SOME RECURRENCE RELATIONS AND SERIES FOR THE GENERALISED LAPLACE TRANSFORM

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1. Introductory. The Laplace transform

$$f(p) = p\int_0^\infty e^{-px}h(x)\,dx \quad (\mathrm{Re}\,p > 0) \qquad \text{.........................(1.1)}$$

has been generalised by Varma [4] by the relation

$$\phi(p) = p\int_0^\infty e^{-\frac{1}{2}px}(px)^{m-\frac{1}{2}}W_{k,m}(px)h(x)\,dx \quad (\mathrm{Re}\,p > 0), \qquad \text{..............(1.2)}$$

which reduces to (1.1) when $k = -m+\frac{1}{2}$ by virtue of the identity

$$W_{-m+\frac{1}{2},m}(x) = x^{-m+\frac{1}{2}}e^{-\frac{1}{2}x}. \qquad \text{..................................(1.3)}$$

We shall define $\phi_{k,m,\lambda}(p)$ by the relation

$$\phi_{k,m,\lambda}(p) = p\int_0^\infty e^{-\frac{1}{2}px}(px)^{m-\frac{1}{2}}W_{k,m}(px)x^\lambda h(x)\,dx \quad (\mathrm{Re}\,p > 0). \qquad \text{...........(1.4)}$$

The object of this paper is to obtain some recurrence formulae and series for $\phi_{k,m,\lambda}(p)$ and to use them to obtain recurrence formulae and series for MacRobert's E-function.

2. Formulae required in the proof. We have [5, p. 352]

$$W_{k,m}(z) = z^{\frac{1}{2}}W_{k-\frac{1}{2},m-\frac{1}{2}}(z) + (\tfrac{1}{2}-k+m)W_{k-1,m}(z), \qquad \text{........................(2.1)}$$

$$W_{k,m}(z) = z^{\frac{1}{2}}W_{k-\frac{1}{2},m+\frac{1}{2}}(z) + (\tfrac{1}{2}-k-m)W_{k-1,m}(z) \qquad \text{.......................(2.2)}$$

and

$$zW'_{k,m}(z) = (k-\tfrac{1}{2}z)W_{k,m}(z) - \{m^2-(k-\tfrac{1}{2})^2\}W_{k-1,m}(z). \qquad \text{..................(2.3)}$$

We have also [3, p. 201]

$$\frac{d}{dz}[z^{m-\frac{1}{2}}e^{-\frac{1}{2}z}W_{k,m}(z)] = -z^{m-1}e^{-\frac{1}{2}z}W_{k+\frac{1}{2},m-\frac{1}{2}}(z). \qquad \text{........................(2.4)}$$

It will be observed that (2.2) can be obtained from (2.1) by using the property

$$W_{k,-m}(z) = W_{k,m}(z). \qquad \text{...(2.5)}$$

From (2.3) we also observe that

$$W'_{k,-m}(z) = W'_{k,m}(z). \qquad \text{...(2.6)}$$

Harishanker has obtained the following series for $W_{k,m}(z)$

$$W_{k+n,m}(z) = (-1)^n\,\Gamma(m+k+n+\tfrac{1}{2})\,n!\sum_{r=0}^{n}\frac{(-1)^r z^{r/2}W_{k+r/2,m+r/2}(z)}{(n-r)!\,r!\,\Gamma(m+k+r+\tfrac{1}{2})} \quad (\mathrm{Re}(\tfrac{1}{2}-k+m) > 0), \qquad \text{......(2.7)}$$

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and

$$W_{k-n,m}(z) = \frac{(-1)^n \Gamma(m+k+n+\frac{1}{2})n!}{\Gamma(m+k+\frac{1}{2})} \sum_{r=0}^{n} \frac{(-1)^r z^{r/2} W_{k-r/2,\, m+r/2}(z)}{(n-r)!\,r!} \quad (\mathrm{Re}(\tfrac{1}{2}-k+m)>0). \qquad \text{.................(2.8)}$$

3. Recurrence formulae for the Whittaker's confluent hypergeometric function. Eliminate $W_{k-1,m}(z)$ between (2.1) and (2.2), divide by $z^{\frac{1}{2}}$, replace k by $k+\frac{1}{2}$ and m by $m-\frac{1}{2}$ to obtain

$$(m-k-\tfrac{1}{2})W_{k,m}(z) = (\tfrac{1}{2}-k-m)W_{k,m-1}(z)+(2m-1)z^{-\frac{1}{2}}W_{k+\frac{1}{2},\,m-\frac{1}{2}}(z). \qquad \text{.........(3.1)}$$

This has been otherwise obtained by Rathie [**2**, p. 392].

In (2.2) replace m by $m-1$ and eliminate $z^{\frac{1}{2}}W_{k-\frac{1}{2},\,m-\frac{1}{2}}(z)$ from this relation and (2.1) to get

$$W_{k,m-1}(z)+(\tfrac{1}{2}-k+m)W_{k-1,m}(z) = (\tfrac{3}{2}-k-m)W_{k-1,m-1}(z)+W_{k,m}(z). \qquad \text{.........(3.2)}$$

Equating the values of $W'_{k,m}(z)$ from (2.3) and (2.4), we obtain

$$(m+k-z-\tfrac{1}{2})W_{k,m}(z) = \{m^2-(k-\tfrac{1}{2})^2\}W_{k-1,m}(z)-z^{\frac{1}{2}}W_{k+\frac{1}{2},\,m-\frac{1}{2}}(z). \qquad \text{...........(3.3)}$$

Simplifying (2.4), we get

$$zW'_{k,m}(z) = (\tfrac{1}{2}z-m+\tfrac{1}{2})W_{k,m}(z)-z^{\frac{1}{2}}W_{k+\frac{1}{2},\,m-\frac{1}{2}}(z). \qquad \text{.....................(3.4)}$$

Using (2.6) with (3.4), we obtain

$$zW'_{k,m}(z) = (\tfrac{1}{2}z+m+\tfrac{1}{2})W_{k,m}(z)-z^{\frac{1}{2}}W_{k+\frac{1}{2},\,m+\frac{1}{2}}(z). \qquad \text{......................(3.5)}$$

4. Recurrence formulae for the generalised Laplace transform $\phi_{k,m,\lambda}(p)$. Using (3.2), we get

$$p\phi_{k,m-1,\lambda+1}(p)+(\tfrac{1}{2}-k+m)\phi_{k-1,m,\lambda}(p) = p(\tfrac{3}{2}-k-m)\phi_{k-1,m-1,\lambda+1}(p)+\phi_{k,m,\lambda}(p). \qquad \text{........(4.1)}$$

Using (3.3), we get

$$(m+k-\tfrac{1}{2})\phi_{k,m,\lambda}(p)-p\phi_{k,m,\lambda+1}(p) = \{m^2-(k-\tfrac{1}{2})^2\}\phi_{k-1,m,\lambda}(p)-p\phi_{k+\frac{1}{2},\,m-\frac{1}{2},\,\lambda+1}(p). \qquad \text{......(4.2)}$$

5. Recurrence formulae for MacRobert's E-function. If

$$x^{\lambda}h(x) = x^{\lambda-1}E\left(\alpha_1,\ldots,\alpha_{r-2}:\ \beta_1,\ldots,\beta_{s-1}:\ \frac{1}{x}\right),$$

then [**2**, p. 392]

$$\phi_{k,m,\lambda}(p) = p^{1-\lambda}E\begin{pmatrix}\alpha_1,\ldots,\alpha_{r-2},\ \lambda,\ \lambda+2m: & p\\ \beta_1,\ldots,\beta_{s-1},\ \lambda+m-k+\frac{1}{2}: & \end{pmatrix}. \qquad \text{.....................(5.1)}$$

The formulae (4.1) and (4.2), on replacing λ by α_{r-1}, $\lambda+2m$ by α_r and $\lambda+m-k+\frac{1}{2}$ by β_s, then give us

$$E\begin{pmatrix}\alpha_1,\ldots,\alpha_{r-2},\alpha_{r-1}+1,\alpha_r-1: & p\\ \beta_1,\ldots,\beta_{s-1},\beta_s: & \end{pmatrix}+(\beta_s-\alpha_{r-1})\,E\begin{pmatrix}\alpha_1,\ldots,\alpha_r: & p\\ \beta_1,\ldots,\beta_{s-1},\beta_s+1: & \end{pmatrix}$$

$$= (1+\beta_s-\alpha_r)\,E\begin{pmatrix}\alpha_1,\ldots,\alpha_{r-1}+1,\alpha_r-1: & p\\ \beta_1,\ldots,\beta_{s-1},\beta_s+1: & \end{pmatrix}+E\begin{pmatrix}\alpha_1,\ldots,\alpha_r: & p\\ \beta_1,\ldots,\beta_s: & \end{pmatrix}, \qquad \text{...........(5.2)}$$

and

$$(\alpha_r-\beta_s)E\begin{pmatrix}\alpha_1,\ldots,\alpha_r: & p\\ \beta_1,\ldots,\beta_s: & \end{pmatrix}-E\begin{pmatrix}\alpha_1,\ldots,\alpha_{r-2},\alpha_{r-1}+1,\alpha_r+1: & p\\ \beta_1,\ldots,\beta_{s-1},\beta_s+1: & \end{pmatrix}$$

$$= (\beta_s-\alpha_{r-1})(\alpha_r-\beta_s)E\begin{pmatrix}\alpha_1,\ldots,\alpha_r: & p\\ \beta_1,\ldots,\beta_{s-1},\beta_s+1: & \end{pmatrix}-E\begin{pmatrix}\alpha_1,\ldots,\alpha_{r-2},\alpha_{r-1}+1,\alpha_r: & p\\ \beta_1,\ldots,\beta_s: & \end{pmatrix}. \qquad \text{...(5.3)}$$

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6. Series for the generalised Laplace transform. Using the results (2.7) and (2.8), we obtain the following series for the generalised Laplace transform, $\phi_{k,m,\lambda}(p)$,

$$\phi_{k+n,m,\lambda}(p) = (-1)^n \Gamma(k+m+n+\tfrac{1}{2})\, n! \sum_{r=0}^{n} \frac{(-1)^r \phi_{k+r/2,\, m+r/2,\, \lambda}(p)}{(n-r)!\, r!\, \Gamma(m+k+r+\tfrac{1}{2})} \quad (\mathrm{Re}(\tfrac{1}{2}-k+m) > 0), \qquad \text{.........(6.1)}$$

and

$$\phi_{k-n,m,\lambda}(p) = \frac{n!\, \Gamma(m+k-n+\tfrac{1}{2})(-1)^n}{\Gamma(m+k+\tfrac{1}{2})} \sum_{r=0}^{n} \frac{(-1)^r \phi_{k-r/2,\, m+r/2,\, \lambda}(p)}{(n-r)!\, r!} \quad (\mathrm{Re}(\tfrac{1}{2}-k+m) > 0). \qquad \text{......(6.2)}$$

7. Series for the MacRobert's *E*-function. Using (5.1) with the results (6.1) and (6.2), we obtain the following finite series involving MacRobert's E-function

$$E\begin{pmatrix}\alpha_1, \dots, \alpha_r : & p \\ \beta_1, \dots, \beta_s - n : & \end{pmatrix} = (-1)^n n!\, \Gamma(1+\alpha_r-\beta_s+n) \sum_{t=0}^{n} \frac{(-1)^t E\begin{pmatrix}\alpha_1, \dots, \alpha_{r-1}, \alpha_r+t : & p \\ \beta_1, \dots, \beta_s : & \end{pmatrix}}{(n-t)!\, t!\, \Gamma(1+\alpha_r-\beta_s+t)} \qquad \text{.........(7.1)}$$

and

$$E\begin{pmatrix}\alpha_1, \dots, \alpha_r : & p \\ \beta_1, \dots, \beta_{s-1}, \beta_s+n : & \end{pmatrix} = \frac{(-1)^n n!\, \Gamma(1+\alpha_r-\beta_s-n)}{\Gamma(1+\alpha_r-\beta_s)} \sum_{t=0}^{n} \frac{(-1)^t E\begin{pmatrix}\alpha_1, \dots, \alpha_{r-1}, \alpha_r+t : & p \\ \beta_1, \dots, \beta_{s-1}, \beta_s+t : & \end{pmatrix}}{(n-t)!\, t!}. \qquad \text{............(7.2)}$$

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