J. Appl. Prob. Spec. Vol. **51A,** 311–332 (2014)

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Part 7. Stochastic geometry

PERCOLATION ON STATIONARY TESSELLATIONS: MODELS, MEAN VALUES, AND SECOND-ORDER STRUCTURE

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APPLIED PROBABILITY TRUST
DECEMBER 2014

PERCOLATION ON STATIONARY TESSELLATIONS: MODELS, MEAN VALUES, AND SECOND-ORDER STRUCTURE

BY GÜNTER LAST AND EVA OCHSENREITHER

Abstract

We consider a stationary face-to-face tessellation X of $\mathbb{R}^d$ and introduce several percolation models by colouring some of the faces black in a consistent way. Our main model is cell percolation, where cells are declared black with probability p and white otherwise. We are interested in geometric properties of the union Z of black faces. Under natural integrability assumptions, we first express asymptotic mean values of intrinsic volumes in terms of Palm expectations associated with the faces. In the second part of the paper we focus on cell percolation on normal tessellations and study asymptotic covariances of intrinsic volumes of $Z \cap W$, where the observation window W is assumed to be a convex body. Special emphasis is given to the planar case where the formulae become more explicit, though we need to assume the existence of suitable asymptotic covariances of the face processes of X. We check these assumptions in the important special case of a Poisson–Voronoi tessellation.

Keywords: Tessellation; percolation; Poisson–Voronoi tessellation; intrinsic volume; the Euler characteristic; asymptotic mean; asymptotic covariance

2010 Mathematics Subject Classification: Primary 60D05
Secondary 60G55; 60K35

1. Introduction

Let X be a *face-to-face tessellation* of $\mathbb{R}^d$, meaning, a random collection of convex and bounded polytopes with nonempty interior (called *cells*) covering the whole space and such that, for any different cells C and $C' \in X$, the intersection $C \cap C'$ is either empty, or a face of both C and C'. Assume that any bounded subset of $\mathbb{R}^d$ is intersected by only finitely many cells. We interpret X as a *particle process*, and assume that X is *stationary*, meaning that the distribution of X coincides with that of $\{C + x : C \in X\}$ for all $x \in \mathbb{R}^d$. For $k \in \{0, \dots, d\}$, let X_k denote the particle process of k-dimensional faces of cells in X, and assume throughout that the intensity measure of X_k is *locally finite*. For more details on stationary tessellations, we refer the reader to [12, Chapter 10] and the next section.

For $p \in [0, 1]$ and $n \in \{0, \dots, d\}$, we define *n-percolation* on X as follows. Given X, we colour the polytopes in X_n independently *black* with probability p. All other polytopes in X_n are *white*. If $n \le d - 1$ and $k \in \{n + 1, \dots, d\}$, then we colour $F \in X_k$ black if all its $(k - 1)$-faces are black. We are interested in the union Z of all black faces of X. This is a stationary random closed set (see [12, Chapter 2]). In the $n = d$ case we refer to this as *cell percolation* and in the $n = 0$ case as *vertex percolation*. In the planar case, $d = 2$, we refer to 1-percolation as *edge percolation*. In the general case we also speak of *face percolation*. As is common, we set $q = 1 - p$ for $0 \le p \le 1$.

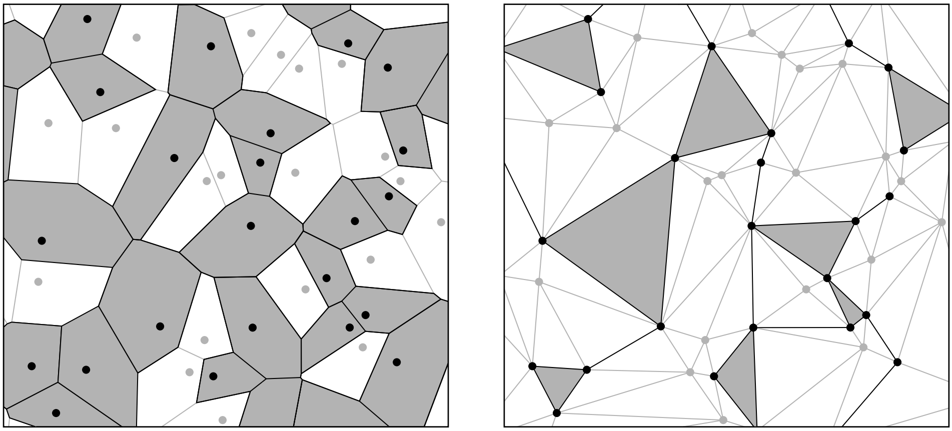

FIGURE 1: Cell percolation on a Poisson–Voronoi tessellation and vertex percolation on a Poisson–Delaunay tessellation.

Cell percolation on a Poisson–Voronoi tessellation (see Figure 1) was studied in [3], where it was shown that the *critical probability* of this model of *continuum percolation* is $\frac{1}{2}$. The present paper was motivated by [10], introducing vertex, edge, and cell percolation on several planar lattices. The authors of [10] noticed that in many models the only nontrivial zero of the mean Euler characteristic is a remarkably accurate approximation of the critical probability.

Our first aim in this paper is to establish n-percolation on X as an interesting model of stochastic geometry and continuum percolation. Our main aim is to study first- and second-order geometric properties of the black phase Z. In Section 2 we collect some preliminaries on stationary tessellations and Palm probabilities, and define face percolation. Asymptotic mean values of intrinsic volumes of $Z \cap W$ are studied in Section 3, where we assume that the *observation window* W is a convex polytope. Asymptotic covariances of intrinsic volumes in the case of cell percolation on a normal tessellation are treated in Section 4 assuming that the asymptotic covariances of intrinsic volumes of face processes exist. Theorem 4.1 shows that these covariances are polynomials in the colouring probability p, and the coefficients are determined by both the global fluctuation of the intrinsic volumes within the face processes and the local geometry of X. The important special case of cell percolation on a planar and *normal* tessellation is discussed in Section 5. In Section 6 we check that a Poisson–Voronoi tessellation satisfies all assumptions required for our general results. All asymptotic covariances are then given by fairly explicit integral formulae. For cell percolation in the planar Poisson–Voronoi case, the asymptotic variance of the Euler characteristic is determined by the intensity and second moment of the number of vertices of a typical cell and has a global maximum at the critical threshold $p = \frac{1}{2}$ (see Corollary 6.1). In Appendix A we give some integrability properties of a Poisson–Voronoi tessellation.

2. Notation and preliminaries

2.1. Palm calculus

It is convenient to follow [8, 9] by assuming the basic sample space $(\Omega, \mathcal{F}, \mathbb{P})$ to be equipped with a *measurable flow* $\theta_x : \Omega \to \Omega,\ x \in \mathbb{R}^d$, i.e. $(\omega, x) \mapsto \theta_x \omega$ is measurable, $\theta_{x+y} = \theta_x \circ \theta_y$ for all $x, y \in \mathbb{R}^d$, and θ_0 is the identity on Ω. Assume also that $\mathbb{P}$ is *stationary*, i.e. $\mathbb{P} \circ \theta_x = \mathbb{P},\ x \in \mathbb{R}^d$.

https://doi.org/10.1239/jap/1417528483 Published online by Cambridge University Press

A *random measure* μ on $\mathbb{R}^d$ is a kernel from Ω to $\mathbb{R}^d$ such that $\mu(\omega, \cdot)$ is locally finite for all $\omega \in \Omega$. If $\mu(\omega, B)$ is integer valued for all bounded Borel sets $B \subset \mathbb{R}^d$ then μ is a *point process*; it is a *simple* point process if $\mu(\{x\}) \le 1$ for all $x \in \mathbb{R}^d$, in which case we identify μ with its support $\{x \in \mathbb{R}^d : \mu(\{x\}) > 0\}$.

A random measure μ is *invariant* if, for any Borel set $B \subset \mathbb{R}^d$,

$$\mu(\theta_x \omega, B - x) = \mu(\omega, B), \qquad x \in \mathbb{R}^d,\ \omega \in \Omega.$$

It then follows that μ is *stationary*, i.e. the distribution of $\mu(\cdot + x)$ is independent of $x \in \mathbb{R}^d$. When μ is invariant, $\gamma_\mu := \mathbb{E}[\mu([0, 1]^d)]$ is the *intensity* of μ. If $0 < \gamma_\mu < \infty$ then the *Palm probability measure* $\mathbb{P}_\mu^0$ of μ is defined by

$$\mathbb{P}_\mu^0(A) := \gamma_\mu^{-1} \iint \mathbf{1}_A(\theta_x \omega) \mathbf{1}\{x \in [0, 1]^d\}\, \mu(\omega, \mathrm{d}x)\, \mathbb{P}(\mathrm{d}\omega), \qquad A \in \mathcal{F}.$$

It satisfies the *refined Campbell theorem*

$$\iint f(\theta_x \omega, x)\, \mu(\mathrm{d}x)\, \mathbb{P}(\mathrm{d}\omega) = \gamma_\mu \iint f(\omega, x)\, \mathrm{d}x\, \mathbb{P}_\mu^0(\mathrm{d}\omega) \tag{2.1}$$

for all measurable $f : \Omega \times \mathbb{R}^d \to [0, \infty)$. This can also be written as

$$\mathbb{E}\biggl[\int f(\theta_x, x)\, \mu(\mathrm{d}x)\biggr] = \gamma_\mu \mathbb{E}_\mu^0\biggl[\int f(\theta_0, x)\, \mathrm{d}x\biggr],$$

where $\mathbb{E}_\mu^0$ denotes the expectation with respect to $\mathbb{P}_\mu^0$.

For ease of reference, we state *Neveu's exchange formula*; it is used frequently in this paper. This formula also goes under the name *mass-transport principle* (see [8, 9] for a brief discussion).

Proposition 2.1. (Neveu's exchange formula.) *Let μ and μ' be invariant random measures on $\mathbb{R}^d$ with positive finite intensities, and let $h : \Omega \times \mathbb{R}^d \to [0, \infty)$ be measurable. Then*

$$\gamma_\mu \mathbb{E}_\mu^0\biggl[\int h(\theta_x, -x)\, \mu'(\mathrm{d}x)\biggr] = \gamma_{\mu'} \mathbb{E}_{\mu'}^0\biggl[\int h(\theta_0, x)\, \mu(\mathrm{d}x)\biggr].$$

2.2. Coloured tessellations and face percolation

We start by introducing some basic terminology for tessellations, referring to [12] for further detail. Let $\mathcal{K}^d$ denote the space of convex bodies (convex and compact subsets of $\mathbb{R}^d$), and equip it with the Borel σ-field associated with the *Hausdorff distance*. A *polytope* is a finite intersection of half-spaces which is bounded and nonempty. The system $\mathcal{P}^d$ of all such polytopes is a measurable subset of $\mathcal{K}^d$. A *tessellation* (of $\mathbb{R}^d$) is a countable system φ of polytopes with nonempty interior (cells) covering the whole space such that any two different elements of φ have disjoint interior and any bounded subset of $\mathbb{R}^d$ is intersected by only finitely many cells. Let $k \in \{0, \dots, d-1\}$. A *k-face* of $C \in \mathcal{P}^d$ is a k-dimensional intersection of C with a *supporting hyperplane* of C. We let $\mathcal{F}_k(C)$ denote the system of all k-faces of C, and define $\mathcal{F}_d(C) := \{C\}$ if C is a full-dimensional polytope and $\mathcal{F}_d(C) := \varnothing$ otherwise. A tessellation φ is *face-to-face* if, for C and $C' \in \varphi$, the intersection $C \cap C'$ is either empty, or a face of both C and C'. Let $\boldsymbol{T}$ denote the set of all face-to-face tessellations. We define the system of k-faces of $\varphi \in \boldsymbol{T}$ by

$$\mathcal{F}_k(\varphi) := \bigcup_{C \in \varphi} \mathcal{F}_k(C)$$

https://doi.org/10.1239/jap/1417528483 Published online by Cambridge University Press

and the system of faces of φ by

$$\mathcal{F}(\varphi) := \bigcup_{k=0}^{d} \mathcal{F}_k(\varphi).$$

Note that $\mathcal{F}_d(\varphi) = \varphi$.

In this paper we define a *coloured tessellation* to be a tuple $\psi = (\varphi, \varphi_0, \dots, \varphi_d)$, where $\varphi \in \boldsymbol{T}$ and $\varphi_k \subset \mathcal{F}_k(\varphi)$ such that $\mathcal{F}_{k-1}(F) \subset \varphi_{k-1}$ whenever $k \geq 1$ and $F \in \varphi_k$. Any face in $\bigcup_{k=0}^{d} \varphi_k$ is called *black*, while the other faces of φ are called *white*. If $F \in \mathcal{F}(\varphi)$ is black then, by definition, all its faces are black as well. Write $X(\psi) := \varphi$ and $X_k^1(\psi) := \varphi_k$.

Let $\boldsymbol{T}_{\mathrm{c}}$ denote the space of all coloured tessellations, and let $\mathcal{T}_{\mathrm{c}}$ denote the smallest σ-field on $\boldsymbol{T}_{\mathrm{c}}$ such that, for all measurable $H \subset \mathcal{P}^d$, the numbers

$$|\{C \in X(\psi) \colon C \in H\}|, \qquad |\{C \in X_0^1(\psi) \colon C \in H\}|, \dots, |\{C \in X_k^1(\psi) \colon C \in H\}|$$

become a measurable function of $\psi \in \boldsymbol{T}_{\mathrm{c}}$. Here and later $|A|$ denotes the cardinality of a set A. The σ-field $\mathcal{T}$ on $\boldsymbol{T}$ is defined similarly.

A *random coloured tessellation* Ψ is a measurable mapping from the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $(\boldsymbol{T}_{\mathrm{c}}, \mathcal{T}_{\mathrm{c}})$. We note that $X(\Psi)$ and $X_0^1(\Psi), \dots, X_d^1(\Psi)$ are *particle processes* in the sense of [12]. The same is true for $\mathcal{F}_0(X(\Psi)), \dots, \mathcal{F}_d(X(\Psi))$. We are interested in the union

$$Z := \bigcup_{k=0}^{d} \bigcup_{F \in X_k^1(\Psi)} F \tag{2.2}$$

of all black faces. It can be shown that Z is a *random closed set* (see [12] for a definition of this concept). We shall always assume that Ψ is stationary, that is,

$$\Psi + x \overset{\mathrm{D}}{=} \Psi, \qquad x \in \mathbb{R}^d, \tag{2.3}$$

where, for $\psi = (\varphi, \varphi_0, \dots, \varphi_d) \in \boldsymbol{T}_{\mathrm{c}}$, $\psi + x := (\varphi + x, \varphi_0 + x, \dots, \varphi_d + x)$, $H + x := \{F + x \colon F \in H\}$ for $H \subset \mathcal{K}^d$, and $A + x := \{y + x \colon y \in A\}$ for $A \subset \mathbb{R}^d$. Then Z is stationary as well, that is,

$$Z + x \overset{\mathrm{D}}{=} Z, \qquad x \in \mathbb{R}^d.$$

We are mainly concerned with what we call n-percolation (or face percolation) on a stationary tessellation. To introduce this concept, assume that we are given a random face-to-face tessellation X, i.e. a random element of the space $\boldsymbol{T}$. Assume that X is stationary, i.e. that the distribution of $X + x$ does not depend on $x \in \mathbb{R}^d$. A coloured tessellation Ψ is an *n-percolation* on X with (percolation) parameter p if $X(\Psi) = X$, and the particle process $X_n^1(\Psi)$ is a *p-thinning* of $\mathcal{F}_n(X)$, that is, given X, the faces in $\mathcal{F}_n(X)$ are included in $X_n^1(\Psi)$ independently of each other with probability p; see [7] for more detail on thinnings. The complete vector $(X_0^1(\Psi), \dots, X_d^1(\Psi))$ of black faces is then defined as follows. For $k < n$, the system $X_k^1(\Psi)$ is the union of all $\mathcal{F}_k(F)$ for $F \in X_n^1(\Psi)$. For $k > n$, the system $X_k^1(\Psi)$ is defined recursively. A polytope $F \in X_k$ belongs to $X_k^1(\Psi)$ if and only if $\mathcal{F}_{k-1}(F) \subset X_{k-1}^1(\Psi)$. In the $n = d$ case we speak of *cell percolation* and in the $n = 0$ case of *vertex percolation*.

Now fix a coloured tessellation Ψ such that

$$\Psi(\theta_x \omega) = \Psi(\omega) - x, \qquad \omega \in \Omega,\ x \in \mathbb{R}^d. \tag{2.4}$$

Then Ψ is stationary in the sense of (2.3). Throughout we use the following shorthand notation for the systems of all (respectively all black) k-faces for $k \in \{0, 1, \dots, d\}$:

$$X_k := \mathcal{F}_k(X), \qquad X_k^1 := X_k^1(\Psi).$$

The invariance assumption (2.4) implies that, for $\omega \in \Omega$ and $x \in \mathbb{R}^d$,

$$(X_k(\theta_x \omega), X_k^1(\theta_x \omega)) = (X_k(\omega) - x, X_k^1(\omega) - x). \tag{2.5}$$

For $k \in \{0, \dots, d\}$, let

$$\eta^{(k)} := \{s(F) \colon F \in X_k\} \tag{2.6}$$

be the point process of *Steiner points* of the faces in $X_k = \mathcal{F}_k(X)$ (see [12] for the definition of the Steiner point $s(K)$ of a nonempty $K \in \mathcal{K}^d$). Since $s(K + x) = s(K) + x$ for all $x \in \mathbb{R}^d$, (2.5) implies that $\eta^{(k)}$ is invariant. By assumption on X, $\eta^{(k)}$ contains infinitely many points, so the intensity $\gamma_k := \gamma_{\eta^{(k)}} = \mathbb{E}[\eta^{(k)}([0, 1]^d)]$ is positive. We assume that $\gamma_k < \infty$, so the Palm probability measure $\mathbb{P}_k^0 := \mathbb{P}_{\eta^{(k)}}^0$ is well defined. The expectation with respect to $\mathbb{P}_k^0$ is denoted by $\mathbb{E}_k^0$. Note that, under $\mathbb{P}_k^0$, the origin is almost surely in the relative interior of some k-dimensional face.

Let $\psi = (\varphi, \varphi_0, \dots, \varphi_d)$ be a coloured tessellation, and let $x \in \mathbb{R}^d$. Since φ is face-to-face, there is unique $F \in \mathcal{F}(\varphi)$ such that x is in the relative interior of F. We then write $F(\psi, x) \equiv F(\varphi, x) = F$. To treat the local neighbourhood of $x \in \mathbb{R}^d$, we introduce the set $\mathcal{S}_l(\psi, x) \equiv \mathcal{S}_l(\varphi, x)$ for $l \in \{0, \dots, d\}$ as follows. Let k be the dimension of $F(\psi, x)$. If $l \geq k$ (respectively $l < k$) then we let $\mathcal{S}_l(\psi, x)$ be the set of all faces $G \in \mathcal{F}_l(\varphi)$ such that $F(\psi, x) \subset G$ (respectively $G \subset F(\psi, x)$). It is convenient to abbreviate

$$(F(x), \mathcal{S}_l(x)) := (F(\Psi, x), \mathcal{S}_l(\Psi, x)), \qquad x \in \mathbb{R}^d. \tag{2.7}$$

Hence, $F(x)$ is the face of X containing x in its relative interior, while $\mathcal{S}_l(x)$ is the system of all l-faces containing $F(x)$ or contained in $F(x)$, respectively. Taking into account the translation covariance of $F(\cdot, \cdot)$, it follows from (2.4) that

$$\mathbb{P}_k^0(F(0) \in \cdot) = \gamma_k^{-1} \mathbb{E} \int \mathbf{1}\{x \in [0, 1]^d,\ F(x) - x \in \cdot\}\, \eta^{(k)}(\mathrm{d}x)$$

is the distribution of a *typical k-face*. The next result is a version of [12, Theorem 10.1.1]. The proof can easily be given with Neveu's exchange formula (see also [2]).

Proposition 2.2. *Let $k, l \in \{0, \dots, d\}$, and let $g \colon \mathcal{P}^d \times \mathcal{P}^d \to [0, \infty)$ be a measurable function. Then*

$$\gamma_k \mathbb{E}_k^0 \sum_{G \in \mathcal{S}_l(0)} g(F(0), G - s(G)) = \gamma_l \mathbb{E}_l^0 \sum_{F \in \mathcal{S}_k(0)} g(F - s(F), F(0)). \tag{2.8}$$

3. Mean value analysis

Let X be a stationary face-to-face tessellation, i.e. a random element of $\boldsymbol{T}$. For $k \in \{0, \dots, d\}$, let $X_k = \mathcal{F}_k(X)$ denote the particle process of k-faces of X. We assume that

$$\sum_{k=0}^{d} \mathbb{E} \sum_{F \in X_k} \mathbf{1}\{F \cap K \neq \varnothing\} < \infty, \qquad K \in \mathcal{K}^d, \tag{3.1}$$

a common assumption in stochastic geometry [12]. It is easy to see that (3.1) implies that $\gamma_k := \mathbb{E}\eta^{(k)}([0, 1]^d) < \infty$, where the point process $\eta^{(k)}$ is defined by (2.6). The refined

Campbell theorem (2.1) allows us to rewrite (3.1) as

$$\sum_{k=0}^{d} \mathbb{E}_k^0 V_d(F(0)+K) < \infty, \qquad K \in \mathcal{K}^d, \tag{3.2}$$

where V_d denotes the volume and where we have used the fact that $(A+x)\cap B \neq \varnothing$ for $A, B \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$ if and only if $x \in B - A := \{y - z\colon y \in A,\ z \in B\}$. Recall that $F(x) \in \mathcal{F}(X)$ is the unique face that contains $x \in \mathbb{R}^d$ in its relative interior. Often we have to assume that

$$\sum_{i,k=0}^{d} \mathbb{E}_k^0[[V_i(F(0))]^2] < \infty, \tag{3.3}$$

where V_i denotes the ith intrinsic volume. Note that (3.2) is a consequence of (3.3), the Steiner formula (see [12]), and the Cauchy–Schwarz inequality.

For $n \in \{0, \dots, d\}$, we consider n-percolation Ψ on X. It is no restriction of generality to assume that (2.4) holds. Let the stationary random closed set Z be given by (2.2). The *density* of the ith intrinsic volume of Z is defined by the limit

$$\delta_i(p) := \lim_{t\to\infty} V_d(W_t)^{-1}\mathbb{E}[V_i(Z\cap W_t)], \tag{3.4}$$

where $W_t := t^{1/d}W$ and $W \in \mathcal{P}^d$ is assumed to have volume one and to contain the origin in its interior. For cell percolation, $\delta_d(p) = p$ is the volume fraction of Z. We shall show below that the limits (3.4) exist and do not depend on W.

Our first aim in this paper is to derive a polynomial formula for these densities. That this formula is based on the joint distribution of $(V_i(F(0)), |\mathcal{S}_n(0)|)$ under the measures $\mathbb{P}_k^0$ should be no surprise (see (2.7) for notation).

Theorem 3.1. *For n-percolation on X satisfying (3.3), the limit for $\delta_i(p)$ in (3.4) exists for any $i \in \{0, \dots, d\}$ and $p \in [0,1]$, and equals*

$$\sum_{k=i}^{n-1}(-1)^{i+k}\gamma_k\mathbb{E}_k^0[(1-q^{|\mathcal{S}_n(0)|})V_i(F(0))] + \sum_{k=n}^{d}(-1)^{i+k}\gamma_k\mathbb{E}_k^0[p^{|\mathcal{S}_n(0)|}V_i(F(0))]. \tag{3.5}$$

In particular, for cell percolation and $i < d$,

$$\delta_i(p) = \sum_{k=i}^{d}(-1)^{i+k+1}\gamma_k\mathbb{E}_k^0[q^{|\mathcal{S}_d(0)|}V_i(F(0))].$$

Before proving Theorem 3.1 we give some geometric preliminaries. Intrinsic volumes can be defined for convex bodies by the Steiner formula. By additivity, they can then be extended to finite unions of convex bodies (see, e.g. [12]). Groemer [4] defined intrinsic volumes for a much wider class of *approximable sets* containing the relative interior of convex bodies and the intersection of a relative open polytope with the boundary of a convex body, such that they are still additive and rigid motion invariant. In particular, denoting the relative interior of a set B by $\mathrm{relint}(B)$,

$$V_i(\mathrm{relint}(K)) = (-1)^{i+\dim(K)}V_i(K), \qquad K \in \mathcal{K}^d. \tag{3.6}$$

Let $\mathrm{int}(B)$ and ∂B denote the interior and the boundary of B. We can write $Z \cap W_t$ as a disjoint union

$$\begin{aligned} Z \cap W_t &= (Z \cap \mathrm{int}(W_t)) \cup (Z \cap \partial W_t) \\ &= \bigcup_{k=0}^{d} \bigcup_{F \in X_k^1} (\mathrm{relint}(F) \cap \mathrm{int}(W_t)) \ \cup \ \bigcup_{k=0}^{d} \bigcup_{F \in X_k^1} (\mathrm{relint}(F) \cap \partial W_t). \end{aligned}$$

Since the tessellation X is stationary, the intersection of a k-face F with W_t is almost surely (a.s.) empty if $\mathrm{relint}(F) \cap \mathrm{int}(W_t) = \varnothing$. Thus, $\mathrm{relint}(F) \cap \mathrm{int}(W_t) = \mathrm{relint}(F \cap W_t)$ a.s. It follows that both $Z \cap \mathrm{int}(W_t)$ and $Z \cap \partial W_t$ are approximable. The additivity of the intrinsic volumes and (3.6) a.s. yield

$$V_i(Z \cap \mathrm{int}(W_t)) = \sum_{k=0}^{d} \sum_{F \in X_k^1} V_i(\mathrm{relint}(F \cap W_t)) = \sum_{k=0}^{d} (-1)^{i+k} \sum_{F \in X_k^1} V_i(F \cap W_t) \tag{3.7}$$

because $\dim(F \cap W_t) = k$ a.s. for $F \in X_k$ if the intersection is nonempty. Since the observation window W is a polytope, we can partition ∂W in the relative interior of the lower-dimensional faces of W and obtain

$$\begin{aligned} V_i(Z \cap \partial W_t) &= \sum_{k=0}^{d} \sum_{l=0}^{d-1} \sum_{U \in \mathcal{F}_l(W)} \sum_{F \in X_k^1} V_i(\mathrm{relint}(F) \cap \mathrm{relint}(U_t)) \\ &= \sum_{k=0}^{d} \sum_{l=0}^{d-1} \sum_{U \in \mathcal{F}_l(W)} \sum_{F \in X_k^1} V_i(\mathrm{relint}(F \cap U_t)), \end{aligned}$$

where, much as before, $U_t := t^{1/d} U$ denotes the scaled face and the last equation holds a.s. because of the stationarity of the tessellation, and because the intersection of F and U_t is a.s. empty if $\mathrm{relint}(F) \cap \mathrm{relint}(U_t) = \varnothing$. Using (3.6), it follows that

$$V_i(Z \cap \partial W_t) = \sum_{k=0}^{d} \sum_{l=0}^{d-1} \sum_{U \in \mathcal{F}_l(W)} \sum_{F \in X_k^1} (-1)^{i+\dim(F \cap U_t)} V_i(F \cap U_t) \quad \text{a.s.} \tag{3.8}$$

Proof of Theorem 3.1. First, we show that

$$\lim_{t \to \infty} t^{-1} \mathbb{E}[V_i(Z \cap \partial W_t)] = 0, \tag{3.9}$$

for which, because of representation (3.8), it is enough to show that

$$\lim_{t \to \infty} t^{-1} \mathbb{E}\Bigg[\sum_{F \in X_k^1} (-1)^{i+\dim(F \cap U_t)} V_i(F \cap U_t) \Bigg] = 0$$

for $k \in \{0, \dots, d\}$ and $U \in \mathcal{F}(W)$ with $\dim(U) < d$. From n-percolation, by definition,

$$\begin{aligned} &\mathbb{E}\Bigg[\sum_{F \in X_k^1} (-1)^{i+\dim(F \cap U_t)} V_i(F \cap U_t) \Bigg] \\ &\quad = \sum_{r=1}^{\infty} (\mathbf{1}\{k < n\}(1 - q^r) + \mathbf{1}\{k \ge n\} p^r) \\ &\qquad \times \mathbb{E} \int (-1)^{i+\dim(F(x) \cap U_t)} V_i(F(x) \cap U_t) \mathbf{1}\{|\mathcal{S}_n(x)| = r\} \, \eta^{(k)}(\mathrm{d}x). \end{aligned}$$

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Using the monotonicity of intrinsic volumes, we obtain

$$\begin{aligned}\bigg|\mathbb{E}\bigg[\sum_{F\in X_k^1}(-1)^{i+\dim(F\cap U_t)}V_i(F\cap U_t)\bigg]\bigg| &\le \mathbb{E}\int V_i(F(x)\cap U_t)\,\eta^{(k)}(\mathrm{d}x)\\ &\le \mathbb{E}\int V_i(F(x))\,\mathbf{1}\{F(x)\cap\partial W_t\ne\varnothing\}\,\eta^{(k)}(\mathrm{d}x)\\ &= \gamma_k\mathbb{E}_k^0\int V_i(F(0))\,\mathbf{1}\{(F(0)+x)\cap\partial W_t\ne\varnothing\}\,\mathrm{d}x,\end{aligned}$$

where the refined Campbell theorem (2.1) justifies the final identity. We claim that

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\int V_i(F(x))\,\mathbf{1}\{F(x)\cap\partial W_t\ne\varnothing\}\,\eta^{(k)}(\mathrm{d}x)=0,\qquad k\in\{0,\dots,d\}. \tag{3.10}$$

Indeed, $\lambda_d(\partial W-t^{-1/d}K)\to\lambda_d(\partial W)=0$ as $t\to\infty$ for any $K\in\mathcal{K}^d$, where λ_d denotes the Lebesgue measure on $\mathbb{R}^d$; see the proof of Theorem 4.1.3 of [12]. Moreover, as in that proof, $\lambda_d(\partial W-t^{-1/d}K)\le c\lambda_d(B^d+K)$ for all $t\ge 1$ and all convex bodies K, where B^d is the unit ball and $c>0$ does not depend on K. Hence, (3.10) follows from the Steiner formula, the Cauchy–Schwarz inequality, our assumption (3.3), and dominated convergence. In particular, (3.9) holds.

Now we treat the main terms (3.7). The definition of n-percolation implies that

$$\begin{aligned}\mathbb{E}V_i(Z\cap\operatorname{int}(W_t)) &= \sum_{k=0}^{d}\sum_{r=1}^{\infty}(\mathbf{1}\{k<n\}(1-q^r)+\mathbf{1}\{k\ge n\}p^r)\\ &\qquad\times\mathbb{E}\int V_i(F(x)\cap W_t)\,\mathbf{1}\{|\mathcal{S}_n(x)|=r\}\,\eta^{(k)}(\mathrm{d}x).\end{aligned}$$

Since, for $k\in\{0,\dots,d\}$ and $x\in\eta^{(k)}$,

$$|V_i(F(x)\cap W_t)-V_i(F(x))\,\mathbf{1}\{x\in W_t\}|\le V_i(F(x))\,\mathbf{1}\{F(x)\cap\partial W_t\ne\varnothing\},$$

(3.10) implies that

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}\mathbb{E}V_i(Z\cap\operatorname{int}(W_t))\\ &\quad=\lim_{t\to\infty}\frac{1}{t}\sum_{k=0}^{d}\sum_{r=1}^{\infty}(\mathbf{1}\{k<n\}(1-q^r)+\mathbf{1}\{k\ge n\}p^r)\\ &\qquad\qquad\times\mathbb{E}\int V_i(F(x))\,\mathbf{1}\{x\in W_t,\ |\mathcal{S}_n(x)|=r\}\,\eta^{(k)}(\mathrm{d}x).\end{aligned} \tag{3.11}$$

But the refined Campbell theorem (2.1) yields

$$\frac{1}{t}\mathbb{E}\int V_i(F(x))\,\mathbf{1}\{x\in W_t,\ |\mathcal{S}_n(x)|=r\}\,\eta^{(k)}(\mathrm{d}x)=\gamma_k\mathbb{E}_k^0[V_i(F(0))\mathbf{1}\{|\mathcal{S}_n(0)|=r\}].$$

Combining (3.11) with (3.9) yields assertion (3.5).

For cell percolation, (3.5) equals

$$\delta_i(p)=\sum_{k=i}^{d}(-1)^{i+k}\,\gamma_k\,\mathbb{E}_k^0[(1-q^{|\mathcal{S}_d(0)|})V_i(F(0))].$$

Applying Theorem 10.1.4 of [12] gives the second assertion for $i<d$.

The tessellation X is *normal* if, for $0\le k\le d$, any k-face is a.s. contained in $d-k+1$ cells. In this case we have the duality relation below.

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Proposition 3.1. *For cell percolation on a normal tessellation X satisfying (3.3) and $p \in [0, 1]$,*

$$\delta_i(p) = (-1)^{d+i+1}\delta_i(1-p), \qquad i \in \{0, \dots, d-1\}.$$

Proof. To make the dependence on the colouring probability $p \in [0, 1]$ explicit, write Z_p instead of Z. The very definition of cell percolation yields

$$\overline{Z_p^{\mathrm{c}}} \stackrel{\mathrm{D}}{=} Z_{1-p}, \tag{3.12}$$

where B^{c} and $\overline{B}$ denote the complement and the closure of a set $B \subset \mathbb{R}^d$, respectively. Define the set of all white k-faces by $X_k^0 := X_k \setminus X_k^1$. The additivity of intrinsic volumes and (3.6) a.s. yield

$$V_i(Z_p^{\mathrm{c}} \cap \mathrm{int}(W_t)) = \sum_{k=0}^{d} \sum_{F \in X_k^0} V_i(\mathrm{relint}(F) \cap \mathrm{int}(W_t)) = \sum_{k=0}^{d} \sum_{F \in X_k^0} (-1)^{i+k} V_i(F \cap W_t),$$

because we have, a.s., $F \cap W_t = \varnothing$ if $\mathrm{relint}(F) \cap \mathrm{int}(W_t) = \varnothing$ and $\dim(F \cap W_t) = \dim(F)$ if $F \cap W_t \neq \varnothing$. Since X is normal, it follows from the inclusion–exclusion principle that

$$V_i(Z_p^{\mathrm{c}} \cap \mathrm{int}(W_t)) = (-1)^{d+i} V_i(\overline{Z_p^{\mathrm{c}}} \cap W_t).$$

Since $V_i(Z_p^{\mathrm{c}} \cap \mathrm{int}(W_t)) + V_i(Z_p \cap \mathrm{int}(W_t)) = V_i(\mathrm{int}(W_t))$, we obtain

$$V_i(\overline{Z_p^{\mathrm{c}}} \cap W_t) = t^{i/d} V_i(W) + (-1)^{d+i+1} V_i(Z_p \cap \mathrm{int}(W_t)),$$

where we have also used the homogeneity of intrinsic volumes. Combining this with (3.12) and using (3.9) yields the assertion.

Combining Proposition 3.1 with Theorem 3.1 we obtain the following result.

Proposition 3.2. *For cell percolation on a normal tessellation X satisfying (3.3) and $i \in \{0, \dots, d\}$,*

$$\delta_i(p) = \sum_{k=i}^{d} (-1)^{d-k} p^{d-k+1} \gamma_k \mathbb{E}_k^0[V_i(F(0))].$$

In the plane we can use the equation $\gamma_1 = \gamma_0 + \gamma_2$ (see [12]) to rewrite (3.5) in the important special case $i = 0$. We restrict attention to cell percolation, leaving the cases of edge and vertex percolation to the reader. These results generalize [10, Section 2].

Corollary 3.1. *For cell percolation on X in $\mathbb{R}^2$,*

$$\delta_0(p) = \gamma_0 q^2 - \gamma_2 pq - \gamma_0 \sum_{m=3}^{\infty} \mathbb{P}_0^0\{|\mathcal{S}_2(0)| = m\} q^m.$$

4. Second-order properties of cell percolation

In this section we consider cell percolation Ψ on a normal tessellation X. We are interested in the limits

$$\sigma_{i,j}(p) := \lim_{t\to\infty} V_d(W_t)^{-1} \operatorname{cov}(V_i(Z \cap W_t),\ V_j(Z \cap W_t)) \tag{4.1}$$

for $i, j \in \{0, \dots, d\}$, where $W_t := t^{1/d} W$ and $W \in \mathcal{K}^d$ is a fixed convex body of unit volume and which contains the origin in its interior. Note that this definition depends on W.

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Our aim is to establish a set of assumptions guaranteeing that these asymptotic covariances exist. It is not hard to see that the result must be a polynomial in the percolation parameter p. The coefficients, however, are complicated, and are determined by the global fluctuation of the intrinsic volumes within the face processes $X_0, \dots, X_d$ as well as by the local geometry of X, which is independent of W.

We need to assume the existence of the limits

$$\rho_{i,j}^{k,l} := \lim_{t\to\infty} V_d(W_t)^{-1} \operatorname{cov}\Big(\int V_i(F(x)\cap W_t)\,\eta^{(k)}(\mathrm{d}x), \int V_j(F(x)\cap W_t)\,\eta^{(l)}(\mathrm{d}x)\Big) \tag{4.2}$$

for all $i, j, k, l \in \{0, \dots, d\}$. Again, these limits depend on W. To describe the extended local neighbourhood of a point $x \in \mathbb{R}^d$, we take $l \in \{0, \dots, d\}$ and $m \in \{0, \dots, d+1\}$, and define $\mathcal{S}_l^m(x)$ to be the system of all l-dimensional faces sharing m neighbouring cells with the face $F(x)$, that is,

$$\mathcal{S}_l^m(x) := \{G \in X_l \colon |\{C \in X \colon G \subset C,\ F(x) \subset C\}| = m\}.$$

Assume that

$$\sum_{i,j,k=0}^{d} \gamma_d \mathbb{E}_d^0[V_i(\mathcal{S}_k(0))^2 V_j(F(0))] < \infty, \tag{4.3}$$

where, for any finite $\mathcal{S} \subset \mathcal{K}^d$, $V_i(\mathcal{S}) := \sum_{G\in\mathcal{S}} V_i(G)$ is the total ith intrinsic volume of the members of $\mathcal{S}$. For $j = 0$, (4.3) implies (3.3) because (2.8) yields, for $i, k \in \{0, \dots, d\}$,

$$\gamma_k \mathbb{E}_k^0[V_i(F(0))^2] \le \gamma_k \mathbb{E}_k^0[V_i(F(0))^2 V_0(\mathcal{S}_d(0))] \le \gamma_d \mathbb{E}_d^0[V_i(\mathcal{S}_k(0))^2 V_0(F(0))].$$

Theorem 4.1. *For cell percolation Ψ on a normal tessellation X satisfying (4.3) and for which the limits (4.2) exist, for $i, j \in \{0, \dots, d\}$, the limits (4.1) exist, and $\sigma_{i,j}(p)$ equals*

$$\sum_{k=i}^{d}\sum_{l=j}^{d} (-1)^{k+l} p^{2d-k-l+2}\Big(\rho_{i,j}^{k,l} + \sum_{m=1}^{d+1-\max(k,l)} (p^{-m}-1)\gamma_k\, \mathbb{E}_k^0[V_i(F(0))\, V_j(\mathcal{S}_l^m(0))]\Big).$$

Proof. By normality and the inclusion–exclusion principle, we have, a.s.,

$$V_i(Z\cap W_t) = \sum_{k=0}^{d} (-1)^{d-k} \sum_{F\in X_k'} V_i(F\cap W_t), \tag{4.4}$$

where X_k' denotes all $F \in X_k$ that are intersections of $d-k+1$ black cells. The definition of cell percolation gives

$$\mathbb{E} V_i(Z\cap W_t) = \sum_{k=0}^{d} (-1)^{d-k} p^{d-k+1} \mathbb{E} \sum_{F\in X_k} V_i(F\cap W_t). \tag{4.5}$$

As (4.3) implies (3.3), all the expectations in (4.5) are finite. Equation (4.4) yields

$$V_i(Z\cap W_t)V_j(Z\cap W_t) = \sum_{k,l=0}^{d} (-1)^{k+l} \iint V_i(F(x)\cap W_t)V_j(F(y)\cap W_t) \times \mathbf{1}\{F(x)\in X_k',\ F(y)\in X_l'\}\,\eta^{(l)}(\mathrm{d}y)\,\eta^{(k)}(\mathrm{d}x).$$

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Decomposing the inner integration according to $F(y) \in \mathcal{S}_l^m(x)$ and using the definition of cell percolation, we obtain

$$\begin{aligned}
&\mathbb{E}[V_i(Z \cap W_t) V_j(Z \cap W_t)] \\
&\quad = \sum_{k,l=0}^{d} (-1)^{k+l} \sum_{m=0}^{d+1-\max(k,l)} p^{2d-k-l-m+2} \\
&\qquad \times \mathbb{E} \iint V_i(F(x) \cap W_t)\, \mathbf{1}\{F(y) \in \mathcal{S}_l^m(x)\} V_j(F(y) \cap W_t)\, \eta^{(l)}(\mathrm{d}y)\, \eta^{(k)}(\mathrm{d}x).
\end{aligned}$$

Combining this last relation with (4.5) gives

$$\begin{aligned}
&\operatorname{cov}(V_i(Z \cap W_t), V_j(Z \cap W_t)) \\
&\quad = \sum_{k,l=0}^{d} (-1)^{k+l} p^{2d-k-l+2} \\
&\qquad \times \Bigg[\Bigg(\mathbb{E} \iint V_i(F(x) \cap W_t) \mathbf{1}\{F(y) \in \mathcal{S}_l^0(x)\} V_j(F(y) \cap W_t)\, \eta^{(l)}(\mathrm{d}y)\, \eta^{(k)}(\mathrm{d}x) \\
&\qquad\qquad - \mathbb{E}\Bigg[\int V_i(F(x) \cap W_t)\, \eta^{(k)}(\mathrm{d}x) \Bigg] \mathbb{E}\Bigg[\int V_j(F(y) \cap W_t)\, \eta^{(l)}(\mathrm{d}y) \Bigg] \Bigg) \\
&\qquad\qquad + \sum_{m=1}^{d+1-\max(k,l)} p^{-m} \mathbb{E} \iint V_i(F(x) \cap W_t) \mathbf{1}\{F(y) \in \mathcal{S}_l^m(x)\} V_j(F(y) \cap W_t) \\
&\qquad\qquad\qquad \times \eta^{(l)}(\mathrm{d}y)\, \eta^{(k)}(\mathrm{d}x) \Bigg].
\end{aligned}$$

Since $\mathbf{1}\{F(y) \in \mathcal{S}_l^0(x)\} = 1 - \sum_{m=1}^{d+1-\max(k,l)} \mathbf{1}\{F(y) \in \mathcal{S}_l^m(x)\}$, we obtain

$$\begin{aligned}
&\operatorname{cov}(V_i(Z \cap W_t), V_j(Z \cap W_t)) \\
&\quad = \sum_{k,l=0}^{d} (-1)^{k+l} p^{2d-k-l+2} \\
&\qquad \times \Bigg[\operatorname{cov}\Bigg(\int V_i(F(x) \cap W_t)\, \eta^{(k)}(\mathrm{d}x), \int V_j(F(x) \cap W_t)\, \eta^{(l)}(\mathrm{d}x) \Bigg) \\
&\qquad\qquad + \sum_{m=1}^{d+1-\max(k,l)} (p^{-m} - 1)\, \mathbb{E} \iint V_i(F(x) \cap W_t)\, \mathbf{1}\{F(y) \in \mathcal{S}_l^m(x)\} \\
&\qquad\qquad\qquad \times V_j(F(y) \cap W_t)\, \eta^{(l)}(\mathrm{d}y)\, \eta^{(k)}(\mathrm{d}x) \Bigg].
\end{aligned}$$

By assumption (4.2), the sum of the covariance terms on the right-hand side divided by t converges to $\rho_{i,j}^{k,l}$. Using the refined Campbell theorem, we observe that, for $m \geq 1$,

$$\begin{aligned}
&\frac{1}{t} \Bigg| \mathbb{E} \iint V_i(F(x) \cap W_t)\, \mathbf{1}\{F(y) \in \mathcal{S}_l^m(x)\}\, V_j(F(y) \cap W_t)\, \eta^{(l)}(\mathrm{d}y)\, \eta^{(k)}(\mathrm{d}x) \\
&\quad - \mathbb{E} \iint \mathbf{1}\{x \in W_t\}\, V_i(F(x))\, V_j(\mathcal{S}_l^m(x))\, \eta^{(k)}(\mathrm{d}x) \Bigg| \\
&\qquad \leq \frac{1}{t} \mathbb{E} \int \mathbf{1}\{\partial W_t \cap F(z) \neq \varnothing\}\, V_i(\mathcal{S}_k(z))\, V_j(\mathcal{S}_l(z))\, \eta(\mathrm{d}z) \\
&\qquad \leq \gamma_d \mathbb{E}_d^0[\lambda_d(\partial W - t^{-1/d} F(0))\, V_i(\mathcal{S}_k(0))\, V_j(\mathcal{S}_l(0))].
\end{aligned}$$

By the dominated convergence theorem, this tends to 0 as $t \to \infty$. Indeed, for $t \geq 1$,

$$\lambda_d(W - F(0))\, V_i(\mathcal{S}_k(0)) V_j(\mathcal{S}_l(0))$$

is a dominating random variable whose integrability is deduced from the Steiner formula, the Cauchy–Schwarz inequality, and assumption (4.3).

5. On the covariance structure in the plane

In this section we consider cell percolation on a planar and normal tessellation X satisfying (4.3) and for which the limits (4.2) exist. Let $f_0(P)$ denote the number of vertices of a polygon $P \subset \mathbb{R}^2$. For a typical cell $F(0)$, $\mathbb{E}_2^0 f_0(F(0)) = 6$ (see [12, Theorem 10.1.6]). For the second moment

$$\mu_2 := \mathbb{E}_2^0[f_0(F(0))^2],$$

Jensen's inequality gives

$$\mu_2 \geq 36. \tag{5.1}$$

In the case of a Poisson–Voronoi tessellation, numerical integration of exact integral expressions gives $\mu_2 \approx 37.78$ (see [5]), so then $\operatorname{var} f_0(F(0)) \approx (1.33)^2$.

The following result expresses the asymptotic covariance structure in terms of second-order properties of the typical cell and the typical edge. We assume that

$$\lim_{t\to\infty} t^{-1} \operatorname{var}(\varepsilon_t) = 0, \tag{5.2}$$

where, for $t > 0$, $\varepsilon_t := \sum_{F\in X_1} V_0(F \cap \partial W_t)$. Recall the definitions of $\rho_{1,1}^{2,2}$, $\rho_{1,0}^{2,2}$, and $\rho_{0,0}^{2,2}$ at (4.2).

Theorem 5.1. *For a planar normal tessellation X satisfying (4.3) and (5.2), suppose that the limits (4.2) exist. Then the asymptotic covariance structure is given by*

$$\begin{aligned}
\sigma_{2,2}(p) &= pq\gamma_2\mathbb{E}_2^0[V_2(F(0))^2],\\
\sigma_{1,2}(p) &= pq(q-p)\gamma_2\mathbb{E}_2^0[V_2(F(0))V_1(F(0))],\\
\sigma_{0,2}(p) &= pq - p^2q^2\gamma_2\mathbb{E}_2^0[V_2(F(0))\,f_0(F(0))],\\
\sigma_{1,1}(p) &= p^2q^2(\rho_{1,1}^{2,2} + \gamma_1\mathbb{E}_1^0[V_1(F(0))^2]) + pq(q-p)^2\gamma_2\mathbb{E}_2^0[V_1(F(0))^2],\\
\sigma_{0,1}(p) &= p^2q^2(q-p)(\rho_{1,0}^{2,2} - \gamma_2\mathbb{E}_2^0[V_1(F(0))\,f_0(F(0))])\\
&\qquad + pq(q-p)(1+pq)\gamma_2\mathbb{E}_2^0[V_1(F(0))],\\
\sigma_{0,0}(p) &= \gamma_2\mu_2p^3q^3 + \gamma_2pq(1+11pq+10p^2q^2) + \rho_{0,0}^{2,2}p^2q^2(q-p)^2.
\end{aligned}$$

Proof. The formulae for $\sigma_{2,2}$, $\sigma_{1,2}$, and $\sigma_{0,2}$ follow directly from Theorem 4.1 by using $\gamma_2\,\mathbb{E}_2^0[V_2(F(0))] = 1$.

For $\sigma_{1,1}$, first recall that, for any convex body $K \subset \mathbb{R}^2$, $V_1(K) = \frac{1}{2}\mathcal{H}^1(\partial K)$ if K has nonempty interior; otherwise, $V_1(K) = \mathcal{H}^1(K)$, where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure on $\mathbb{R}^2$. It follows that

$$\int V_1(F(x)\cap W_t)\,\eta^{(1)}(\mathrm{d}x) = \int V_1(F(x)\cap W_t)\,\eta^{(2)}(\mathrm{d}x) - \frac{1}{2}\mathcal{H}^1(\partial W_t),$$

and, therefore, $\rho_{1,1}^{1,1} = \rho_{1,1}^{1,2} = \rho_{1,1}^{2,2}$. Proposition 2.2 yields, via a straightforward calculation,

$$\gamma_1 \mathbb{E}_1^0[V_1(F(0))\, V_1(\mathcal{S}_1^1(0))] = 4\gamma_2 \mathbb{E}_2^0[V_1(F(0))^2] - 2\gamma_1 \mathbb{E}_1^0[V_1(F(0))^2],$$
$$\gamma_1 \mathbb{E}_1^0[V_1(F(0))\, V_1(\mathcal{S}_2^1(0))] = 2\gamma_2 \mathbb{E}_2^0[V_1(F(0))^2].$$

Inserting the above formula into Theorem 4.1 yields the asserted formula for $\sigma_{1,1}$ after a simple calculation.

For the remaining covariances $\sigma_{0,1}$ and $\sigma_{0,0}$, consider Euler's formula; it yields

$$(|X_{0,t}| + \varepsilon_t) + (|X_{2,t}| + 1) = (|X_{1,t}| + \varepsilon_t) + 2,$$

where $X_{k,t}$ denotes the set of all k-faces that have nonempty intersection with W_t. Furthermore, by normality we have $2(|X_{1,t}|+\varepsilon_t) = 3(|X_{0,t}|+\varepsilon_t)$. Combining this with the previous equation yields

$$|X_{0,t}| = 2|X_{2,t}| - \varepsilon_t - 2, \qquad |X_{1,t}| = 3|X_{2,t}| - \varepsilon_t - 3. \tag{5.3}$$

With these two relations we determine $\sigma_{0,1}$ and $\sigma_{0,0}$ as follows. Using (5.3) and assumption (5.2), we obtain

$$\rho_{1,0}^{1,0} = \rho_{1,0}^{2,0} = 2\rho_{1,0}^{2,2}, \qquad \rho_{1,0}^{1,1} = \rho_{1,0}^{2,1} = 3\rho_{1,0}^{2,2}, \qquad \rho_{1,0}^{1,2} = \rho_{1,0}^{2,2},$$

while with Proposition 2.2 we obtain

$$\gamma_1 \mathbb{E}_1^0[V_1(F(0))\, V_0(\mathcal{S}_0^1(0))] = 2\gamma_2 \mathbb{E}_2^0[V_1(F(0))\, f_0(F(0))] - 4\gamma_2 \mathbb{E}_2^0[V_1(F(0))],$$
$$\gamma_1 \mathbb{E}_1^0[V_1(F(0))\, V_0(\mathcal{S}_1^1(0))] = 2\gamma_2 \mathbb{E}_2^0[V_1(F(0))\, f_0(F(0))] - 2\gamma_2 \mathbb{E}_2^0[V_1(F(0))],$$
$$\gamma_1 \mathbb{E}_1^0[V_1(F(0))] = \gamma_2 \mathbb{E}_2^0[V_1(F(0))].$$

Together with Theorem 4.1, these observations yield the asserted formula for $\sigma_{0,1}$.

Next, we determine $\rho_{0,0}^{k,l}$. Again, with (5.3) and assumption (5.2), we obtain

$$\rho_{0,0}^{0,0} = 4\rho_{0,0}^{2,2}, \qquad \rho_{0,0}^{0,1} = 6\rho_{0,0}^{2,2}, \qquad \rho_{0,0}^{0,2} = 2\rho_{0,0}^{2,2}, \qquad \rho_{0,0}^{1,1} = 9\rho_{0,0}^{2,2}, \qquad \rho_{0,0}^{1,2} = 3\rho_{0,0}^{2,2}.$$

To treat the second summand of $\sigma_{0,0}$, let $f(k,l,m) := \gamma_k \mathbb{E}_k^0|\mathcal{S}_l^m(0)|$ for $k, l \in \{0,1,2\}$ and $m \in \{1, \dots, 3 - \max(k,l)\}$. Using Proposition 2.2 together with normality yields

$$f(0,0,1) = \gamma_0 \mathbb{E}_0^0\Big[\sum_{G \in \mathcal{S}_2(0)} f_0(G) - 9\Big] = \gamma_2\mu_2 - 9\gamma_0,$$
$$f(1,0,1) = \gamma_1 \mathbb{E}_1^0\Big[\sum_{G \in \mathcal{S}_2(0)} f_0(G) - 4\Big] = \gamma_2\mu_2 - 4\gamma_1,$$
$$f(0,1,1) = \gamma_0 \mathbb{E}_0^0\Big[\sum_{G \in \mathcal{S}_2(0)} f_0(G) - 6\Big] = \gamma_2\mu_2 - 6\gamma_0,$$
$$f(1,1,1) = \gamma_1 \mathbb{E}_1^0\Big[\sum_{G \in \mathcal{S}_2(0)} f_0(G) - 2\Big] = \gamma_2\mu_2 - 2\gamma_1,$$
$$f(0,0,2) = f(0,1,2) = f(2,0,1) = f(0,2,1) = f(2,1,1) = 3\gamma_0,$$
$$f(1,2,1) = f(1,0,2) = 2\gamma_1,$$
$$f(0,0,3) = \gamma_0,$$
$$f(1,1,2) = \gamma_1,$$
$$f(2,2,1) = \gamma_2.$$

Using Theorem 4.1 and the relations $\gamma_0 = 2\gamma_2$ and $\gamma_1 = 3\gamma_2$ completes our proof.

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Clearly, all asymptotic covariances considered in Theorem 5.1 are 0 for $p = 0$ or 1. We continue with a brief discussion of their maxima and minima.

Corollary 5.1. *Let the assumptions of Theorem 5.1 be satisfied. Then the variance* $\sigma_{2,2}$ *has a global maximum at* $p = \frac{1}{2}$, *and* $\sigma_{1,2}$ *has a global maximum at* $p = \frac{1}{2}(1 - 1/\sqrt{3})$ *and a global minimum at* $p = \frac{1}{2}(1 + 1/\sqrt{3})$.

The covariance $\sigma_{0,2}$ *has a global minimum at* $p = \frac{1}{2}$ *and the variance* $\sigma_{1,1}$ *has a global maximum (minimum) at* $p = \frac{1}{2}$ *according to whether*

$$2\gamma_2 \mathbb{E}_2^0[V_1(F(0))^2] < (>) \, \rho_{1,1}^{2,2} + \gamma_1 \mathbb{E}_1^0[V_1(F(0))^2].$$

The variance $\sigma_{0,0}$ *has a strict global maximum (minimum) at* $p = \frac{1}{2}$ *according to whether*

$$\mu_2 > (<) \, \frac{86}{3} + \frac{4\rho_{0,0}^{2,2}}{3\gamma_2}.$$

6. Poisson–Voronoi percolation

In this section we consider cell percolation on the Voronoi tessellation X generated by a stationary Poisson process η in $\mathbb{R}^d$ with intensity $\gamma > 0$. For a formal definition, we introduce the space $\mathcal{N}$ of all locally finite subsets χ of $\mathbb{R}^d$ whose convex hull coincides with $\mathbb{R}^d$ and whose points are in *general quadratic position* (the latter means that no $d + 2$ points of A lie on the boundary of some ball and any $k \in \{2, \dots, d + 1\}$ points in χ do not lie in a $(k - 2)$-dimensional affine subspace of $\mathbb{R}^d$). The *Voronoi cell* $C(\chi, x)$ of $x \in \chi \in \mathcal{N}$ is the set of all $y \in \mathbb{R}^d$ satisfying $|y - x| \le \min\{|y - z| : z \in \chi\}$. The system $\{C(\chi, x) : x \in \chi\}$ of all Voronoi cells with respect to χ is called the *Voronoi tessellation* (of $\mathbb{R}^d$).

We can assume without loss of generality that $\eta(\omega) \in \mathcal{N}$ for all $\omega \in \Omega$. The *Poisson–Voronoi tessellation* $X := \{C(\eta, x) : x \in \eta\}$ is then stationary, face-to-face, and normal (see [12, Theorems 10.2.2 and 10.2.3]).

For $x, y \in \mathbb{R}^d$, set $\eta^x := \eta \cup \{x\}$ and $\eta^{x,y} := \eta \cup \{x, y\}$. To abbreviate our notation, define random variables for $i, k \in \{0, \dots, d\}$ and $x, y \in \mathbb{R}^d$ by

$$V_i^{(k)}(x) := \sum_{F \in \mathcal{F}_k(C(\eta^x, x))} V_i(F) \quad \text{and} \quad V_i^{(k)}(x, y) := \sum_{F \in \mathcal{F}_k(C(\eta^{x,y}, x))} V_i(F).$$

We now show that the assumptions of Theorem 4.1 and Theorem 5.1 are satisfied, and also obtain a more explicit representation for the limits (4.2). In particular, these limits are independent of the observation window W. The special case $i = j = k = l = d - 1$ is basically well known; see [1] for $d = 2$ and [11] for general d.

Theorem 6.1. *For the Poisson–Voronoi tessellation in* $\mathbb{R}^d$, *the finiteness condition (4.3) is satisfied, and the limits (4.2) exist and are given by*

$$\begin{aligned}&(d - k + 1)(d - l + 1)\rho_{i,j}^{k,l} \\ &\quad = \gamma \, \mathbb{E}[V_i^{(k)}(0) V_j^{(l)}(0)] + \gamma^2 \int (\mathbb{E}[V_i^{(k)}(x, 0) \, V_j^{(l)}(0, x)] - \mathbb{E}V_i^{(k)}(0) \, \mathbb{E}V_j^{(l)}(0)) \, \mathrm{d}x. \qquad (6.1)\end{aligned}$$

For a planar Poisson–Voronoi tessellation, the zero limit in (5.2) holds.

Proof. Assumption (4.3) is a consequence of the Cauchy–Schwarz inequality, Lemma A.2, and Lemma A.4.

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We prove (6.1) in two steps. First consider an asymptotic covariance that is similar to $\rho_{i,j}^{k,l}$ but easier to determine. Defining

$$\tau_{i,j}^{k,l}(t) := \operatorname{cov}\bigg(\int_{W_t} V_i^{(k)}(x)\,\eta(\mathrm{d}x), \int_{W_t} V_j^{(l)}(x)\,\eta(\mathrm{d}x)\bigg), \qquad t > 0,$$

we show that $\lim_{t\to\infty} t^{-1}\tau_{i,j}^{k,l}(t)$ equals

$$\gamma\,\mathbb{E}[V_i^{(k)}(0)V_j^{(l)}(0)] + \gamma^2 \int [\mathbb{E}[V_i^{(k)}(x,0)\,V_j^{(l)}(0,x)] - \mathbb{E}V_i^{(k)}(0)\,\mathbb{E}V_j^{(l)}(0)]\,\mathrm{d}x, \tag{6.2}$$

and that this asymptotic covariance is finite. In the second step, we show that the asymptotic covariance in (6.2) equals $\rho_{i,j}^{k,l}$ (up to a constant), specifically,

$$(d-k+1)(d-l+1)\rho_{i,j}^{k,l} = \lim_{t\to\infty} t^{-1}\tau_{i,j}^{k,l}(t). \tag{6.3}$$

We start from the Mecke formula (see, e.g. [12]), so

$$\begin{aligned}
\tau_{i,j}^{k,l}(t) &= \mathbb{E}\int_{W_t} V_i^{(k)}(x)\,V_j^{(l)}(x)\,\eta(\mathrm{d}x) + \mathbb{E}\int_{W_t}\int_{W_t} V_i^{(k)}(x)\,V_j^{(l)}(y)\,\mathbf{1}\{x\neq y\}\,\eta(\mathrm{d}x)\,\eta(\mathrm{d}y)\\
&\quad - \mathbb{E}\int_{W_t} V_i^{(k)}(x)\,\eta(\mathrm{d}x)\,\mathbb{E}\int_{W_t} V_j^{(l)}(x)\,\eta(\mathrm{d}x)\\
&= \gamma\int_{W_t} \mathbb{E}[V_i^{(k)}(x)\,V_j^{(l)}(x)]\,\mathrm{d}x\\
&\quad + \gamma^2\int_{W_t}\int_{W_t} [\mathbb{E}[V_i^{(k)}(x,y)\,V_j^{(l)}(y,x)] - \mathbb{E}V_i^{(k)}(x)\,\mathbb{E}V_j^{(l)}(y)]\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Using the stationarity of η, translation invariance of the functions $V_i^{(k)}(\cdot)$ and $V_i^{(k)}(\cdot,\cdot)$, and a change of variables, we obtain

$$\begin{aligned}
\frac{\tau_{i,j}^{k,l}(t)}{t} &= \gamma\,\mathbb{E}[V_i^{(k)}(0)\,V_j^{(l)}(0)]\\
&\quad + \frac{\gamma^2}{t}\iint [\mathbb{E}[V_i^{(k)}(x,0)\,V_j^{(l)}(0,x)] - \mathbb{E}V_i^{(k)}(0)\,\mathbb{E}V_j^{(l)}(0)]\mathbf{1}\{x+y,\ y\in W_t\}\,\mathrm{d}x\,\mathrm{d}y\\
&= \gamma\,\mathbb{E}[V_i^{(k)}(0)\,V_j^{(l)}(0)]\\
&\quad + \gamma^2\int [\mathbb{E}[V_i^{(k)}(x,0)\,V_j^{(l)}(0,x)] - \mathbb{E}V_i^{(k)}(0)\,\mathbb{E}V_j^{(l)}(0)]\,\frac{V_d(W_t\cap(W_t-x))}{V_d(W_t)}\,\mathrm{d}x.
\end{aligned}$$

By the dominated convergence theorem, this converges to the right-hand side of (6.2) as $t\to\infty$, which is in fact the right-hand side of (6.1). Since $V_d(W_t\cap(W_t-x))/t \le 1$, a dominating function is given by

$$|\mathbb{E}[V_i^{(k)}(x,0)\,V_j^{(l)}(0,x)] - \mathbb{E}V_i^{(k)}(0)\,\mathbb{E}V_j^{(l)}(0)|. \tag{6.4}$$

Indeed, in the following we show that the integral of this function is finite.

Next, we need a technical tool. The *Voronoi flower* of $x\in\eta$ is defined by

$$S(\eta,x) := \bigcup_{y\in C(\eta,x)} B(y, \|y-x\|),$$

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where $B(x, r)$ denotes the closed ball with center $x \in \mathbb{R}^d$ and radius $r \geq 0$. Using this definition and the triangle inequality, we obtain

$$\begin{aligned}
&\int |\mathbb{E}[V_i^{(k)}(x,0)\, V_j^{(l)}(0,x)] - \mathbb{E}V_i^{(k)}(0)\, \mathbb{E}V_j^{(l)}(0)|\, \mathrm{d}x \\
&\quad \leq \int \Big|\mathbb{E}\Big[\mathbf{1}\big\{S(\eta^{0,x}, x) \subset B\big(x, \tfrac{1}{3}\|x\|\big), S(\eta^{0,x}, 0) \subset B\big(0, \tfrac{1}{3}\|x\|\big)\big\}\, V_i^{(k)}(x,0)\, V_j^{(l)}(0,x)\Big] \\
&\qquad\quad - \mathbb{E}V_i^{(k)}(0)\, \mathbb{E}V_j^{(l)}(0)\Big|\, \mathrm{d}x \\
&\qquad + \int \Big|\mathbb{E}\Big[\Big(\mathbf{1}\big\{S(\eta^{0,x}, 0) \not\subset B\big(0, \tfrac{1}{3}\|x\|\big)\big\} + \mathbf{1}\big\{S(\eta^{0,x}, x) \not\subset B\big(x, \tfrac{1}{3}\|x\|\big)\big\} \\
&\qquad\qquad - \mathbf{1}\big\{S(\eta^{0,x}, 0) \not\subset B\big(0, \tfrac{1}{3}\|x\|\big), S(\eta^{0,x}, x) \not\subset B\big(x, \tfrac{1}{3}\|x\|\big)\big\}\Big) \\
&\qquad\quad \times V_i^{(k)}(x,0)\, V_j^{(l)}(0,x)\Big]\Big|\, \mathrm{d}x \\
&\quad =: I_1 + I_2.
\end{aligned}$$

In the following we assume that $x \neq 0$, and later use the identity relation

$$\mathbf{1}\big\{S(\eta^{0,x}, x) \subset B\big(x, \tfrac{1}{3}\|x\|\big)\big\} = \mathbf{1}\big\{S(\eta^{x}, x) \subset B\big(x, \tfrac{1}{3}\|x\|\big)\big\}. \tag{6.5}$$

Indeed, when $S(\eta^{0,x}, x) \subset B(x, \frac{1}{3}\|x\|)$, the origin cannot be contained in $S(\eta^{0,x}, x)$, so it cannot be a neighbour of x with respect to $\eta^{0,x}$. Because $S(\eta^{0,x}, x)$ is determined by x and the neighbours of x with respect to $\eta^{0,x}$, deleting the origin does not change the Voronoi flower of x, i.e. $S(\eta^x, x) = S(\eta^{0,x}, x) \subset B(x, \frac{1}{3}\|x\|)$. In the case $S(\eta^{0,x}, x) \not\subset B(x, \frac{1}{3}\|x\|)$, the Voronoi flower cannot get larger if we add more points to the point process, i.e. $S(\eta^{0,x}, x) \subset S(\eta^x, x)$. This implies that $S(\eta^x, x) \not\subset B(x, \frac{1}{3}\|x\|)$.

Now we use the stopping set property of the Voronoi flowers $S(\eta^x, x)$ and $S(\eta^0, 0)$ (see [14]). Because a Voronoi cell and its corresponding Voronoi flower are determined by the Poisson points contained in the flower, the random variables

$$\mathbf{1}\big\{S(\eta^x, x) \subset B\big(x, \tfrac{1}{3}\|x\|\big)\big\}\, V_i^{(k)}(x, 0), \qquad \mathbf{1}\big\{S(\eta^0, 0) \subset B\big(0, \tfrac{1}{3}\|x\|\big)\big\}\, V_i^{(k)}(0, x)$$

are determined by the restrictions of η to $B(x, \frac{1}{3}\|x\|)$ and $B(0, \frac{1}{3}\|x\|)$, respectively. Using (6.5), an analogous equation for the Voronoi flower of the origin, and the facts that $B(x, \frac{1}{3}\|x\|)$ and $B(0, \frac{1}{3}\|x\|)$ are disjoint and the restrictions of a Poisson process to disjoint sets are independent, we have

$$\begin{aligned}
I_1 = \int &\Big|\mathbb{E}\Big[\mathbf{1}\big\{S(\eta^x, x) \subset B\big(x, \tfrac{1}{3}\|x\|\big)\big\}\, V_i^{(k)}(x)\Big]\, \mathbb{E}\Big[\mathbf{1}\big\{S(\eta^0, 0) \subset B\big(0, \tfrac{1}{3}\|x\|\big)\big\}\, V_j^{(l)}(0)\Big] \\
&- \mathbb{E}V_i^{(k)}(0)\, \mathbb{E}V_j^{(l)}(0)\Big|\, \mathrm{d}x.
\end{aligned}$$

Appealing to stationarity yields

$$\begin{aligned}
I_1 &= \int \Big|\mathbb{E}\Big[\mathbf{1}\big\{S(\eta^0, 0) \subset B\big(0, \tfrac{1}{3}\|x\|\big)\big\}\, V_i^{(k)}(0)\Big]\, \mathbb{E}\Big[\mathbf{1}\big\{S(\eta^0, 0) \subset B\big(0, \tfrac{1}{3}\|x\|\big)\big\}\, V_j^{(l)}(0)\Big] \\
&\qquad - \mathbb{E}V_i^{(k)}(0)\, \mathbb{E}V_j^{(l)}(0)\Big|\, \mathrm{d}x \\
&= \int \Big|\mathbb{E}\Big[\mathbf{1}\big\{S(\eta^0, 0) \not\subset B\big(0, \tfrac{1}{3}\|x\|\big)\big\}\, V_i^{(k)}(0)\Big]\, \mathbb{E}\Big[\mathbf{1}\big\{S(\eta^0, 0) \not\subset B\big(0, \tfrac{1}{3}\|x\|\big)\big\}\, V_j^{(l)}(0)\Big] \\
&\qquad - \mathbb{E}\Big[\mathbf{1}\big\{S(\eta^0, 0) \not\subset B\big(0, \tfrac{1}{3}\|x\|\big)\big\}\, V_i^{(k)}(0)\Big]\, \mathbb{E}V_j^{(l)}(0) \\
&\qquad - \mathbb{E}V_i^{(k)}(0)\, \mathbb{E}\Big[\mathbf{1}\big\{S(\eta^0, 0) \not\subset B\big(0, \tfrac{1}{3}\|x\|\big)\big\}\, V_j^{(l)}(0)\Big]\Big|\, \mathrm{d}x \\
&=: \int |J_1 - J_2 - J_3|\, \mathrm{d}x.
\end{aligned}$$

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To show that I_1 is finite, observe first that $|J_1| \le \min\{|J_2|, |J_3|\}$ by the triangle inequality, and that J_2 and J_3 have the same finiteness property modulo different parameters, so I_1 is finite if $\int |J_2| \, \mathrm{d}x$ is finite. Define the *Voronoi neighbourhood* of a point $x \in \chi$ with respect to χ by the point set

$$N(\chi, x) := \{y \in \chi \setminus \{x\} \colon C(\chi, x) \cap C(\chi, y) \neq \varnothing\}. \tag{6.6}$$

Now J_2 is a product, for which the second factor is

$$\mathbb{E} V_j^{(l)}(0) \le \mathbb{E}[|N(\eta^0, 0)|^{d-l} \, V_j(C(\eta^0, 0))] \le [\mathbb{E}|N(\eta^0, 0)|^{2d-2l} \, \mathbb{E}(V_j(C(\eta^0, 0))^2)]^{1/2},$$

which is finite, while, for the other factor, we have

$$\begin{aligned} &\mathbb{E}\big[\mathbf{1}\big\{S(\eta^0, 0) \not\subset B\big(0, \tfrac{1}{3}\|x\|\big)\big\} \, V_i^{(k)}(0)\big] \\ &\quad \le \mathbb{E}\big[\mathbf{1}\big\{2 \operatorname{diam}(C(\eta^0, 0)) > \tfrac{1}{3}\|x\|\big\} \, |N(\eta^0, 0)|^{d-k} \, V_i(C(\eta^0, 0))\big] \\ &\quad \le \big[\mathbb{P}\big\{\operatorname{diam}(C(\eta^0, 0)) > \tfrac{1}{6}\|x\|\big\}\big]^{1/3} \, [\mathbb{E}|N(\eta^0, 0)|^{3d-3k} \, \mathbb{E}(V_i(C(\eta^0, 0))^3)]^{1/3}. \end{aligned}$$

Since $\int [\mathbb{P}\{\operatorname{diam}(C(\eta^0, 0)) > \frac{1}{6}\|x\|\}]^{1/3} \, \mathrm{d}x$ is finite (cf. Lemma A.1 and Corollary A.1), $\int |J_2| \, \mathrm{d}x < \infty$, and, hence, I_1 is finite.

To show that $I_2 < \infty$, write $I_2 =: \int |\mathbb{E}[(J_1 + J_2 - J_3) V_i^{(k)}(x, 0) V_j^{(l)}(0, x)]| \, \mathrm{d}x$ for the integral defining I_2 earlier. As for I_1, it is enough to verify that the term involving J_1 is finite. This term is

$$\begin{aligned} &\int \mathbb{E}\big[\mathbf{1}\big\{S(\eta^{0,x}, 0) \not\subset B\big(0, \tfrac{1}{3}\|x\|\big)\big\} \, V_i^{(k)}(x, 0) \, V_j^{(l)}(0, x)\big] \, \mathrm{d}x \\ &\quad \le \int \mathbb{E}\big[\mathbf{1}\big\{S(\eta^{0,x}, 0) \not\subset B\big(0, \tfrac{1}{3}\|x\|\big)\big\} |N(\eta^{0,x}, x)|^{d-k} \, V_i(C(\eta^{0,x}, x)) \\ &\qquad\qquad \times |N(\eta^{0,x}, 0)|^{d-l} \, V_j(C(\eta^{0,x}, 0))\big] \, \mathrm{d}x \\ &\quad \le \int \big[\mathbb{P}\big\{\operatorname{diam}(C(\eta^0, 0)) > \tfrac{1}{6}\|x\|\big\}\big]^{1/5} \, \mathrm{d}x \\ &\qquad \times (\mathbb{E}(|N(\eta^0, 0)| + 1)^{5d-5k} \mathbb{E}(|N(\eta^0, 0)| + 1)^{5d-5l} \mathbb{E} V_i(C(\eta^0, 0))^5 \\ &\qquad\quad \times \mathbb{E} V_j(C(\eta^0, 0))^5)^{1/5}. \end{aligned}$$

This is finite by Lemma A.1, Lemma A.3, and Corollary A.1. Hence, the integral of the dominating function given in (6.4) is finite and, therefore, the second summand on the right-hand side of (6.2) is finite, too.

Using the monotonicity of the intrinsic volumes, normality and Hölder's inequality, for the first summand on the right-hand side of (6.2), we obtain

$$\begin{aligned} \mathbb{E}[V_i^{(k)}(0) \, V_j^{(l)}(0)] &\le \mathbb{E}[|\mathcal{F}_k(C(\eta^0, 0))| \, V_i(C(\eta^0, 0)) \, |\mathcal{F}_l(C(\eta^0, 0))| \, V_j(C(\eta^0, 0))] \\ &\le \mathbb{E}[|N(\eta^0, 0)|^{d-k} \, V_i(C(\eta^0, 0)) \, |N(\eta^0, 0)|^{d-l} \, V_j(C(\eta^0, 0))] \\ &\le (\mathbb{E}|N(\eta^0, 0)|^{6d-3k-3l} \, \mathbb{E} V_i(C(\eta^0, 0))^3 \, \mathbb{E} V_j(C(\eta^0, 0))^3)^{1/3}. \end{aligned}$$

This is finite by Lemma A.1 and Corollary A.1. So, the right-hand side of (6.2) is finite.

The next step is to prove (6.3). Because a Poisson–Voronoi tessellation is normal,

$$\int V_i(F(x) \cap W_t) \, \eta^{(k)}(\mathrm{d}x) = \frac{1}{d - k + 1} \int V_i(\mathcal{F}_k(C(\eta, x)) \cap W_t) \, \eta(\mathrm{d}x)$$

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and (6.3) is equivalent to

$$\lim_{t\to\infty}\frac{1}{t}\operatorname{cov}\Big(\int V_i(\mathcal{F}_k(C(\eta,x))\cap W_t)\,\eta(\mathrm{d}x),\int V_j(\mathcal{F}_l(C(\eta,x))\cap W_t)\,\eta(\mathrm{d}x)\Big)$$
$$=\lim_{t\to\infty}\frac{1}{t}\operatorname{cov}\Big(\int V_i^{(k)}(x)\,\mathbf{1}\{x\in W_t\}\,\eta(\mathrm{d}x),\int V_j^{(l)}(x)\,\mathbf{1}\{x\in W_t\}\,\eta(\mathrm{d}x)\Big). \qquad (6.7)$$

We start by showing that

$$\lim_{t\to\infty}\frac{1}{t}\operatorname{var}\Big(\int[V_i(\mathcal{F}_k(C(\eta,x))\cap W_t)\;-\;V_i^{(k)}(x)\,\mathbf{1}\{x\in W_t\}]\,\eta(\mathrm{d}x)\Big)=0. \qquad (6.8)$$

We abbreviate the notation by defining

$$h(\chi,B):=\int[V_i(\mathcal{F}_k(C(\chi,x))\cap B)-V_i(\mathcal{F}_k(C(\chi,x)))\,\mathbf{1}\{x\in B\}]\,\chi(\mathrm{d}x)$$

for $\chi\in\mathcal{N}$ and Borel sets $B\subset\mathbb{R}^d$. From the Poincaré inequality (see [13]) we have

$$\frac{\operatorname{var}h(\eta,W_t)}{t}\le\frac{\gamma}{t}\,\mathbb{E}\int[h(\eta^x,W_t)-h(\eta,W_t)]^2\,\mathrm{d}x. \qquad (6.9)$$

We now determine an upper bound for $|h(\eta^x,W_t)-h(\eta,W_t)|$, using the Voronoi neighbourhood $N(\chi,x)$ of a point $x\in\chi$ defined in (6.6). Because the addition of a point $x\in\mathbb{R}^d$ to η changes only the cells of the points $y\in\eta$ for which $y\in N(\eta^x,x)$,

$$\begin{aligned}
&|h(\eta^x,W_t)-h(\eta,W_t)|\\
&\quad=\Big|V_i(\mathcal{F}_k(C(\eta^x,x))\cap W_t)-V_i(\mathcal{F}_k(C(\eta^x,x)))\,\mathbf{1}\{x\in W_t\}\\
&\qquad+\sum_{y\in N(\eta^x,x)}V_i(\mathcal{F}_k(C(\eta^x,y))\cap W_t)-V_i(\mathcal{F}_k(C(\eta^x,y)))\,\mathbf{1}\{y\in W_t\}\\
&\qquad-\sum_{y\in N(\eta^x,x)}V_i(\mathcal{F}_k(C(\eta,y))\cap W_t)-V_i(\mathcal{F}_k(C(\eta,y)))\,\mathbf{1}\{y\in W_t\}\Big|\\
&\quad\le f(\eta^x,x)\,\mathbf{1}\Big\{\bigcup_{y\in N(\eta^x,x)\cup\{x\}}C(\eta^x,x)\cap\partial W_t\ne\varnothing\Big\},
\end{aligned}$$

where, for $x\in\chi\in\mathcal{N}$ and fixed $k\in\{0,\dots,d\}$,

$$f(\chi,x):=2\Big[V_i(\mathcal{F}_k(C(\chi,x)))+\sum_{y\in N(\chi,x)}V_i(\mathcal{F}_k(C(\chi,y)))+\sum_{y\in N(\chi,x)}V_i(\mathcal{F}_k(C(\chi-\delta_x,y)))\Big].$$

Note that all moments of $f(\eta^0,0)$ exist by Lemma A.4 and Lemma A.1. Using (6.9), the translation covariance of $C(\cdot,\cdot)$, $N(\cdot,\cdot)$, and $\mathcal{F}_k$, the stationarity of η, and the translation

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invariance of the intrinsic volumes, we obtain

$$\begin{aligned}\frac{\operatorname{var}(h(\eta, W_t))}{t} &\le \frac{\gamma}{t}\,\mathbb{E}\int \mathbf{1}\Big\{\bigcup_{y\in N(\eta^x,x)\cup\{x\}} C(\eta^x, x)\cap \partial W_t \neq \varnothing\Big\} f(\eta^x, x)^2\,\mathrm{d}x \\ &= \frac{\gamma}{t}\mathbb{E}\int \mathbf{1}\Big\{\Big(\bigcup_{y\in N(\eta^0,0)\cup\{0\}} C(\eta^0, y)+x\Big)\cap \partial W_t \neq \varnothing\Big\}\, f(\eta^0, 0)^2\,\mathrm{d}x \\ &= \gamma\mathbb{E}\Big[\lambda_d\Big(\partial W - t^{-1/d}\bigcup_{y\in N(\eta^0,0)\cup\{0\}} C(\eta^0, y)\Big)\, f(\eta^0, 0)^2\Big].\end{aligned}$$

This converges to 0 as $t\to\infty$ by the dominated convergence theorem. Using Lemma A.4, Lemma A.1, and the Steiner formula,

$$\sum_{y\in N(\eta^0,0)\cup\{0\}} \lambda_d(W - C(\eta^0, y))\, f(\eta^0, 0)^2$$

is a dominating function. This proves (6.8).

By using (6.8) and the Cauchy–Schwarz inequality we show that (6.7) holds (and, hence, also (6.3)). We abbreviate

$$U_i^k := \int V_i(\mathcal{F}_k(C(\eta, x))\cap W_t)\,\eta(\mathrm{d}x) \quad \text{and} \quad V_i^k := \int V_i(\mathcal{F}_k(C(\eta, x)))\,\mathbf{1}\{x\in W_t\}\,\eta(\mathrm{d}x).$$

The triangle and Cauchy–Schwarz inequalities imply that

$$\begin{aligned} 0 &\le \lim_{t\to\infty} t^{-1}|\operatorname{cov}(U_i^k, U_j^l) - \operatorname{cov}(V_i^k, V_j^l)| \\ &\le \lim_{t\to\infty} t^{-1}|\operatorname{cov}(U_i^k - V_i^k, V_j^l)| + t^{-1}|\operatorname{cov}(U_i^k - V_i^k, U_j^l - V_j^l)| \\ &\quad + t^{-1}|\operatorname{cov}(V_i^k, U_j^l - V_j^l)| \\ &\le \lim_{t\to\infty} \sqrt{t^{-1}\operatorname{var}(U_i^k - V_i^k)}\sqrt{t^{-1}\operatorname{var}(V_j^l)} + \sqrt{t^{-1}\operatorname{var}(U_i^k - V_i^k)}\sqrt{t^{-1}\operatorname{var}(U_j^l - V_j^l)} \\ &\quad + \sqrt{t^{-1}\operatorname{var}(V_i^k)}\sqrt{t^{-1}\operatorname{var}(U_j^l - V_j^l)}. \end{aligned}$$

This limit is 0 by (6.8) and the finiteness of $\lim_{t\to\infty} t^{-1}\operatorname{var}(V_i^k)$, as already shown.

The remaining assumption (5.2) can be shown analogously to (6.8) by using the Poincaré inequality.

Theorem 6.1 and Theorem 4.1 together yield the covariance structure for cell percolation on a Poisson–Voronoi tessellation. The variance of the Euler characteristic in the planar case is worth particular mention.

Corollary 6.1. *For cell percolation on a planar Poisson–Voronoi tessellation, the asymptotic variance $\sigma_{0,0}$ of the Euler characteristic exists and is given by*

$$\sigma_{0,0}(p) = \gamma_2\mu_2 p^3q^3 + \gamma_2 pq(1 - 8pq - 14p^2q^2),$$

for which there is a strict global maximum at $p = \frac{1}{2}$.

Proof. By Theorem 6.1 we can apply Theorem 5.1. Since the asymptotic variance $\rho_{0,0}^{2,2}$ equals the intensity γ_2, we obtain the formula for $\sigma_{0,0}$. The second assertion follows from the corresponding assertion of Theorem 5.1 and (5.1) or from a direct calculation.

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Appendix A. Integrability properties of Poisson–Voronoi tessellations

We consider the Voronoi tessellation X of $\mathbb{R}^d$ generated by a Poisson process η of intensity $\gamma > 0$. Define the *second-order neighbourhood* of a point $x \in \chi$ with respect to $\chi \in \mathcal{N}$ by

$$N_2(\chi, x) := \{y \in \chi : \text{there exists } z \in N(\chi, x) \text{ with } y \in N(\chi, z)\}$$

(cf. definition of $N(\chi, x)$ around (6.6)).

Lemma A.1. *It holds that* $\mathbb{E}|N_2(\eta^0, 0)|^m < \infty$ *for all* $m \in \mathbb{N}$.

Proof. Enumerate η as $Y_1, Y_2, \ldots$ such that $0 < \|Y_1\| < \|Y_2\| < \cdots$. Let $n := |N_2(\eta^0, 0)|$, and assume that $n \geq 2$ is true a.s. Hence, there is a point $x \in N_2(\eta^0, 0)$ with $\|x\| \geq \|Y_n\| =: t$. It is easy to see that $x \in \eta$ is a second-order neighbour of 0 with respect to η^0 if and only if there exist $y \in \eta \setminus \{x\}$ and balls B and B' with $0, y \in \partial B$, $x, y \in \partial B'$, $\operatorname{int}(B) \cap \eta = \varnothing$, and $\operatorname{int}(B') \cap \eta = \varnothing$. Let $\tilde{B}$ denote either B or B', so that, for $\|x\| \geq t$, $\operatorname{diam}(\tilde{B}) \geq \frac{1}{2}t$ and $\tilde{B} \cap B(0, \frac{1}{2}t) \neq \varnothing$.

There exist balls $B_1, \ldots, B_l \subset \operatorname{int}(B(0, t))$ of diameter $\frac{1}{8}t$ such that each ball $\tilde{B}$ of diameter at least $\frac{1}{2}t$ and $\tilde{B} \cap B(0, \frac{1}{2}t) \neq \varnothing$ contains at least one of the balls $B_1, \ldots, B_l$. By a scaling argument, the number l of balls can be chosen independently of t.

So, $|N_2(\eta^0, 0)| \geq n$ implies that $\eta(B_i) = 0$ for at least one $i \in \{1, \ldots, l\}$. From the binomial property of the Poisson process we have

$$\mathbb{P}\{\eta(B_1) = 0 \mid \|Y_n\| = t\} = \left(1 - \frac{\kappa_d (t/16)^d}{\kappa_d t^d}\right)^{n-1} = (1 - 16^{-d})^{n-1},$$

and the conditional probability that at least one ball B_i contains no point of η is at most $l(1 - 16^{-d})^{n-1}$. Denoting the density of $\|Y_n\|$ by f_n, $\mathbb{P}\{|N_2(\eta^0, 0)| \geq n\}$ equals

$$\int \mathbb{P}\{|N_2(\eta^0, 0)| \geq n \mid \|Y_n\| = t\} \, f_n(t) \, \mathrm{d}t \leq \int l(1 - 16^{-d})^{n-1} f_n(t) \, \mathrm{d}t;$$

hence,

$$\mathbb{P}\{|N_2(\eta^0, 0)| \geq n\} \leq l(1 - 16^{-d})^{n-1}.$$

Lemma A.2. *For all* $m \in \mathbb{N}$ *and* $k, n \in \{0, \ldots, d\}$, $\mathbb{E}_n^0 |\mathcal{S}_k(0)|^m < \infty$.

Proof. By normality, it suffices to treat the case $n > k$. Proposition 2.2 implies that

$$(d - n + 1) \, \gamma_n \mathbb{E}_n^0 |\mathcal{S}_k(0)|^m = \gamma_n \, \mathbb{E}_n^0 \biggl[\sum_{F \in \mathcal{S}_d(0)} |\mathcal{S}_k(0)|^m\biggr] = \gamma_d \, \mathbb{E}_d^0 \biggl[\sum_{F \in \mathcal{S}_n(0)} |\mathcal{S}_k(s(F))|^m\biggr].$$

Because $n > k$ we can bound $|\mathcal{S}_k(s(F))|$ for $F \in \mathcal{S}_n(0)$ by $|\mathcal{S}_k(0)|$. Furthermore, each k-face of the typical cell is contained in exactly $d - k$ neighbouring cells of the typical cell and so

$$(d - n + 1) \, \gamma_n \mathbb{E}_n^0 |\mathcal{S}_k(0)|^m \leq \gamma_d \mathbb{E}_d^0 [|\mathcal{S}_n(0)| \, |\mathcal{S}_k(0)|^m] \leq \gamma \, \mathbb{E}[|N(\eta^0, 0)|^{d-n+md-mk}],$$

which is finite by Lemma A.1.

The proof of the next lemma is given in [6, Theorem 2].

Lemma A.3. *There exist constants* c_1 *and* $c_2 > 0$ *such that, for all* $u \geq 0$,

$$\mathbb{P}\{\operatorname{diam}(C(\eta^0, 0)) \geq u\} \leq c_1 \exp(-c_2 u).$$

Corollary A.1. *For all $i \in \{0, \dots, d\}$, $\mathbb{E}V_i(C(\eta^0, 0))^m < \infty$ for every $m \in \mathbb{N}$.*

Proof. We use $C(\eta^0, 0) \subset B(0, \operatorname{diam}(C(\eta^0, 0)))$ and the monotonicity of the intrinsic volumes to obtain, for any $m \in \mathbb{N}$,

$$\mathbb{E}V_i(C(\eta^0, 0))^m \le \mathbb{E}V_i(B(0, \operatorname{diam}(C(\eta^0, 0))))^m = V_i(B(0, 1))^m \,\mathbb{E}\operatorname{diam}(C(\eta^0, 0))^{im}.$$

This is finite by Lemma A.3.

We introduce a modification of the system $\mathcal{S}_l(\varphi, x)$ for a tessellation $\varphi \in \boldsymbol{T}$ and $x \in \mathbb{R}^d$ with $F(x) \in \mathcal{F}_k(\varphi)$, and define

$$\tilde{\mathcal{S}}_l(\varphi, x) := \begin{cases} \mathcal{S}_l(\varphi, x), & \min(k, l) < d, \\ \{G \in \mathcal{F}_k(\varphi) \colon G \cap F(x) \in \mathcal{F}_{k-1}(\varphi)\}, & k = l = d. \end{cases}$$

The proof of the following version of Proposition 2.2 can be easily given with Neveu's exchange formula.

Proposition A.1. *Let $k, l \in \{0, \dots, d\}$, and let $g \colon \mathcal{P}^d \times \mathcal{P}^d \to [0, \infty)$ be a measurable function. Then*

$$\gamma_k \mathbb{E}_k^0 \sum_{G \in \tilde{\mathcal{S}}_l(0)} g(F(0), G - s(G)) = \gamma_l \mathbb{E}_l^0 \sum_{F \in \tilde{\mathcal{S}}_k(0)} g(F - s(F), F(0)). \tag{A.1}$$

Let $R(B)$ be the radius of the circumball of a subset $B \subset \mathbb{R}^d$.

Lemma A.4. *For all $m \in \mathbb{N}$,*

$$\sum_{k=0}^{d} \mathbb{E}_k^0 \left[R\left(F(0) \cup \bigcup_{G \in \tilde{\mathcal{S}}_d(0)} G \right)^m \right] < \infty.$$

Proof. For $k < d$, we have

$$\mathbb{E}_k^0 \left[R\left(F(0) \cup \bigcup_{G \in \tilde{\mathcal{S}}_d(0)} G \right)^m \right] \le \mathbb{E}_k^0 \max_{G \in \tilde{\mathcal{S}}_d(0)} (2R(G))^m \le 2^m \mathbb{E}_k^0 \sum_{G \in \tilde{\mathcal{S}}_d(0)} R(G)^m$$

and in the $k = d$ case

$$\begin{aligned} \mathbb{E}_d^0 R\left(F(0) \cup \bigcup_{G \in \tilde{\mathcal{S}}_d(0)} G \right)^m &\le \mathbb{E}_d^0 (R(F(0)) + 2 \max_{G \in \tilde{\mathcal{S}}_d(0)} R(G))^m \\ &\le 2^m \,\mathbb{E}_d^0 \left[R(F(0))^m + 2^m \sum_{G \in \tilde{\mathcal{S}}_d(0)} R(G)^m \right]. \end{aligned}$$

Owing to the fact that $R(F(0)) \le 2 \operatorname{diam}(C(\eta^0, 0))$ and Lemma A.3, it is in both cases enough to show that

$$\mathbb{E}_k^0 \sum_{G \in \tilde{\mathcal{S}}_d(0)} R(G)^m < \infty.$$

https://doi.org/10.1239/jap/1417528483 Published online by Cambridge University Press

Using (A.1) with $g(F, G) := R(G)^m$ and the Cauchy–Schwarz inequality, we obtain

$$\gamma_k \mathbb{E}_k^0 \sum_{G \in \tilde{\mathcal{S}}_d(0)} R(G)^m = \gamma_d \mathbb{E}_d^0[|\tilde{\mathcal{S}}_k(0)|\, R(F(0))^m] \le \gamma_d \mathbb{E}_d^0[|\tilde{\mathcal{S}}_k(0)|^2]^{1/2}\, \mathbb{E}_d^0[R(F(0))^{2m}]^{1/2}.$$

As above, the second factor is finite. For $k < d$, it follows by normality that $|\tilde{\mathcal{S}}_k(0)| = d - k + 1$, and, for $k = d$, we have $|\tilde{\mathcal{S}}_k(0)| \le |N_2(\eta^0, 0)|$, so the first factor is finite by Lemma A.1.

Acknowledgement

The authors gratefully acknowledge support from the German Research Foundation (DFG) through the research unit 'Geometry and Physics of Spatial Random Systems' under the grant LA 965/7-1.

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