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On the distance of non-reflexive spaces to the collection of all conjugate spaces

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We show that there exists a conjugate Banach space $E = B^*$ with a basis, such that the distance from E to the collection of all conjugate Banach spaces can be made arbitrarily large, by suitable renorming of E. This solves a problem raised by B.V. Godun, *Dokl. Akad. Nauk SSSR* 236 (1977), 18-20.

The Banach-Mazur distance of two Banach spaces E and F is defined by

$$(1) \qquad d(E, F) = \begin{cases} \inf\limits_u \|u\|\|u^{-1}\| & \text{if } E \text{ is isomorphic to } F , \\ +\infty & \text{if } E \text{ is not isomorphic to } F , \end{cases}$$

where the infimum is taken over all isomorphisms u of E onto F. Recently, Godun [5] has introduced, for a Banach space E, the number

$$(2) \quad D(E) = \sup_{|||\cdot|||\in A} \inf_{X^*\in C} d\big((E, |||\cdot|||), X^*\big) = \sup_{|||\cdot|||\in A} d\big((E, |||\cdot|||), C\big) ,$$

where C denotes the collection of all conjugate Banach spaces X^* and A denotes the collection of all equivalent norms $|||\cdot|||$ on E. Godun has shown ([5], Theorem) that for every non-reflexive Banach space E we have $D(E) \geq 2$ and has raised the following problem: does there exist a conjugate Banach space $E = B^*$ such that $D(E) = \infty$? In the present note

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we shall show that the answer is affirmative and, moreover, one can even find such an E with a basis. Our proof is very short, but uses deep results of Enflo [3], Lindenstrauss [8], Figiel and Johnson [4], and Grothendieck [6].

We recall that a Banach space E is said to have

(a) *the approximation property*, if for every compact subset Q of E and every $\varepsilon > 0$ there exists a continuous linear operator $v : E \to E$ with $\dim v(E) < \infty$, such that $\|x-v(x)\| < \varepsilon \quad (x \in Q)$;

(b) *the* λ-*approximation property*, if one can find v with the above properties and satisfying, in addition, $\|v\| \le \lambda$.

The 1-approximation property is also called *the metric approximation property*.

Now we are ready to give the following

EXAMPLE. By Enflo's negative solution of the approximation problem [3] and a result of Lindenstrauss [8], there exists a conjugate Banach space $E = B^*$ with a basis, such that $E^* = B^{**}$ does not have the approximation property. Then, by a theorem of Figiel and Johnson [4], there exists a sequence $\{|||\cdot|||_n\}_{n=1}^{\infty}$ of equivalent norms on E, so that $(E, |||\cdot|||_n)$ does not have the n-approximation property. Now let X^* be any conjugate Banach space.

Case 1°. X^* is separable and has the approximation property. Then, as has been observed by Johnson, Rosenthal, and Zippin ([7], Remark 4.11) the results of Grothendieck [6] imply that X^* has the metric approximation property. Hence,

$$d\left((E, |||\cdot|||_n), X^*\right) \ge n \quad (n = 1, 2, \dots) \tag{3}$$

(since otherwise it would follow that $(E, |||\cdot|||_n)$ has the n-approximation property).

Case 2°. X^* is separable and does not have the approximation property, or X^* is non-separable. In this case,

$$d\left((E, |||\cdot|||_n), X^*\right) = \infty \tag{4}$$

$\big($since $(E, |||\cdot|||_n)$ has a basis, whence also the approximation property, so $(E, |||\cdot|||_n)$ is not isomorphic to X^* $\big)$.

Hence, since X^* was an arbitrary conjugate Banach space, it follows that

$$\inf_{X^* \in C} d\big((E, |||\cdot|||_n), X^*\big) \geq n \quad (n = 1, 2, \ldots) , \tag{5}$$

so $D(E) = \infty$.

REMARK. Godun [5] has achieved the proof that $D(E) \geq 2$ for every non-reflexive Banach space, by showing the following stronger result. *For every non-reflexive Banach space* E *and each* $\varepsilon > 0$ *there exists an equivalent norm* $|||\cdot|||$ *on* E *such that there exists no* projection p of norm $|||p||| < 2 - \varepsilon$ of E^{**} onto $\kappa(E)$, the canonical image of E in E^{**} . Since Godun [5] did not mention the paper [2], let us observe that the weaker result in which $|||p||| < 2 - \varepsilon$ is replaced by $|||p||| = 1$, had been proved in [2], Theorem 2.1 (giving an affirmative answer to a problem of Davis and Johnson [1]) and that the above result of Godun solves Problem 2.1 of [2]. Also, in [2] it was observed that for any equivalent norm $|||\cdot|||$ on a quasi-reflexive space E of order 1 $\big($that is, with $\dim E^{**}/\kappa(E) = 1$ $\big)$ and for any $\varepsilon > 0$ there exists a projection p of E^{**} onto $\kappa(E)$ of norm $|||p||| < 2 + \varepsilon$. However, the following problem remains open:

PROBLEM. Let E be a non-reflexive Banach space. Does there exist an equivalent norm $|||\cdot|||$ on E such that there exists no projection p of norm $|||p||| \leq 2$ of E^{**} onto $\kappa(E)$?

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