SETS OF EQUALLY INCLINED SPHERES

J. G. MAULDON

1. Introduction. In n-dimensional Euclidean space E_n, where we shall throughout assume that $n \geqslant 2$, the maximum number of $(n - 1)$-dimensional spheres which can be mutually orthogonal is $n + 2$, and it is well known that the sum of the squares of the reciprocals of their radii is zero, so that the spheres cannot be all real. The maximum number of such spheres which can be mutually tangent is also $n + 2$, and in 1936 Soddy **(7)** indicated the beautiful relation connecting their radii. These two formulae are the particular cases $\gamma = 0$, $\gamma = -1$ of Theorem 1 below, which gives the relation connecting the radii of a set of $n + 2$ such spheres when every pair is inclined at a given non-zero angle θ, where γ is written for $\cos\theta$. The formula is applicable even in the case when real spheres do not intersect in real points, so that the angle θ is not real—in fact γ, defined by (1), is always real for real spheres, and can take any real value except* $+1$.

Let us denote a sphere of dimension n by S_n. Then, in Euclidean space of any dimension, if a set of spheres S_{n-1} is such that every pair intersects in an S_{n-2}, we find that either (a) the same S_{n-2} is common to all the S_{n-1} or (b) the S_{n-1} all lie in some n-dimensional space E_n or (c) they all lie on the n-dimensional surface of a sphere S_n. The easiest way of proving this result is to invert the whole figure with respect to a common point of two of the spheres S_{n-1}, and this method readily leads to an analysis of the case (a), which will not be further considered here except (§ 9) in the degenerate case $\gamma = 1$.

If a set of $n + 2$ spheres S_{n-1} are such that each pair intersects in an S_{n-2} and each pair is equally inclined, with inclination $\gamma = \cos\theta$, then case (b) is covered by Theorems 1 to 3 and case (c) by Theorem 4.

The results are also applicable in elliptic space, and in § 10 they are extended to hyperbolic space.

2. Notation and definitions. The *inclination* γ of two spheres S_{n-1} in E_n with finite non-zero radii r_1 and r_2 is defined by the equation

$$d^2 = r_1^2 + r_2^2 - 2\gamma r_1 r_2, \tag{1}$$

where d is the distance between their centres. Thus for real spheres γ is always real, if they intersect in real points at an angle θ then $\gamma = \cos\theta$, and the

Received June 28, 1961. Prepared with the partial support of the National Science Foundation, Grant G-14648.

*The value $\gamma = +1$ yields a degenerate case, which is excluded from the main body of the paper and covered briefly in § 9.

sign of the radius has relevance, so that for concentric spheres with radii $\pm r$ we have $\gamma = -1$. We shall not consider spheres of zero radius, but (1) can be extended in a straightforward way to the case of spheres of infinite radius. The underlying idea is to assign to the normal at each point of a sphere a definite (continuous) orientation, to define θ as the angle between the oriented normals at a common point, and to take $\gamma = \cos\theta$.

For two spheres S_{n-1} in spherical n-space (that is, on a sphere S_n) their inclination γ is given by the well-known result of spherical trigonometry

$$\cos\phi = \cos\alpha_1\cos\alpha_2 + \gamma\sin\alpha_1\sin\alpha_2, \tag{2}$$

where α_1 and α_2 are their angular radii and ϕ is the angular distance between their centres.

The inclination γ of two real spheres S_{n-1} in E_n or S_n can alternatively be expressed in terms of the cross-ratio χ of the two pairs of extremities of collinear diameters, oriented in the same or opposite sense according as the S_{n-1} have radii of the same or opposite sign (for definiteness, the angular radius of a real S_{n-1} in S_n will always be taken between $-\pi$ and $+\pi$). In fact, we have $\gamma = (1+\chi)/(1-\chi)$.

3. The principal results.

THEOREM 1. *If N spheres S_{n-1} in n-dimensional space E_n are such that each pair of spheres has the same inclination $\gamma \neq 1$ then $N \leqslant n+2$ and, if $N = n+2$, we have*

$$\left\{\sum_{i=1}^{n+2} r_i^{-1}\right\}^2 = (n+1+\gamma^{-1})\sum_{i=1}^{n+2} r_i^{-2}, \tag{3}$$

where $r_1, r_2, \ldots, r_{n+2}$ are the radii of the spheres.

The formula (3) is of the general type considered in the latter part of **(1)**.

If the r_i are real, the Schwarz inequality shows at once that we must have $\gamma^{-1} \leqslant 1$, but it is rather more surprising that for real spheres we must have $\gamma < 0$:

THEOREM 2. *If the $n+2$ spheres in Theorem 1 are all real, then*

$$\gamma \leqslant -(n+1)^{-1}. \tag{4}$$

We may notice that if $\theta = \operatorname{arc\,cos}\gamma$ is real, then (4) is equivalent to

$$\pi - \operatorname{arc\,sec}(n+1) \leqslant \theta \leqslant \pi. \tag{5}$$

Conversely, we have

THEOREM 3. *If γ satisfies (4), then there exist real numbers $r_1, r_2, \ldots, r_{n+2}$ satisfying (3), and for any such set of real numbers r_i there exists a set of $n+2$ real spheres S_{n-1} (with radii r_i) satisfying the conditions of Theorem 1.*

Subject to (4), *the* r_i *may always be chosen to be not all of the same sign. On the other hand, they may be chosen to be all of the same sign* (*positive, say*) *if and only if* $\gamma < -n^{-1}$.

For spherical space S_n we have

THEOREM 4. *If* $N \geqslant 3$ *spheres* S_{n-1} *in spherical n-space* S_n *are such that each pair of the* S_{n-1} *has the same inclination* $\gamma \neq 1$ *then* $N \leqslant n+2$. *If* $N = n+2$, *we have*

$$\left\{ \sum_{i=1}^{n+2} \cot \alpha_i \right\}^2 = (n+1+\gamma^{-1}) \left\{ 1 - \gamma + \sum_{i=1}^{n+2} \cot^2 \alpha_i \right\}, \tag{6}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_{n+2}$ *are the angular radii of the* S_{n-1}, *and if the* $n+2$ *spheres* S_{n-1} *are all real, then* (4) *holds.*

Conversely, if γ *and* $\cot \alpha_i$ *are real numbers subject to* (4) *and* (6), *then there exists a set of* $n+2$ *real* S_{n-1} *in* S_n *with angular radii* α_i *such that each pair of the* S_{n-1} *has the same inclination* γ.

Regarding the spherical n-space S_n as a sphere embedded in E_{n+1} we obtain the same result for right circular cones with a common vertex (at the centre of S_n). Identifying diametrically opposite points we reach the same formula for elliptic space. The corresponding results for hyperbolic space are given in § 10.

4. Proof of the formulae (3) and (6). Although (3) is readily deduced from (6) by letting the radius of the spherical n-space tend to infinity, it is convenient to prove (3) first, since our proof of (6) makes use of (3).

Elementary considerations **(6)** with rectangular cartesian co-ordinates, regarding all but the first sphere as fixed, show that the required relation must be algebraic in r_1 (indeed, quadratic, but we shall not use this fact). Hence it is algebraic in r_1^{-1} and it is clearly symmetric in $r_1^{-1}, \ldots, r_{n+2}^{-1}$, so that it can be expressed as a polynomial relation in the elementary symmetric functions $p_1, p_2, \ldots, p_{n+2}$ of the quantities r_i^{-1}.

Confining attention for the moment to a proof of Gosset's result **(5)**, which is the particular case $\theta = \pi$ (or $\gamma = -1$) of (3), consider the special case $r_1^{-1} = x$, $r_2^{-1} = r_3^{-1} = \ldots = r_n^{-1} = y$, $r_{n+1}^{-1} = r_{n+2}^{-1} = 0$. Then the two hyperplanes must be parallel and a distance $2y^{-1}$ apart, so that the equation for x must be

$$(x-y)^2 = 0, \tag{7}$$

which is of degree 2 in the variables x and y together. Now the required polynomial in $p_1, \ldots, p_{n+2}$ must clearly be homogeneous in the r_i^{-1} and, since it reduces to (7) in the particular case under consideration, it must be of weight 2. Hence it depends only on p_1 and p_2, and the only possible form of relation is $p_1^2/p_2 =$ constant, or equivalently

https://doi.org/10.4153/CJM-1962-042-6 Published online by Cambridge University Press

$$\left\{\sum_{i=1}^{n+2} r_i^{-1}\right\}^2 = C\sum_{i=1}^{n+2} r_i^{-2}, \tag{8}$$

where the constant $C = n$ is evaluated by again considering the special case, with $x = y$.

In the general case $\gamma \neq -1$ it turns out that there is no advantage in taking two of the spheres to have infinite radius, and as a result the proof is a little more complicated. Taking $r_1^{-1} = x$, $r_2^{-1} = r_3^{-1} = \ldots = r_{n+2}^{-1} = y$, we find that the centres of the spheres of radius y^{-1} are situated at the vertices of a regular simplex of side $\{2y^{-2}(1 - \gamma)\}^{\frac{1}{2}}$ whose circumradius is

$$\{x^{-2} + y^{-2} - 2\gamma x^{-1}y^{-1}\}^{\frac{1}{2}}.$$

Now (**3**, p. 158) the ratio of these two quantities is $\{2(n+1)/n\}^{\frac{1}{2}}$ and hence $(n+1)(x^2 + y^2 - 2\gamma xy) = nx^2(1-\gamma)$, which is equivalent to

$$\{x + (n+1)y\}^2 = (n + 1 + \gamma^{-1})\{x^2 + (n+1)y^2\}. \tag{9}$$

Since this particular case of the relation is of weight 2, it follows as before that the required general relation is of the form (8), where the constant $C = n + 1 + \gamma^{-1}$, exhibited explicitly in (9), is determined by considering the same special case.

For the proof of (6) let us take for granted, for the moment, the fact that the required general relation is algebraic in $\cot \alpha_1$. Then, being symmetric, it must be expressible as a polynomial in the elementary symmetric functions $p_1, \ldots, p_{n+2}$ of the quantities $\cot \alpha_i$. Since, in the limit as the radius of the spherical n-space tends to infinity, it reduces to (3), it must be of maximum weight 2, and it is therefore dependent only on p_1 and p_2 or, equivalently, on $\sum \cot \alpha_i$ and $\sum \cot^2 \alpha_i$. Finally, the special case $\alpha_1 = \xi$, $\alpha_2 = \alpha_3 = \ldots = \alpha_{n+2} = \eta$ yields a regular spherical simplex of side

$$\sigma = \arccos(\cos^2 \eta + \gamma \sin^2 \eta) \tag{10}$$

and of circumradius

$$\rho = \arccos(\cos \xi \cos \eta + \gamma \sin \xi \sin \eta) \tag{11}$$

where (an immediate deduction from (**3**, p. 158)) we have

$$\sin^2(\tfrac{1}{2}\sigma)/\sin^2 \rho = (n+1)/(2n). \tag{12}$$

Eliminating ρ and σ from (10), (11), (12) we have

$$\{\cot \xi + (n+1)\cot \eta\}^2 = (n + 1 + \gamma^{-1})\{1 - \gamma + \cot^2 \xi + (n+1)\cot^2 \eta\} \tag{13}$$

whence the general result (6) follows exactly as before.

It only remains to prove that the required result is in fact algebraic in $\cot \alpha_1$ and for this we suppose all but the first sphere fixed. In E_{n+1} with origin at the centre of the spherical n-space S_n, let the axes of the $n+2$ spheres S_{n-1} have direction cosines l_{ik}, where $i = 1, \ldots, n+2$ and $k = 1, \ldots, n+1$. Then for the $n+2$ unknowns $l_{11}, l_{12}, \ldots, l_{1,n+1}$ and α_1 we have the $n+1$ equations linear in l_{1k}, $\cos\alpha_1$ and $\sin\alpha_1$:

$$\sum_{k=1}^{n+1} l_{1k}l_{jk} = \cos\alpha_1\cos\alpha_j + \gamma\sin\alpha_1\sin\alpha_j \qquad (j = 2, \ldots, n+2) \tag{14}$$

together with the quadratic equation

$$\sum_{k=1}^{n+1} l_{1k}^2 = \cos^2\alpha_1 + \sin^2\alpha_1. \tag{15}$$

Solving (14) for l_{1k}, substituting in (15) and dividing by $\sin^2\alpha_1$ yields an algebraic (indeed, quadratic) equation in $\cot\alpha_1$, as required.

5. An elementary proof of Theorem 2 and the corresponding result in spherical space. Take any three of the spheres S_{n-1} and effect a real inversion so that one becomes a hyperplane A and the other two become spheres B, C with positive radii. If $\gamma > 1$, the smaller of B, C would lie wholly within the larger.

Since γ is the same for each of the pairs (A, B), (A, C), it follows that the distances from A of B and C must be proportional to their radii, so that B cannot lie wholly within C nor C within B. Combining these results, we have $\gamma < 1$.

The Schwarz inequality applied to (3), and an *a fortiori* argument applied to (6), yield $\gamma^{-1} \leqslant 1$ and hence $\gamma < 0$.

Finally, the result $n + 1 + \gamma^{-1} \geqslant 0$ is now trivial, and this completes the proof.

6. An algebraic lemma. Although the methods of §§4 and 5 are interesting, they are not powerful enough to prove the converse theorems. For these we shall need the lemma below, which can be used to yield the forward theorems as well.

LEMMA. *If* $\mathbf{M}$ *is the symmetric matrix of order* $N \geqslant 3$ *whose elements* m_{ij}, *for* $i, j = 1, 2, \ldots, N$, *are defined by*

$$m_{ij} = \begin{cases} 1, & i = j, \\ \cos\alpha_i\cos\alpha_j + \gamma\sin\alpha_i\sin\alpha_j, & i \neq j, \end{cases} \tag{16}$$

where $\gamma \neq 1$, *then the rank of* $\mathbf{M}$ *is at least* $N - 1$ *and, in the case when* $\mathbf{M}$ *is real and singular, its characteristic roots are all non-negative if and only if*

$$\gamma \leqslant -(N-1)^{-1}. \tag{17}$$

Proof. Write

$$m_{ii} = \cos^2\alpha_i + \gamma\sin^2\alpha_i + k\sin^2\alpha_i, \qquad i = 1, 2, \ldots, N, \tag{18}$$

https://doi.org/10.4153/CJM-1962-042-6 Published online by Cambridge University Press

where $k = 1 - \gamma$, and expand $|\mathbf{M}|$ in powers of k. The coefficient of k^{N-r} is a sum of terms each of which contains a factor which can be written as the determinant of the matrix product $\mathbf{K'UK}$, where $\mathbf{K'}$ is the transpose of $\mathbf{K}$,

$$\text{(19)} \qquad \mathbf{U} = \begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix} \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} \cos\alpha_{i_1} \ldots \cos\alpha_{i_r} \\ \sin\alpha_{i_1} \ldots \sin\alpha_{i_r} \end{bmatrix}$$

so that the rank of $\mathbf{K'UK}$ does not exceed 2. Hence the coefficient of k^{N-r} is zero unless $r = 0$ or 1 or 2. Taking out these three cases separately, we find that the condition for singularity can be written

$$\text{(20)} \quad k^2 + k \sum_{i=1}^{N} \operatorname{cosec}^2 \alpha_i \,(\cos^2 \alpha_i + \gamma \sin^2 \alpha_i)$$
$$+ \gamma \sum_{i<j}\sum \operatorname{cosec}^2 \alpha_i \operatorname{cosec}^2 \alpha_j \sin^2 (\alpha_i - \alpha_j) = 0,$$

where a factor $k^{N-2}\Pi\sin^2\alpha_i$ has been removed from the determinant. The substitution $k = 1 - \gamma$ yields

$$\text{(21)} \qquad \left\{ \sum_{i=1}^{N} \cot\alpha_i \right\}^2 = (N - 1 + \gamma^{-1})\left\{1 - \gamma + \sum_{i=1}^{N} \cot^2 \alpha_i \right\},$$

which is thus the condition that the rank of the matrix $\mathbf{M}$ should not exceed $N - 1$. Now, if the rank of $\mathbf{M}$ were $N - 2$ or less, we should have not only $|\mathbf{M}| = 0$, implying (21), but also the cofactor of each diagonal element m_{ii} would be zero, implying the validity of (21) with N replaced by $N - 1$ and any of the α_i (say α_1) deleted. An immediate consequence would be

$$\text{(22)} \qquad (N - 1 + \gamma^{-1}) \cot\alpha_1 = \sum_{i=1}^{N} \cot\alpha_i,$$

and similarly $\cot\alpha_i = \cot\alpha_1$ (all i), leading to a contradiction unless $\gamma = 1$, which is excluded by hypothesis. Hence the rank of $\mathbf{M}$ is at least $N - 1$.

We now turn attention to the case when $\mathbf{M}$ is real and we notice that (21), regarded as a locus in the N-dimensional space of real cartesian co-ordinates $x_i = \cot\alpha_i$, represents a central quadric variety Q. Define the "outside" U of Q as that component of the complement of Q which contains the points $\alpha_i = \alpha$ for $i = 1, 2, \ldots, N$, where α is real and small, but not necessarily positive. At such a point the characteristic roots of $\mathbf{M}$ are an $(N - 1)$-fold root $= \sin^2\alpha(1 - \gamma)$ and a simple root nearly equal to N. These roots are all positive if $\gamma < 1$, while if $\gamma > 1$ at least two of them are negative, and by continuity the same results hold throughout U. Since Q is the boundary of U and since only one of the characteristic roots of $\mathbf{M}$ vanishes on Q, it follows that the characteristic roots of $\mathbf{M}$ are all non-negative on Q if and only if $\gamma < 1$. For real non-singular $\mathbf{M}$ the argument of the last five lines of § 5 shows that this condition is equivalent to (17) as required.

7. Proof of Theorem 4 with its converse. Just as for (14), let the axes of the N spheres S_{n-1} have direction cosines l_{ik}, where $i = 1, \ldots, N$ and

https://doi.org/10.4153/CJM-1962-042-6 Published online by Cambridge University Press

$k = 1, \ldots, n+1$. Then the resulting equations like (14) and (15) can be written in matrix form

$$\mathbf{L}'\mathbf{L} = \mathbf{M}, \tag{23}$$

where $\mathbf{L}$ is the $(n+1) \times N$ matrix whose elements are l_{ki}, $\mathbf{L}'$ is its transpose, and $\mathbf{M}$ is defined by (16). From the lemma of § 6 and (23) it follows that

$$N - 1 \leqslant \operatorname{rk} \mathbf{M} \leqslant \operatorname{rk} \mathbf{L} \leqslant n + 1 \tag{24}$$

and hence $N \leqslant n+2$, with equality only if (21) holds.

The reduction of $\mathbf{M}$ to diagonal form by a real orthogonal transformation shows that, if $\mathbf{M}$ is real, then the real representation (23) is possible if and only if $\operatorname{rk} \mathbf{M} \leqslant n+1$ and $\mathbf{M}$ has no negative characteristic root. In the case $N = n+2$ these conditions are equivalent to (21) and (17), which are identical with (6) and (4).

8. Proofs of Theorems 1, 2, 3. The question of the existence and sign of the r_i in Theorem 3 is readily settled by writing $r_i^{-1} = u_i$ and considering the intersection of the hyperplane $\sum u_i = (n+1+\gamma^{-1})^{\frac{1}{2}}$ with the sphere $\sum u_i^2 = 1$. The remaining theorems may be obtained from Theorem 4 by letting the radius of the spherical n-space tend to infinity and noticing that the characteristic roots of $\mathbf{M}$ are continuous functions of the elements of $\mathbf{M}$, so that Theorem 4 has a certain "stability."

9. The degenerate case $\gamma = 1$. In this case, we observe that the matrix $\mathbf{M}$ of § 6 can be expressed in the form $\mathbf{K}'\mathbf{K}$, where $\mathbf{K}$ is defined in (19) and $r = N$, so that $\operatorname{rk} \mathbf{M} \leqslant 2$. It then follows from (23) that $\operatorname{rk} \mathbf{L} \leqslant 2$, so that the axes of the spheres S_{n-1} in Theorem 4 all lie in the same (ordinary two-dimensional) plane, and hence the spheres S_{n-1} all touch at the same point, and their number and radii are arbitrary. The same applies to the spheres S_{n-1} of Theorem 1 and it is for this reason that the case $\gamma = 1$ may be described as degenerate.

10. Further results. The submaximal case $N < n+2$ can be dealt with rather easily by the methods of §§ 6 and 7. Of more interest is the fact that Theorem 4 can be immediately transferred from spherical space to hyperbolic space.* Using canonical coordinates (**4**, p. 281) we see that the inclination γ of two spheres is defined by the equation corresponding to (2):

$$\cosh \phi = \cosh \alpha_1 \cosh \alpha_2 - \gamma \sinh \alpha_1 \sinh \alpha_2 \tag{25}$$

and the result corresponding to (6) is

$$\left\{ \sum_{i=1}^{n+2} \coth \alpha_i \right\}^2 = (n+1+\gamma^{-1}) \left\{ \sum_{i=1}^{n+2} \coth^2 \alpha_i + \gamma - 1 \right\}. \tag{26}$$

*Suggested by the referee.

https://doi.org/10.4153/CJM-1962-042-6 Published online by Cambridge University Press

If we consider* two equidistant surfaces S_{n-1} whose "radii" (constant distance from the axis) are β_1 and β_2, where ϕ is the angle between their axes, then the formula for their inclination γ is

$$\cos\phi = \gamma\cosh\beta_1\cosh\beta_2 - \sinh\beta_1\sinh\beta_2 \tag{27}$$

and the result corresponding to (26) is obtained by writing $\tanh\beta_i$ instead of $\coth\alpha_i$.

Finally, still in hyperbolic n-space for a mixed system consisting of spheres, horospheres and equidistant surfaces, all of the type S_{n-1} and mutually inclined with the same inclination γ, we have the relation

$$\left\{\sum_{i=1}^{n+2}\kappa_i\right\}^2 = (n+1+\gamma^{-1})\left\{\sum_{i=1}^{n+2}\kappa_i^2+\gamma-1\right\}, \tag{28}$$

where κ_i is defined as $\coth\alpha_i$ for a sphere, $\tanh\beta_i$ for an equidistant surface, and unity for a horosphere.

A study of Poincaré's conformal model (**2**, p. 302) of hyperbolic space in Euclidean space shows that here also, for the reality of such a system, the condition (4), $\gamma \leqslant -(n+1)^{-1}$, is necessary and sufficient. This result may also, of course, be obtained by the methods of §§ 6 and 7.

CORRECTION

In the present paper, appears the erroneous assertion that in spherical n-space no set of $n+3$ oriented spheres (with radii r_j) can have the same non-zero mutual inclination γ. This is false if and only if

$$\gamma = -(n+2)^{-1}, \quad \sum\cot r_j = 0 \quad \text{and} \quad \sum\cot^2 r_j = -(n+3)/(n+2).$$

It is true for sets of $n+4$ spheres. Similar corrections apply to elliptic, hyperbolic and Euclidean space.

References

1. H. S. M. Coxeter, *Interlocked rings of spheres*, Scripta Mathematica, *18* (1952), 113–121.
2. ——— *An introduction to geometry* (New York, 1961).
3. ——— *Regular polytopes* (London, 1948).
4. ——— *Non-Euclidean geometry* (Toronto, 1957).
5. T. Gosset, *The kiss precise*, Nature, *139* (1937), 62.
6. J. G. Mauldon, *Equally inclined spheres*, Proc. Camb. Phil. Soc. (to appear).
7. F. Soddy, *The kiss precise*, Nature, *137* (1936), 1021.

University of California at Berkeley
Corpus Christi College, Oxford

*In this paper we distinguish between the two branches of the usual concept (**2**, p. 303) of "equidistant surface," regarding them as two distinct equidistant surfaces with oppositely oriented axes and with radii of opposite sign. The inclination of two such equidistant surfaces (cf. the first paragraph of § 2) is then given by $\gamma = -1$. Changing the orientation of the axis *or* the sign of the radius (but not both) yields the region obtained by reflecting Poincaré's half-plane model in its boundary.

https://doi.org/10.4153/CJM-1962-042-6 Published online by Cambridge University Press