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Integral Sets and the Center of a Finite Group

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Abstract. We give a description of the atoms in the Boolean algebra generated by the integral subsets of a finite group.

1 Integral Sets

Suppose G is a finite group; let $\widehat{G}$ be the set of characters of representations of G over the complex numbers.

For any subset $A \subseteq G$ and any $\chi \in \widehat{G}$, let $\chi(A) = \sum_{a \in A} \chi(a)$. We call A *integral* if $\chi(A) \in \mathbb{Z}$ for every $\chi \in \widehat{G}$; it suffices to check integrality of a set using only the irreducible characters.

Let $\mathbb{B}(\mathcal{I}_G)$ denote the Boolean algebra *generated* by the integral sets; let $\mathcal{P}(G)$ denote the power set of G.

For any $a \in G$ the set $[a] = \{b \mid \langle b \rangle = \langle a \rangle\}$, that is, the set of generators of the cyclic subgroup generated by a, is an integral subset of G. For any subgroup H of the finite group G, we have $\mathbb{B}(\mathcal{I}_H) \subseteq \mathbb{B}(\mathcal{I}_G)$ [1].

Proposition 1.1 *Every non-central element of the finite group G is an atom in $\mathbb{B}(\mathcal{I}_G)$.*

Proof Consider a non-central element a, of order m, of the finite group G. We know that the set $[a]$ is integral. However, these elements break up into disjoint subsets according to their conjugacy class. It is easy to see that the conjugacy classes of these elements have the same number of elements, which is divisor of $\phi(m)$. To see this, say a is conjugate to a^i via b; now suppose a^j is in a different conjugacy class, then a^{ij} also belongs to that class via conjugation by b.

Suppose that $[a]$ is distributed across r conjugacy classes, each containing s elements, $rs = \phi(m)$. Thus $\chi([a]) = s \sum_{j=1}^{r} \chi(a^{i_j})$, where the a^{i_j} are in distinct classes. Thus $\sum_{j=1}^{r} \chi(a^{i_j})$ is a rational number that is also an algebraic integer, and so is integral also.

Suppose now that for the given element $[a]$, the number r of its distinct classes is $2 \leq r < \phi(m)$. Since we are considering the non-central element a, then all of the conjugacy classes of a^{i_j}, $j = 1, \dots, r$ (let $i_1 = 1$) have at least two elements. Let us consider a second element different from a^{i_j}, denoted b_j. Then the sets

$$A = \{a^{i_j}, j = 1, \dots, r\} \quad \text{and} \quad B = \{a, b_2, \dots, b_r\}$$

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are both integral. Hence the singleton $\{a\} = A \cap B$ is an atom in $\mathbb{B}(\mathcal{I}_G)$.

If $[a]$ belongs to only one conjugacy class, then $\chi([a]) = \phi(m)\chi(a)$, so also $\chi(a)$ is an integer.

Hence every non-central element is an atom of $\mathbb{B}(\mathcal{I}_G)$. ■

Corollary 1.2 *Let D be the center of the finite group G. Then $\mathcal{P}(G - D) \subseteq \mathbb{B}(\mathcal{I}_G)$.*

2 Boolean Algebra of Integral Sets

Let $\mathbb{B}(\mathcal{F}_G)$ be the Boolean algebra generated by $\mathcal{F}_G$, the family of subgroups of the finite group G. The sets $[a]$ are the atoms for $\mathbb{B}(\mathcal{F}_G)$ [1]. Thus for any finite group G, $\mathbb{B}(\mathcal{F}_G) \subseteq \mathbb{B}(\mathcal{I}_G) \subseteq \mathcal{P}(G)$, and the atoms of $\mathbb{B}(\mathcal{I}_G)$ are contained in the atoms of $\mathbb{B}(\mathcal{F}_G)$.

We proved in [1] that $\mathbb{B}(\mathcal{F}_G) = \mathbb{B}(\mathcal{I}_G)$ if and only if G is abelian. In Corollary 2.3 we reprove one direction of this result without appealing to Dedekind's Theorem [2] on Hamiltonian groups as in [1].

Theorem 2.1 *Let D be the center of the finite group G. Then $\mathbb{B}(\mathcal{I}_G)$ is the (internal) direct product of the Boolean algebras $\mathbb{B}(\mathcal{F}_D)$ and the power set $\mathcal{P}(G-D)$ of non-central elements.*

Proof By Corollary 1.2, the Boolean algebra $\mathcal{P}(G - D)$ is contained in $\mathbb{B}(\mathcal{I}_G)$.

Say $[G{:}D] = N$; then for any character χ on D, let $\operatorname{Ind}(\chi)$ denote the induced character to G, then $\operatorname{Ind}(\chi)(x) = N\chi(x)$ for all $x \in D$.

Suppose S is an integral subset of G. For every character χ of D with induced character $\operatorname{Ind}(\chi)$ to G, $\operatorname{Ind}(\chi)(S)$ is integral.

But for any $x \in G$, the value of $\operatorname{Ind}(\chi)(x) = \sum \chi(r^{-1}xr)$, where r ranges over those elements in a set of left coset representatives of D in G for which $r^{-1}xr \in D$. But for the center D, $r^{-1}xr \in D$ if and only if $x \in D$. So $\operatorname{Ind}(\chi)(x) = 0$ if $x \notin D$.

Let $S' = S \cap D$. Thus $\operatorname{Ind}(\chi)(S) = \operatorname{Ind}(\chi)(S') = N\chi(S')$, so $\chi(S')$ is (rational and hence) integral for every character on D. Thus S' is a union of atoms for $\mathbb{B}(\mathcal{F}_D)$ by [1].

Thus S is a disjoint union of atoms in $\mathbb{B}(\mathcal{F}_D)$ and atoms in $\mathcal{P}(G - D)$. ■

The next result follows immediately from Theorem 2.1.

Corollary 2.2 *$\mathbb{B}(\mathcal{I}_G) = \mathcal{P}(G)$ if and only if the center of G is an elementary abelian 2-group.*

Corollary 2.3 *If $\mathbb{B}(\mathcal{F}_G) = \mathbb{B}(\mathcal{I}_G)$, then G is abelian.*

Proof Any non-central element must be of order 2 by Theorem 2.1. With this assumption the sets $[a]$ are integral atoms. However by Theorem 2.1 it now follows that all elements of order bigger than 2 must be central.

It now follows immediately that G is abelian. Suppose by way of contradiction that x, y are non-central elements and $xy \neq yx$. If xy is central, then $x(xy) = y$ also commute with x, a contradiction; if xy is non-central, then $xy = (xy)^{-1} = y^{-1}x^{-1} = yx$, a contradiction. ■

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