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Nagoya Math. J.
Vol. 167 (2002), 95–100

ON THE AVERAGE OF CENTRAL VALUES OF SYMMETRIC SQUARE L-FUNCTIONS IN WEIGHT ASPECT

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Abstract. It is proved that the central values of symmetric square L-functions of normalized Hecke eigenforms for the full modular group on average satisfy an analogue of the Lindelöf hypothesis in weight aspect, under the assumption that these values are non-negative.

§1. Introduction

Let S_k be the space of cusp forms of even integral weight $k \geq 12$ with respect to $SL_2(\mathbf{Z})$. According to [8] (cf. also [6]) the central values of Hecke L-functions of normalized Hecke eigenforms in S_k on average satisfy an analogue of the Lindelöf hypothesis when the weight varies.

The purpose of this note is to show a similar result for symmetric square L-functions, under the assumption that their central values are non-negative.

For the proof we use a "kernel function" for the symmetric square L-function as given by Zagier in [10] and then proceed in a similar way as in [5,6,8], exploiting the bounds for Petersson norms implied by the work of Iwaniec [4]. Note that the kernel function of [10] was used in [5] to prove some non-vanishing results for symmetric square L-functions inside the critical strip.

Recall that by the work of Gelbart and Jacquet [3] the symmetric square L-function (up to a variable shift) also is the standard L-function of a cuspidal automorphic form on $GL(3)$. From this point of view it is therefore quite natural and important to study the Lindelöf hypothesis (actually in all aspects) for those L-functions (compare [7]).

Received December 25, 2000.
2000 Mathematics Subject Classification: 11F03.

https://doi.org/10.1017/S0027763000025447 Published online by Cambridge University Press

§2. Statement of result

We let $\mathcal{F}_k$ be the set of normalized Hecke eigenforms in S_k. For $f \in \mathcal{F}_k$ we denote by $D_f(s)\,(s \in \mathbf{C})$ the symmetric square L-function of f defined by analytic continuation of the Euler product

$$\prod_p (1-\alpha_p^2 p^{-s})^{-1}(1-\alpha_p\beta_p p^{-s})^{-1}(1-\beta_p^2 p^{-s})^{-1} \qquad (\mathrm{Re}(s) > k)$$

where α_p, β_p are defined by

$$\alpha_p + \beta_p = a(p),\ \ \alpha_p\beta_p = p^{k-1}$$

and $a(p)$ is the p-th Fourier coefficient of f. Recall that the modified function

$$D_f^*(s) := 2^{-s}\pi^{-3s/2}\Gamma(s)\Gamma\Big(\frac{s-k+2}{2}\Big)D_f(s)$$

is invariant under $s \mapsto 2k-1-s$ [9,10].

According to the generalized Riemann hypothesis, all the zeroes of $D_f^*(s)$ should lie on the critical line $\mathrm{Re}(s) = k - \frac{1}{2}$. In particular, since $D_f(s)$ is real on the real line, one would expect that $D_f(k-\frac{1}{2}) \geq 0$.

Theorem. *Suppose that $D_f(k-\frac{1}{2}) \geq 0$ for all $f \in \mathcal{F}_k$ and all k. Then*

$$\sum_{f\in\mathcal{F}_k} D_f(k-\frac{1}{2}) <<_\epsilon k^{1+\epsilon} \qquad (k \to \infty)$$

for any $\epsilon > 0$ where the implied constant in $<<_\epsilon$ depends only on ϵ.

Proof. Again as in [5], our starting point is Zagier's identity [10]

$$(1)\ \sum_{f\in\mathcal{F}_k} \frac{D_f(s+k-1)}{||f||^2}$$
$$= c_k^{-1} \cdot \frac{(4\pi)^{s+k-1}}{\Gamma(s+k-1)}\Big(\sum_{t\in\mathbf{Z}} (I_k(t^2-4,t;s) + I_k(t^2-4,-t;s))L(s,t^2-4)$$
$$+\frac{(-1)^{k/2}\Gamma(s+k-1)\zeta(2s)}{2^{2s+k-3}\pi^{s-1}\Gamma(k)}\Big) \qquad (2-k < \mathrm{Re}(s) < k-1)$$

where

$$c_k := \frac{(-1)^{k/2}\pi}{2^{k-3}(k-1)}$$

and $||f||$ denotes the usual Petersson norm of f. Furthermore, for t an integer and $\Delta := t^2 - 4$, in the range $2 - k < \mathrm{Re}(s) < k - 1$ we have put

$$I_k(\Delta, t; s) := \begin{cases} \frac{\Gamma(k-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(k)} \int_0^\infty \frac{y^{k+s-2}}{(y^2 + ity - \frac{1}{4}\Delta)^{k-1/2}} dy, & \text{if } \Delta \neq 0 \\ e^{\operatorname{sign} t \cdot \frac{\pi i}{2}(s-k)} \sqrt{\pi} \frac{\Gamma(s-\frac{1}{2})\Gamma(k-s)}{\Gamma(k)} |t|^{s-k}, & \text{if } \Delta = 0. \end{cases}$$

Finally

$$L(s, \Delta) := \begin{cases} \zeta(2s-1), & \text{if } \Delta = 0 \\ L_D(s) \sum_{d|f} \mu(d) \Big(\frac{D}{d}\Big) \sigma_{1-2s}\Big(\frac{f}{d}\Big), & \text{if } \Delta \neq 0 \end{cases}$$

where if $\Delta \neq 0$ we have set $\Delta = Df^2$ with $f \in \mathbf{N}$ and D the discriminant of $\mathbf{Q}(\sqrt{\Delta})$, $L_D(s)$ is the associated Dirichlet L-function and $\sigma_\nu(m) := \sum_{d|m} d^\nu \quad (m \in \mathbf{N}, \nu \in \mathbf{C})$.

We now specialize (1) to the case $s = \frac{1}{2}$. Note that on the right-hand side of (1) the sum of the terms corresponding to $t = \pm 2$ in the sum over t has a simple pole at $s = \frac{1}{2}$ and also the single term outside the sum over t has a simple pole at $s = \frac{1}{2}$. Both poles cancel and a short calculation reveals that these terms altogether give the contribution

$$\lim_{s \to \frac{1}{2}} \Big(2((I_k(0, 2; s) + I_k(0, -2; s))\zeta(2s-1) + \frac{(-1)^{k/2}\Gamma(s+k-1)\zeta(2s)}{2^{2s+k-3}\pi^{s-1}\Gamma(k)} \Big)$$
$$= \frac{(-1)^{k/2}\sqrt{\pi}}{2^{k-2}\Gamma(k)} (C_1 \Gamma'(k - \frac{1}{2}) + C_2 \Gamma(k - \frac{1}{2}))$$

where C_1 and C_2 are absolute constants.

We therefore obtain from (1) that

$$(2) \quad \sum_{f \in \mathcal{F}_k} \frac{D_f(k-\frac{1}{2})}{||f||^2}$$
$$= \frac{(-1)^{k/2}(k-1)2^{3k-3}\pi^{k-3/2}}{\Gamma(k-\frac{1}{2})}$$
$$\times \Big(\sum_{t \geq 1, t \neq 2} (I_k(t^2-4, t; \frac{1}{2}) + I_k(t^2-4, -t; \frac{1}{2}))L(\frac{1}{2}, t^2-4)$$
$$+ I_k(-4, 0; \frac{1}{2})L(\frac{1}{2}, -4) + \frac{(-1)^{k/2}\sqrt{\pi}}{2^{k-2}\Gamma(k)}(C_1 \Gamma'(k-\frac{1}{2}) + C_2 \Gamma(k - \frac{1}{2})) \Big).$$

We shall now estimate the right-hand side of (2) in k-aspect using arguments similar to those used earlier in [5].

First of all, given $\epsilon' > 0$, for any $t \in \mathbf{Z}$, $t \neq \pm 2$ one has

$$L(\frac{1}{2}, t^2 - 4) << _{\epsilon'} |t^2 - 4|^{1/2+\epsilon'} \tag{3}$$

where the constant implied in $<<_{\epsilon'}$ only depends on ϵ' and not on t [2, chap. 12, problem 22 (b)].

Also the inequalities

$$\begin{aligned} I_k(t^2-4, t; \frac{1}{2}) &+ I_k(t^2-4, -t; \frac{1}{2}) \\ &<< \begin{cases} (t^2-4)^{-1/4} \left(\dfrac{t - \sqrt{t^2-4}}{t+\sqrt{t^2-4}}\right)^{\frac{k-1}{2}} \dfrac{\Gamma(k-\frac{1}{2})^2}{2^k \Gamma(k)^2}, & \text{if } t \geq 3 \\ \dfrac{\Gamma(k-\frac{1}{2})^2}{2^k\Gamma(k)^2}, & \text{if } t = 1 \end{cases} \end{aligned} \tag{4}$$

hold where the constants involved in $<<$ are absolute [5, pp. 1644–1645].

Finally

$$I_k(-4, 0; \frac{1}{2}) = \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{k}{2} - \frac{1}{4})^2}{\Gamma(k)} \tag{5}$$

[5, formula (5)].

Choosing ϵ' in (3) small enough, we then see combining (3), (4) and (5) that the right-hand side of (2) is

$$\begin{aligned} &<< \frac{2^{3k}\pi^k (k-1)}{\Gamma(k-\frac{1}{2})} \left(\frac{\Gamma(k-\frac{1}{2})^2}{2^k\Gamma(k)^2} + \frac{\Gamma(\frac{k}{2}-\frac{1}{4})^2}{\Gamma(k)} + \frac{\Gamma(k-\frac{1}{2})}{2^k\Gamma(k)} + \frac{\Gamma'(k-\frac{1}{2})}{2^k\Gamma(k)} \right) \\ &= \frac{(4\pi)^k(k-1)}{\Gamma(k)} \left(\frac{\Gamma(k-\frac{1}{2})}{\Gamma(k)} + \frac{\Gamma(\frac{k}{2}-\frac{1}{4})^2\, 2^k}{\Gamma(k-\frac{1}{2})} + 1 + \frac{\Gamma'(k-\frac{1}{2})}{\Gamma(k-\frac{1}{2})} \right) \end{aligned}$$

where the constant in $<<$ is absolute.

By Legendre's duplication formula one has

$$2^{k-3/2}\Gamma\left(\frac{k}{2} - \frac{1}{4}\right) = \sqrt{\pi}\, \frac{\Gamma(k-\frac{1}{2})}{\Gamma(\frac{k}{2}+\frac{1}{4})}.$$

Also

$$\frac{\Gamma'(k-\frac{1}{2})}{\Gamma(k-\frac{1}{2})} = \log(k - \frac{1}{2}) + \mathcal{O}(\frac{1}{k}) \qquad (k \to \infty)$$

[1, 6.3.18].

Therefore we see that

$$\sum_{f\in\mathcal{F}_k} \frac{D_f(k-\frac{1}{2})}{||f||^2} << \frac{(4\pi)^k(k-1)}{\Gamma(k)}\Big(\frac{\Gamma(k-\frac{1}{2})}{\Gamma(k)} + \frac{\Gamma(\frac{k}{2}-\frac{1}{4})}{\Gamma(\frac{k}{2}+\frac{1}{4})} + \log k\Big). \tag{6}$$

Observe that

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} \sim x^{a-b} \quad (x\to\infty) \tag{7}$$

for any fixed real numbers a and b as follows immediately from Stirling's formula.

Using (7) on the right-hand side of (6) with $a=-\frac{1}{4}$, $b=\frac{1}{4}$ and $x=k-\frac{1}{4}$ resp. $x=\frac{k}{2}$ we therefore finally obtain that

$$\sum_{f\in\mathcal{F}_k} \frac{D_f(k-\frac{1}{2})}{||f||^2} << \frac{(4\pi)^k(k-1)}{\Gamma(k)}(\frac{1}{\sqrt{k}} + \log k). \tag{8}$$

On the other hand, given $\epsilon>0$, from [4] we infer that

$$||f||^2 <<_\epsilon \frac{\Gamma(k)k^\epsilon}{(4\pi)^k} \tag{9}$$

for all $f\in\mathcal{F}_k$ where the implied constant depends only on ϵ and not on k (cf. the comments made in [8] and [6, sect. 3]).

We now use our assumption that $D_f(k-\frac{1}{2})\geq 0$ for all $f\in\mathcal{F}_k$ and all k. Inserting (9) into (8) we then obtain

$$\sum_{f\in\mathcal{F}_k} D_f(k-\frac{1}{2}) <<_\epsilon k^{1+\epsilon} \qquad (k\to\infty)$$

for any given $\epsilon>0$ as claimed.

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