$

which proves the lemma.

For some applications, it is desirable to assume that the linear functional f, in the definition of strong smoothability, separates x and $c\ell A$. The following lemma shows that we can, in fact, make a stronger assumption.

2.7. LEMMA. *For a convex subset A of X, the following statements are equivalent; (i) A is strongly smoothable; (ii) given any ε, $0<\varepsilon<1$, there is an $x\in X\backslash c\ell(A)$ such that $S(f, 1-\varepsilon, B(X))\subset k(x, A)$ whenever f is any norm one functional with $\inf f(A)\ge f(x)$.*

Proof. We only need to prove that (i) implies (ii).

Suppose that A is strongly smoothable and let $0<\varepsilon<1$. For $\delta=\varepsilon/3$, there is $x\in X\backslash c\ell(A)$ and $g\in S(X^*)$ such that $\{y\in B(X):g(y)\ge\delta\}\subset k(x, A)$. Let $f\in S(X^*)$ with $\inf f(A)\ge f(x)$.

It follows that f is non-negative on $\{y\in B(X):g(y)\ge\delta\}$, or, in the terminology of [14] $f\ge 0$ on $K(g, \delta)$. Then by Corollary 2.3 of [14] we have $\|f-g\|\le 2\delta$. Letting $v\in D_1\equiv g^{-1}(\delta)\cap B(X)$, we then have $f(v)\le 3\delta$ so that $\sup f(D_1)=d\le\varepsilon$. To complete the lemma it suffices to show that $k(x, A)$ contains the set $D_2\equiv\{z\in B(X):f(z)=d\}$. Let $y\in B(X)$ with $g(y)>3\delta$ and suppose that $z\in D_2\backslash k(x, A)$. Then, $g(z)<\delta$, and so the point $w=\frac{1}{2}(y-z)$ is interior to $k(x, A)$

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(because $w \in B(X)$ and $g(w) > \delta$). Then $f(w) > 0$, that is, $f(y) > f(z)$. Since $g(z) < \delta < 3\delta < g(y)$ it follows that the straight line segment $[z, y]$ intersects D_1, say at the point v. But then $\sup f(D_1) \geq f(v) > f(z) = d$, contradicting the definition of d. Thus, we cannot have $z \in D_2 \backslash k(x, A)$, which completes the lemma.

Proof of Theorem 2.3. The proofs of both statements closely parallel each other, so we shall prove only (i). By hypotheses on A both A and A^0 are closed bounded convex sets whose norm interior contains the origin. Assuming that A is dentable, then, since A has nonempty norm interior and contains slices of arbitrarily small diameter, for any $\varepsilon > 0$ there is a slice $S(f, \alpha, A)$ of A and $x \in S(X)$ such that $k(0, S(f, \alpha, A)) \subset k(x, \varepsilon B(X))$. We assume, without loss of generality, that $\sup f(A) > 1$. Taking polars and applying Lemma 2.4 and Corollary 2.5, we obtain $k(f, A^0) \supset S(x, 1-\varepsilon, B(X^*))$. Consequently, A^0 is (weak*) strongly smoothable.

Since A^0 is weak* closed, given any $f \in X^* \backslash A^0$, there exists $x \in S(X)$ with $\inf x(A^0) \geq x(f)$. Lemma 2.7 now shows that if A^0 is strongly smoothable, then it is also weak* strongly smoothable. We may assume then, that given $0 < \varepsilon < 1$ we have $x \in S(X)$ and $f \in X^* \backslash A^0$ such that $k(f, A^0) \supset S(x, 1-\varepsilon, B(X^*))$. Then, with $\alpha = \sup f(A) - 1$, $k(0, S(f, \alpha, A)) \subset k(x, \varepsilon B(X))$. Noting that $k(\lambda f, A^0) \supset k(f, A^0)$ for all λ with $\sup f(A) > \lambda^{-1} \geq 1$, we may assume that $0 < \alpha < \varepsilon$ and also that $\|f\| \leq 1 + \sup\{\|g\| : g \in A^0\} \equiv N < \infty$. Further, we have $\sup\{\|y\| : y \in A\} = M < \infty$. Applying Lemma 2.6 it follows that diam $S(f, \alpha, A) \leq 8\varepsilon NM^2 + 2\varepsilon M$. Thus, A is dentable, and the theorem is proved.

The following theorem shows how the relationship between dentability and weak* dentability compares to that between strong smoothability and weak* strong smoothability.

2.8. THEOREM. *Let A be a non-empty convex subset of X, let B be a non-empty weak* closed convex subset of X^*, and let C be the weak* closure of the canonical image of A in X^{**}. Then*: (i) *B is dentable whenever B is weak* dentable*; (ii) *B is strongly smoothable if and only if B is weak* strongly smoothable*; (iii) *C is weak* dentable if and only if A is dentable*; (iv) *C is weak* strongly smoothable whenever A is strongly smoothable.*

Proof. Statements (i) and (iii) are obvious. Statement (ii) was verified for the case $B = A^0$ in the proof of Theorem 2.3. The proof in the general case is the same.

To prove (iv), let Q be the canonical embedding of X into X^{**}. Suppose A is strongly smoothable, that is, given $\varepsilon > 0$ there is some $x \in X \backslash c\ell(A)$ and $f \in S(X^*)$ such that $k(x, A) \supset S(f, 1-\varepsilon, B(X))$. Since $Q(B(X))$ is weak* dense in $B(X^{**})$, we have $w^* - c\ell(k(Q(x), C)) \supset S(f, 1-\varepsilon, B(X^{**}))$. By Lemma 2.7, we may assume that f strictly separates x from A, so $Q(x) \notin C$. For $y^{**} \in C$, it follows that $Qx - y^{**} \notin C - y^{**}$ and by Lemma 2.4 $k(Qx - y^{**}, C - y^{**})$ is the polar of some cone in X^*. Thus, $k(Qx - y^{**}, C - y^{**})$, and hence $k(Qx, C)$ is weak* closed, completing the proof.

2.9. Remarks. (i) In the next section we show that the converses of statements (i) and (iv) in the above theorem are not true in general.

(ii) In Theorem 2.8 we do not assume that the sets A and B are bounded or have nonvoid interiors. There appear to be difficulties, however, if these assumptions are removed from the hypothesis of Theorem 2.3.

3. **An example.** In this section we give an example which, as mentioned in the introduction, answers negatively a question of Kemp [10], viz., if $B(X^*)$ is dentable, must $B(X)$ be smoothable?

3.1. Definitions. If A is a bounded subset of X, following the definition of Edelstein [7] as reformulated by Kemp [10], we say that A is *smoothable* if for every $\varepsilon > 0$ there exists an $f \in S(X^*)$ and some closed ball $B \subset X$ such that $\sup f(B) < \sup f(A)$ and $A \backslash S(f, \varepsilon, A) \subset B$.

A point $x \in A$ is a *strongly exposed point* of A (resp., *weak* strongly exposed point* of A) if there is an $f \in S(X^*)$ such that for every $\varepsilon > 0$, there is an $\alpha > 0$ such that $x \in S(f, \alpha, A)$ and diam $S(f, \alpha, A) < \varepsilon$. (resp., if $X = Y^*$ and f can be chosen in Y).

It is easily seen that if A has a (weak*) strongly exposed point then A is (weak*) dentable.

3.2. Example. Let $X = C(S)$, S a compact Hausdorff space and recall that X^* is isometrically isomorphic to $rca(S)$, the space of all regular signed measures on Σ, the Borel subsets of S, with the variation norm (cf. [6, p. 265]).

First we observe that $B(X^*)$ has a strongly exposed point and, hence, is dentable. For, let $s_0 \in S$, ϕ_0 be the characteristic function of $\{s_0\}$ and μ_0 be the Dirac measure at s_0 (i.e., for every $E \in \Sigma$, $\mu_0(E) = 1$, if $s_0 \in E$, $\mu_0(E) = 0$ otherwise). Now in a straightforward manner it can be seen that $\phi_0 \in B(X^{**})$ and μ_0 is strongly exposed by ϕ_0. Further, if s_0 is an isolated point in S then evidently $\phi_0 \in B(X)$ and μ_0 is a weak*-strongly exposed point. Conversely, if $B(X^*)$ has a weak* strongly exposed point, then the norm of X has a point of Fréchet differentiability (cf. [2, p. 450]); and it has been shown by Cox and Nadler [3] (and more recently by Kemp [10]) that this can happen only if S has an isolated point.

On the other hand, if S has no isolated points, we observe that $B(X^*)$ is not weak* dentable. For, let $f \in S(X)$ and $\mu \in S(X^*)$ such that $f(\mu) = 1$, then for $0 < \beta < 1$ we will exhibit a measure $\nu \in S(f, \beta, B(X^*))$ for which $\|\nu - \mu\| \geq 1$. Since S is a perfect set and $N = \{s \in S \colon |f(s)| > 1 - \beta\}$ is a nonempty open subset of S, $c\ell(N)$ is uncountable [9, p. 88]. Now the variation $v(\mu, S)$ is finite, so there exists $s_1 \in c\ell(N)$ such that $v(\mu, \{s_1\}) = 0$. Define $\nu \in X^* = rca(S)$ to be the measure whose value at each $E \in \Sigma$ is given by $\nu(E) = \operatorname{sgn} f(s_1)$ whenever $s_1 \in E$, and by $\nu(E) = 0$ otherwise. Then $\nu \in S(f, \beta, B(X^*))$ and $\|\nu - \mu\| \geq v(\nu, \{s_1\}) = 1$.

This example yields the following

3.3 THEOREM. *There exists a real Banach space X such that (i) $B(X^*)$ is dentable and $B(X)$ fails to be smoothable; and, (ii) $B(X^{**})$ is (weak*) strongly smoothable and $B(X)$ fails to be strongly smoothable.*

Proof. Let $X = C(S)$ where S is a compact Hausdorff space with no isolated points. (i) $B(X^*)$ is dentable as shown above and Kemp [10, Prop. 3.2] has shown that $B(X)$ is not smoothable.

(ii) $B(X^*)$ is not weak* dentable but is dentable as seen above, so by Theorem 2.3 $B(X^{**})$ is (weak*) strongly smoothable and $B(X)$ is not strongly smoothable.

3.4. REMARK. It may be interesting to observe that for $X = C(S)$ the following are equivalent:

(i) S has an isolated point;
(ii) the sup norm on X is Fréchet differentiable at some nonzero point;
(iii) $B(X)$ is smoothable;
(iv) $B(X)$ is strongly smoothable;
(v) $B(X^*)$ is weak* dentable;
(vi) $B(X^*)$ has a weak* denting point;
(vii) $B(X^*)$ has a weak* strongly exposed point.

4. **Strongly smoothable spaces.** A Banach space X is called a *strongly smoothable space* whenever all of its convex bodies are strongly smoothable. (By a *convex body* we mean a bounded convex subset with nonempty interior.)

First, we show that strongly smoothable spaces are separably determined:

4.1. THEOREM. *If X fails to be strongly smoothable, then every separable subspace of X is contained in a separable subspace that fails to be strongly smoothable.*

Proof. Let $A \subset X$ be a closed convex body with $0 \in \operatorname{int} A$ which fails to be strongly smoothable. By Lemma 2.7 this is equivalent to the statement:

(4.1.1) $\exists \varepsilon$, $0 < \varepsilon < 1$, such that given $x \notin A$, there is $f \in S(X^*)$

with $\inf f[A] \geq f(x)$ and $S(f, 1-\varepsilon, B(X)) \not\subset k(x, A)$.

Let E_1 be any separable subspace of X. We inductively construct a sequence of separable spaces $E_1 \subset E_2 \subset \cdots \subset E_k \subset \cdots$ as follows: set $A_k = E_k \cap A$ and choose $\{x_i^k\colon i = 1,2,\ldots\}$ to be a dense sequence in the boundary of A_k. Set $x_{ij}^k = (1+j^{-1})x_i^k$, $j = 1,2,\ldots$. Clearly $x_{ij}^k \notin A$, so by (4.1.1) there exists $f_{ij}^k \in S(X^*)$ such that $\inf f_{ij}^k[A] \geq f_{ij}^k(x_{ij}^k)$ and there exists $y_{ij}^k \notin k(x_{ij}^k, A)$, $y_{ij}^k \in S(X^*)$ and

$f_{ij}^k(y_{ij}^k) \geq \varepsilon$. We now define E_{k+1} to be the closed linear span of $E_k \cup \{y_{ij}^k : i,j = 1,2,\ldots\}$ and set $Y = c\ell[\bigcup_{k \in N} E_k]$.

Clearly Y is separable and we claim that the closed convex body $A_0 = A \cap Y$ fails to be strongly smoothable in Y. To show this, let $y \in Y \backslash A_0$ and let K be the convex cone with vertex y generated by A_0 (i.e. $K = y + k(y, A_0)$). Now $\text{dist}(y, A_0) = d > 0$, so $U = \text{int}\, K \cap \{y + dB(Y))$ is a nonempty open subset of Y and $U \cap A_0 = \phi$.

Since $\cup E_k$ is dense in Y there is some k such that $E_k \cap U \neq \phi$. By the density of the x_i^k's in the boundary of A_k there is some x_i^k such that $\lambda x_i^k \in U$ for some $\lambda > 1$. So, for some $j \geq 1$, $x_{ij}^k \in \text{int}\, K$ and $[y, x_{ij}^k] \cap A = \phi$. Note that this implies that $x_{ij}^k \in [y, z]$ for some $z \in \text{int}\, A_0$.

Now $f_{ij}^k(x_{ij}^k) \leq \inf f_{ij}^k[A] \leq \inf f_{ij}^k[A_0]$ and since $z \in \text{int}\, A_0$, we have $f_{ij}^k(y) \leq f_{ij}^k(x_{ij}^k)$. Setting $g = f_{ij}^k \mid Y$, we have $g(y) \leq \inf g[A_0]$, $g(y_{ij}^k) \geq \varepsilon$, $y_{ij}^k \in S(Y) \backslash k(y, A_0)$. Since $\|g\| \leq 1$, all of these assertions remain valid if g is replaced by $\|g\|^{-1} \cdot g$, so by (4.1.1) A_0 is not strongly smoothable in Y.

Following Namioka and Phelps [13], we call X an *Asplund space* if every continuous convex function defined on an open convex subset of X of Fréchet differentiable on a dense G_δ subset of its domain. Such spaces were first introduced by Asplund [1] and were called "strong differentiability spaces". We next show that strongly smoothable spaces are precisely Asplund spaces.

4.2. THEOREM. *A real Banach space is strongly smoothable if and only if it is an Asplund space.*

Proof. Suppose X is an Asplund space, then Asplund has shown [1, Prop. 5] that every weak* compact convex subset of X^* is weak* dentable. However, if C is a bounded subset of X^* and D, the weak* closed convex hull of C, is weak* dentable then C is weak* dentable. Thus, it follows from Theorem 2.3 that X is strongly smoothable.

Conversely, suppose that X is a strongly smoothable space. Now C, a weak* compact convex subset of X^*, is weak* dentable, if the weak* closed symmetric convex body $D = B(X^*) + (C - C)$ is weak* dentable; which is so by Theorem 2.3. Thus by Lemma 3 and Theorem 6 of [13], X is an Asplund space.

The following Corollary is an immediate consequence of Theorems 4.1 and 4.2.

4.3. COROLLARY. (Namioka, Phelps, [16]). *Let X be a real Banach space and suppose that every separable subspace of X is an Asplund space, then X is an Asplund space.*

The converse of Corollary 4.3 holds as shown by Namioka and Phelps [13, Thm. 12].

A real Banach space X has the *Radon-Nikodým Property* (RNP) if every bounded subset C of X is dentable. That the dual of an Asplund space has the RNP is essentially due to Asplund [1]; that the converse is also true is due to Stegall [18]:

4.4. THEOREM. (Stegall, [18]). *A real Banach space X is an Asplund space whenever X^* has the RNP.*

Proof. By a deep theorem of Stegall [17], every separable subspace of X^* has a separable dual. Asplund [1, Thm. 1] shows that, in this case, every separable subspace is an Asplund space, and, by Corollary 4.3, X is an Asplund space.

4.5. THEOREM. (Ekeland, Lebourg [8]). *If the norm on X is Fréchet differentiable (except at the origin) then X is an Asplund space.*

Proof. (For then, X^* has the RNP [4], [11]). Alternatively, the continuity of the differential map together with the Bishop-Phelps density theorem shows that every separable subspace has a separable dual and hence is an Asplund space [1].

5. **Strong smoothability and malleability.** The norm in X is said to be *malleable* (Sullivan [19]) if for each $\varepsilon>0$ there exists $x\in S(X)$ and $\delta>0$ such that for every $\lambda, 0<\lambda<\delta$, and for all $y\in B(X)$ it follows that $\|x+\lambda y\|+\|x-\lambda y\|-2<\varepsilon\lambda$.

Sullivan showed that every bounded subset of X is smoothable whenever the norm is malleable. Here, we show that malleability of the norm is equivalent to strong smoothability of the unit ball.

5.1. THEOREM. *For a Banach space X the following are equivalent*:

(i) *The norm in X is malleable*;

(ii) *$B(X^*)$ is weak* dentable*;

(iii) *$B(X)$ is strongly smoothable.*

Proof. The equivalence of (ii) and (iii) follows from Theorem 2.3.

To show that (i) $\Rightarrow$ (ii), assume that (ii) is false. Then there exists an $\varepsilon>0$ such that for every α, $0<\alpha<1$, and $x\in S(X)$ we have diam $S(x, \alpha, B(X^*))\geq\varepsilon$. In particular, given $\lambda>0$, for $\alpha=\min\{\frac{1}{2}, \varepsilon\lambda/4(1+\lambda)\}$ there exist $f, g\in S(x, \alpha, B(X^*))$ such that $\|f-g\|\geq\varepsilon-\alpha$. Choose $y\in B(X)$ so that $(f-g)(y)\geq\|f-g\|-\alpha\geq\varepsilon-2\alpha$. Then it may be verified that

$$\|x+\lambda y\|+\|x-\lambda y\|-2\geq f(x+\lambda y)+g(x-\lambda y)-2\geq\frac{\lambda\varepsilon}{2}.$$

Thus the norm in X fails to be malleable.

For the converse, assume (i) is false. Then there is $\varepsilon>0$ such that for every $x\in S(X)$ and each α, $0<\alpha<1$, there exist λ, $0<\lambda<\alpha$, and $y\equiv y(x, \alpha)\in B(X)$ such that $\|x+\lambda y\|+\|x-\lambda y\|-2\geq\varepsilon\lambda$.

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By the Hahn-Banach theorem, there exist $f, g \in S(X^*)$ such that $\|x+\lambda y\| = f(x+\lambda y)$ and $\|x-\lambda g\| = g(x-\lambda y)$. Consequently,

$$\varepsilon\lambda \leq f(x+\lambda y)+g(x-\lambda g)-2 \leq \lambda(f-g)(y),$$

so $\|f-g\| \geq \varepsilon$. Further,

$$f(x) \geq f(x+\lambda y)-\lambda \geq \|x\|-\lambda\|y\|-\lambda \geq 1-2\lambda > 1-2\alpha,$$

so $f \in S(x, 2\alpha, B(X^*))$. Similarly, $g \in S(x, 2\alpha, B(X^*))$, so diam $S(x, 2\alpha, B(X^*)) \geq \varepsilon$. Thus $B(X^*)$ fails to be weak* dentable, which completes the proof of the theorem.

Using the result of Sullivan [19] referred to above, we obtain

5.2. Corollary. *If $B(X)$ is strongly smoothable (equivalently, if $B(X^*)$ is weak* dentable), then every bounded subset of X is smoothable.*

We do not know if the converse of 5.2 is true. However, we mention that smoothability of $B(X)$ by itself does not imply that it is strongly smoothable; this can be easily seen using the example in [12]. These comments suggest the following

5.3. Problem. Suppose that every equivalent unit ball for the Banach space X is smoothable. Is X an Asplund space?

We remark that the question has an affirmative answer if every equivalent unit ball is strongly smoothable (or if every equivalent norm is malleable) for under these circumstances every symmetric weak* compact convex subset of X^* would be dentable, i.e., X^* would have the RNP, and so X would be an Asplund space by Theorem 4.4.

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