Canad. Math. Bull Vol. 62 (1), 2019 pp. 179–182
http://dx.doi.org/10.4153/CMB-2017-065-7

On a Property of Real Plane Curves of Even Degree

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Abstract. F. Cukierman asked whether or not for every smooth real plane curve $X \subset \mathbb{P}^2$ of even degree $d \geqslant 2$ there exists a real line $L \subset \mathbb{P}^2$ such $X \cap L$ has no real points. We show that the answer is yes if $d = 2$ or 4 and no if $n \geqslant 6$.

1 Introduction

F. Cukierman asked whether or not for every smooth real plane curve $X \subset \mathbb{P}^2$ there exists a real line $L \subset \mathbb{P}^2$ such that the intersections $X \cap L$ has no real points. In other words, can we see all real points of X in some affine space of the form $\mathbb{A}^2 = \mathbb{P}^2 \smallsetminus L$?

Note that if d is odd, then the answer is no for trivial reasons: $X \cap L$ is cut out by an odd degree polynomial on L, and hence, always has a real point. On the other hand, in the case where $d = 2$, the answer is readily seen to be yes. Indeed, given a real conic X in $\mathbb{P}^2$, choose a complex point $z \in X(\mathbb{C}) \smallsetminus X(\mathbb{R})$ that is not real and let L be the (real) line passing through z and its complex conjugate $\overline{z}$. If X is smooth, then L is not contained in X. Hence, the intersection $(X \cap L)(\mathbb{C}) = \{z, \overline{z}\}$ contains no real points.

The main result of this note, Theorem 1.1, asserts that the answer to Cukierman's question is yes if $d = 2$ or 4 and no if $n \geqslant 6$.

Theorem 1.1 (i) *Suppose $d = 2$ or 4. Then for every smooth plane curve $X \subset \mathbb{P}^2$ of degree d defined over the reals, there exists a real line $L \subset \mathbb{P}^2$ such that $(X \cap L)(\mathbb{R}) = \varnothing$.*

(ii) *Suppose $d \geqslant 6$ is an even integer. Then there exists a smooth plane curve $X \subset \mathbb{P}^2$ of degree d defined over the reals, such that $(X \cap L)(\mathbb{R}) \neq \varnothing$ for every real line $L \subset \mathbb{P}^2$.*

The proof of Theorem 1.1 presented in Sections 3 and 4 uses deformation arguments. These arguments, in turn, rely on the preliminary material in Section 2.

2 Continuity of Minimizer and Maximizer Functions

Lemma 2.1 *Let V, W, and F be topological manifolds. Assume that F is compact, $\pi: V \to W$ is an F-fibration, and $f: V \to \mathbb{R}$ is a continuous function. Then the minimizer $\mu(w) := \min\{f(v) \mid \pi(v) = w\}$ and the maximizer $\nu(w) := \max\{f(v) \mid \pi(v) = w\}$ are continuous functions $W \to \mathbb{R}$.*

Received by the editors June 15, 2017; revised October 3, 2017.
Published electronically November 2, 2017.
The author was partially supported by NSERC Discovery Grant 253424-2017.
AMS subject classification: **14P05**, 14H50.
Keywords: real algebraic geometry, plane curve, maximizer function, bitangent.

Proof Since F is compact, f assumes its minimal and maximal values on every fiber $\pi^{-1}(w)$. Hence, the functions μ and ν are well defined. Note also that if we replace f by $-f$, we will change $\mu(w)$ to $-\nu(w)$. Thus, it suffices to show that μ is continuous. Finally, to show that μ is continuous at $w \in W$, we can replace W by a small neighborhood of w and thus assume that $V = W \times F$ and $\pi\colon V \to W$ is projection to the first factor. In this special case, the continuity of μ is well known; see, *e.g.*, [Wo] (*cf.* also [Da]). ∎

Corollary 2.2 *Let $d \geqslant 2$ be an even integer, let Pol_d be the affine space of homogeneous polynomials of even degree d in 3 variables, and let $\check{\mathbb{P}}^2$ be the dual projective plane parametrizing the lines in $\mathbb{P}^2$. Then the functions*

$$m_p(L) \quad \text{and} \quad M_p(L)\colon \mathrm{Pol}_d(\mathbb{R}) \times \check{\mathbb{P}}^2(\mathbb{R}) \longrightarrow \mathbb{R}$$

given by $m_p(L) = \min\{p(x) \mid x \in L(\mathbb{R})\}$ and $M_p(L) = \max\{p(x) \mid x \in L(\mathbb{R})\}$ are well defined and continuous.

Note that a polynomial $p(x, y, z)$ of even degree d gives rise to a continuous function $\mathbb{P}^2(\mathbb{R}) \to \mathbb{R}$ given by

$$(x : y : z) \longrightarrow \frac{p(x, y, z)}{(x^2 + y^2 + z^2)^{d/2}}. \tag{2.1}$$

By a slight abuse of notation, we will continue to denote this function by p.

Proof of Corollary 2.2 We will apply Lemma 2.1 in the following setting. Let

$$W := \mathrm{Pol}_d \times \check{\mathbb{P}}^2 \quad \text{and} \quad V := \{(p, L, a) \mid a \in L\} \subset \mathrm{Pol}_d \times \check{\mathbb{P}}^2 \times \mathbb{P}^2.$$

In other words, $V = \mathrm{Pol}_d \times \mathrm{Flag}(1, 2)$, where $\mathrm{Flag}(1, 2)$ is the flag variety of $(1, 2)$-flags in a 3-dimensional vector space. Clearly V and W are smooth algebraic varieties defined over $\mathbb{R}$. Their sets of real points, $V(\mathbb{R})$ and $W(\mathbb{R})$, are topological manifolds and the projection $\pi\colon V(\mathbb{R}) \to W(\mathbb{R})$ to the first two components is a topological fibration with compact fiber $F = \mathbb{P}^1(\mathbb{R})$.

Applying Lemma 2.1 to the continuous function $f\colon V(\mathbb{R}) \to \mathbb{R}$ given by $f(p, L, a) := p(a)$, where $p(a)$ is evaluated as in (2.1), we deduce the continuity of the real-valued functions $m_p(L) = \mu(p, L)$ and $M_p(L) = \nu(p, L)$ on $\mathrm{Pol}_d(\mathbb{R}) \times \check{\mathbb{P}}^2(\mathbb{R})$. ∎

Proposition 2.3 *Let $p \in \mathbb{R}[x, y, z]$ be a homogeneous polynomial of even degree and $X \subset \mathbb{P}^2$ be the zero locus of p. Set*

$$m(p) := \max_{L \in \check{\mathbb{P}}^2} m_p(L) \quad \text{and} \quad M(p) := \min_{L \in \check{\mathbb{P}}^2} M_p(L),$$

where L ranges over the real lines in $\mathbb{P}^2$.

(i) *$m(p)$ and $M(p)$ are well defined continuous functions $\mathrm{Pol}_d(\mathbb{R}) \to \mathbb{R}$;*
(ii) *$m(p) \leqslant M(p)$;*
(iii) *$(X \cap L)(\mathbb{R}) \neq \varnothing$ for every real line $L \subset \mathbb{P}^2$ if and only if $m(p) \leqslant 0 \leqslant M(p)$;*
(iv) *p assumes both positive and negative values on each real line $L \subset \mathbb{P}^2$ if and only if $m(p) < 0 < M(p)$;*
(v) *If $m(p) = M(p) = 0$, then X is not a smooth curve.*

https://doi.org/10.4153/CMB-2017-065-7 Published online by Cambridge University Press

Proof By Corollary 2.2, $M_p(L)$ and $m_p(L)$ are continuous functions $\mathrm{Pol}_d(\mathbb{R}) \times \check{\mathbb{P}}^2(\mathbb{R}) \to \mathbb{R}$. Since $\check{\mathbb{P}}^2(\mathbb{R})$ is compact, Lemma 2.1 tells us that the functions $m(p)$ and $M(p)\colon \mathrm{Pol}_d(\mathbb{R}) \to \mathbb{R}$ are well defined and continuous. This proves (i).

(iii) and (iv) are immediate consequences of the definition of $m(p)$ and $M(p)$.

To prove (ii) and (v), choose lines $L_1, L_2 \subset \mathbb{P}^2$ such that $m_p(L)$ attains its maximal value $m(p)$ at $L = L_1$ and $M_p(L)$ attains its minimal value $M(p)$ at $L = L_2$. If L_1 and L_2 intersect at a point $a \in \mathbb{P}^2(\mathbb{R})$, then

$$m(p) = m_p(L_1) \leqslant p(a) \leqslant M_p(L_2) = M(p). \tag{2.2}$$

This proves (ii).

In part (v), where we further assume that $m(p) = M(p) = 0$, the inequalities (2.2) tell us that $p(a) = 0$ is the maximal value of p on $L_1(\mathbb{R})$ and the minimal value of p on $L_2(\mathbb{R})$. Hence, a lies on X, and both L_1 and L_2 are tangent to X at a. We want to show that X cannot be a smooth curve. Assume the contrary. Then X has a unique tangent line at a. Thus, $L_1 = L_2$, and $0 = m_p(L_1) = M_p(L_2) = M_p(L_1)$. We conclude that p is identically zero on $L_1(\mathbb{R}) = L_2(\mathbb{R})$. Consequently, $L_1 = L_2 \subset X$, contradicting our assumption that X is a smooth curve. ■

3 Proof of Theorem 1.1(i)

The case where $d = 2$ was handled in the introduction; we will thus assume that $d = 4$.

Lemma 3.1 *Let $p \in \mathbb{R}[x, y, z]$ be a homogeneous polynomial of degree* 4 *cutting out a smooth quartic curve X in $\mathbb{P}^2$. Then either $m(p) \geqslant 0$ or $M(p) \leqslant 0$.*

Proof By a theorem of H. G. Zeuthen [Zeu], X has a real bitangent line $L \subset \mathbb{P}^2$. (For a modern proof of Zeuthen's theorem, we refer the reader to [Ru, Corollary 4.11]; *cf.* also [PSV].) The restriction of $p(x, y, z)$ to L is a real quartic polynomial with two double roots, *i.e.,* a polynomial of the form $\pm q(s,t)^2$, where s and t are linear coordinates on L, and $q \in \mathbb{R}[s, t]$ is a binary form of degree 2. In particular, p does not change sign on L, *i.e.*, either (i) $p(a) \geqslant 0$ for every $a \in L(\mathbb{R})$ or (ii) $p(a) \leqslant 0$ for every $a \in L(\mathbb{R})$. In case (i), $m(p) \geqslant m_p(L) \geqslant 0$ and in case (ii), $M(p) \leqslant M_p(L) \leqslant 0$. ■

We are now ready to finish the proof of Theorem 1.1(i) for $d = 4$. The geometric idea is to move a bitangent line off the quartic curve. To turn this idea into a proof, we argue by contradiction. Assume the contrary: there exists a smooth real quartic curve $X \subset \mathbb{P}^2$ such that $(X \cap L)(\mathbb{R}) \neq \varnothing$ for every real line $L \subset \mathbb{P}^2$. Let $p \in \mathbb{R}[x, y, z]$ be a defining polynomial for X. By Proposition 2.3(iii), $m(p) \leqslant 0 \leqslant M(p)$. In view of Lemma 3.1, after possibly replacing p by $-p$, we can assume that $m(p) = 0$. Proposition 2.3(v) now tells us that $m(p) = 0 < M(p)$. Let

$$p_t(x, y, z) = p(x, y, z) - t(x^2 + y^2 + z^2)^2,$$

where t is a real parameter, and let $X_t \subset \mathbb{P}^2$ be the quartic curve cut out by p_t. Note that X_t can be singular for only finitely many values of $t \in \mathbb{R}$. Thus, we can choose $t \in (0, M(p))$ so that X_t is smooth. Since $x^2 + y^2 + z^2$ is identically 1 on $\mathbb{P}^2(\mathbb{R})$ (*cf.* (2.1)), we have

$$m(p_t) = m(p) - t < 0 < M(p) - t = M(p_t).$$

This contradicts Lemma 3.1, which asserts that $m(p_t) \geqslant 0$ or $M(p_t) \leqslant 0$. ∎

4 Proof of Theorem 1.1(ii)

Given an even integer $d \geqslant 6$, set $p(x,y,z) := (x^3+y^3+z^3)^2(x^2+y^2+z^2)^{(d-6)/2}$ and

$$p_t(x,y,z) = p(x,y,z) - t(x^d+y^d+z^d),$$

where t is a real parameter. In view of Proposition 2.3(iii), it suffices to show that if $t > 0$ is sufficiently small, then (i) X_t is smooth and (ii) $m(p_t) < 0 < M(p_t)$.

Since the Fermat curve, $x^d + y^d + z^d = 0$, is smooth, X_t is singular for only finitely many values of t, and (i) follows.

To prove (ii), note that p is non-negative but is not identically 0 on any real line $L \subset \mathbb{P}^2$. Thus, $M_p(L) > 0$ and consequently, $M(p) > 0$. By Proposition 2.3(i), $M(p_t) > 0$ for small t. On the other hand, for every real line $L \subset \mathbb{P}^2$, the cubic polynomial $x^3 + y^3 + z^3$ vanishes at some real point a of L. Hence, for every $t > 0$, we have $p_t(a) < 0$ and thus $m_{p_t}(L) < 0$. We conclude that $m(p_t) < 0$, as desired. ∎

Acknowledgments The author is grateful to Fernando Cukierman, Corrado de Concini, Kee Yuen Lam, Grigory Mikhalkin, and Benedict Williams for stimulating discussions and to the anonymous referees for their helpful comments.

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