- A mini-game consisting of an odd number of total points, where B is favoured, is less likely to occur than one consisting of an even number of total points where A is favoured. Yet, B is favoured overall.

To put it simply, but perhaps not convincingly, B is favoured to win the mini-game because the proportion of mini-games B wins of those consisting of an odd number of total points far exceeds the proportion of mini-games A wins of those consisting of an even number of total points. This is quite apparent from the probabilities given in Table 1.

From Figure 1, $A(p)$ is minimised near $p = 0.80$. There $A(p) \approx 0.48$, giving B only a slight advantage (0.52 probability) to win the mini-game. If the mini-game were played in reality, the first receiver's advantage would surely go unnoticed and might even be denied, if brought to the players' attention. For this admittedly contrived scenario, a mathematical analysis is required to reveal that which would otherwise be masked by misleading intuition.

Each day we make hundreds of choices based on the situation before us. Some are unimportant. Coffee or tea? Turn left here? Others have significant consequences. Most decisions are made instinctively for convenience, with no in-depth analysis.

Listen to our intuition? Of course!

Trust our intuition? Not always a good idea!

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107.46 Illustrating complex mappings with Excel

Introduction

Just recently I became aware that Excel has most of the elementary functions of complex analysis included in its library of functions. I guess there are more people out there than me who are not familiar with this feature of Excel, so I would like to give a couple of applications. Since the graphical capabilities of Excel are great, these complex functions could be a good starting point for illustrating complex mappings. The fact that use of Excel is so widespread makes simple complex calculations easily accessible for most students.

The complex functions in Excel can by no means make up for a computer algebra system (CAS). Still I find it interesting that an old workhorse like Excel can do complex calculations.

A simple complex mapping

In Figure 1 we illustrate how a curve in the z-plane is mapped to a curve in the w-plane by a mapping $w = f(z)$. The graphs are based on tables just as we are used to when making ordinary graphs in Excel. In this example we let the mapping f be $f(z) = az + b$ where a and b are complex constants. The curve in the z-plane is given by $z = x + iy$. In this case we let $y = -2x^2 + 2$, that is a simple parabola.

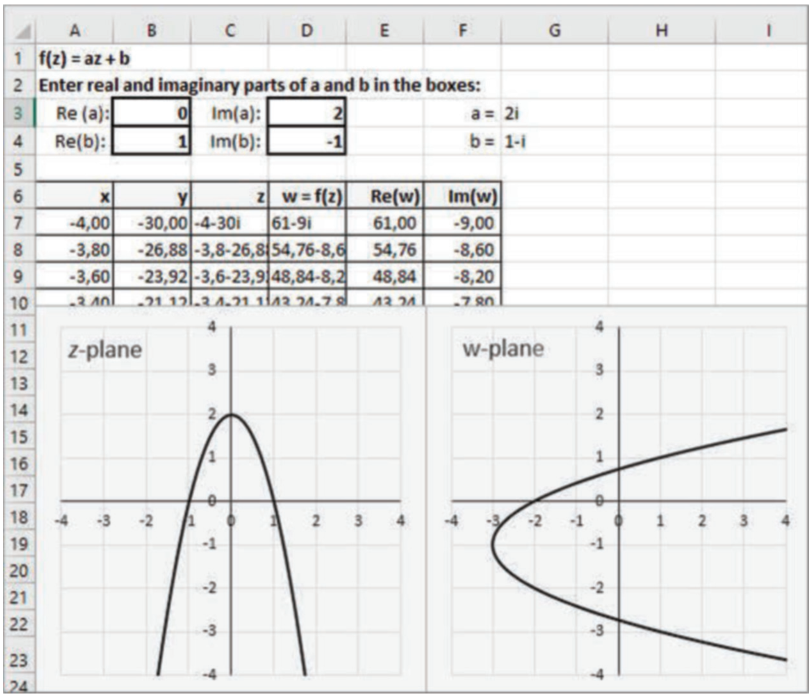

FIGURE 1: A parabola is mapped by the function of the form $f(z) = az + b$

The upper part of the figure illustrates how the real and imaginary parts of a and b can be entered. In cell G3 the real and imaginary parts of a are combined into a complex number by the function COMPLEX (B3,D3). The same is done for b in cell G4. Further down, the start of the underlying table for the graphs is shown. In column C in the table x and y are combined into the complex number z, and in cell D7 the value of $w = f(z)$ is calculated. The complex expressions in Excel become somewhat involved, in this case: IMSUM (IMPRODUCT(G3, C7), G4). Here we have first multiplied the values in cells G3 and C7 by using the function IMPRODUCT( ). Then the value in cell G4 is added with the help of the function IMSUM( ). To get

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the correct copying down the column we use absolute references. Therefore the \$-sign appears in the formula. Now we can extract the real and imaginary parts of w by using the functions IMREAL and IMAGINARY in columns E and F. Then it is simple routine work to make the graphs. One can experiment with different values for a and b to see what happens to the image of the curve under this so-called linear mapping.

This simple spreadsheet described here can be downloaded from [1].

How could the complex functions in Excel be used in teaching? In the example that we have just shown, experimenting with different values for the parameters a and b could give rich possibilities in discovery-based teaching. It will be striking how different parameters can result in magnifications, rotations or translations. Another point is that what happens in a complex mapping becomes more transparent here than in a CAS-application. In this Excel-based mapping you can almost see how a curve is mapped point by point through the table. On the contrary, in CAS you would just write a command for the mapping.

Excel-based mappings could also be used in connection with ordinary textbook problems, to check or illustrate the solution. A more comprehensive student assignment could be to construct a spreadsheet that illustrates how different curves are mapped by a complex function.

To get an overview of the complex functions in Excel, Excel's built-in help system is useful. In his book *Excel for Engineers and Scientists* the author S. C. Bloch devotes one chapter to complex functions [2]. (This chapter is on the book's accompanying CD.)

A spreadsheet that illustrates the images of different curves under various mappings

In the following I will give a brief description of an Excel workbook that shows how different types of curves are mapped by a few elementary functions. The curves chosen are straight line, circle/rounded rectangle and equilateral triangle. Figure 2 shows a sample from the workbook where the mapping is $f(z) = z^2$.To get an understanding of how the rounded rectangle or the triangle is mapped, one can experiment with the mapping of lines that correspond to their sides. The slanting line in the figure corresponds to the right side of the triangle. In the image of the line the parts in the first, second and fourth quadrants can be recognised in the image of the triangle. It looks much like an upside down fishhook. Similarly, the upright “fishhook” in the first, third and fourth quadrants is the image of the left side of the triangle. One can look at the image of a horizontal line to understand that the image of the base line of the triangle is the large arc in the second and third quadrants. (Part of a parabola).

The workbook described here can be downloaded from [1].

In this workbook one can choose between four types of elementary functions for the mapping $w = f(z)$ (linear, sine, exponential and power functions). These mappings are placed in different worksheets. In Figure 2 we have chosen the power function $f(z) = kz^n$ with $k = 1$ and $n = 2$. There is also an option where the user can easily enter their own function. (The worksheet named $f(z)$). In this case the function formula is entered into indicated cells in the table below the graphs. Finally, the formula must be copied down the columns.

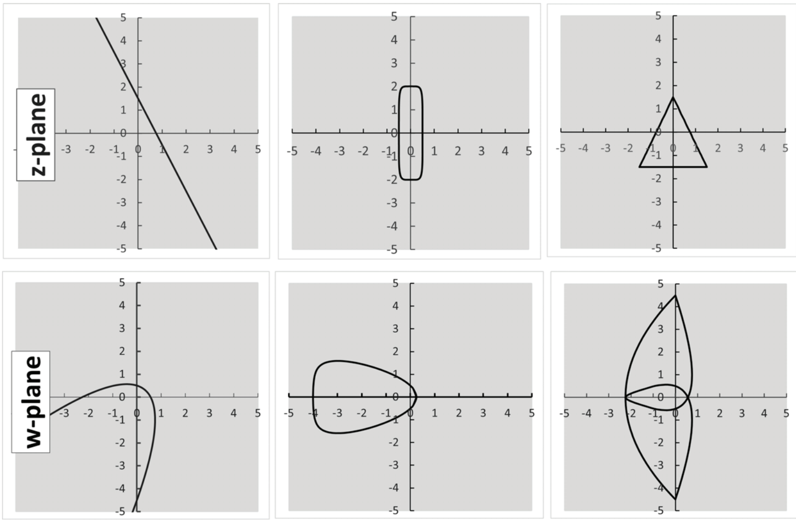

FIGURE 2: Mapping of various curves by the function $f(z) = z^2$

To create the circle/rounded rectangle we used a superellipse [3, 4] given by the parametrisation

$$x(t) = |\cos t|^{2/m} a\, sgn(\cos t),\ y(t) = |\sin t|^{2/m} b\, sgn(\sin t),\ 0 \leqslant t < 2\pi.$$

The superellipse can be viewed as a generalisation of the familiar ellipse given by $x(t) = a \cos t$, $y(t) = b \sin t$, $0 \leqslant t < 2\pi$. Due to the exponent $2/m$ we must take care especially about situations when the trigonometric functions give a negative value. Therefore the absolute value and the sign functions show up in the formula. In the workbook we have restricted m to integer values from 2 to 6 where 2 gives a circle or ellipse and 6 makes the figure look much like a rectangle. In Figure 2 the value of m is 6. As one would expect, a and b give the extreme points of the superellipse. The superellipse is a very versatile tool for creating geometric figures in that one formula can give both circles/ellipses and rounded rectangles.

The triangle is constructed just by a parametrisation of its sides.

This spreadsheet could have been made more user-friendly by making use of the programming facilities in Excel (Visual Basic for Applications). But here it was a point intended to keep things simple. Moreover one avoids the potential risks of macro viruses if the spreadsheet is distributed.

A classic application from aerodynamics

In the same spirit as in Figures 1 and 2 we could easily construct a spreadsheet that shows the image of a circle under the mapping

$$f(z) = z + \frac{k^2}{z},$$

the so called *Joukowski transformation*. (See e.g. [5]). We choose $k = 1$ and let the circle pass through the point $z = 1$ and let the point $z = -1$ be inside the circle. In Figure 3 you can see the resulting *Joukowski airfoil* for a chosen centre position and radius of the circle. In this case the Excel formula for $f(z)$ will be IMSUM(D21, IMDIV(1, D21)), where the value of z is placed in cell D21. The spreadsheet can be downloaded from [1]. The Joukowski transform has been very important in airfoil design. Experimenting with different centre positions and radii students can have many questions for investigation. The reason why it is crucial whether the circle passes through the points $z = \pm 1$ is because $f'(z) = 0$ at these points, so the transformation is not conformal there.

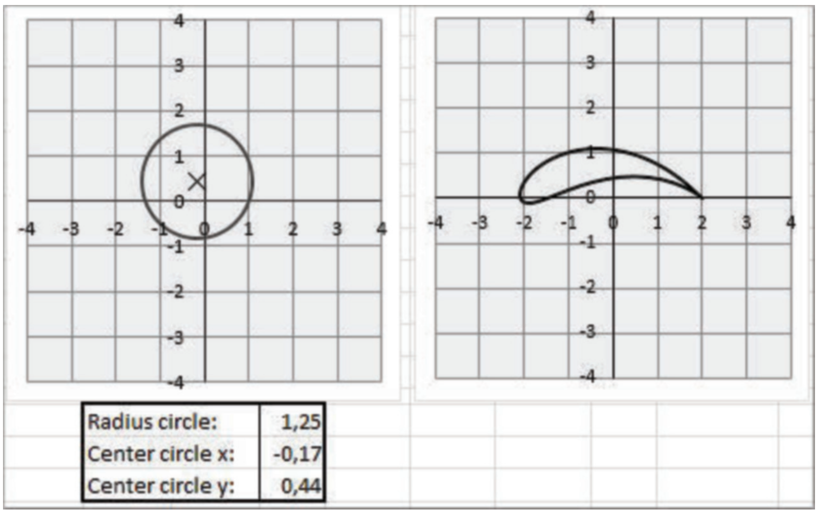

FIGURE 3: Mapping of a circle by the Joukowski transform. The cross shows the centre of the circle

Summary

It seems not to be well known that Microsoft Excel has most of the elementary functions of complex analysis included in its library of functions. This gives easy access to complex arithmetic for a broad audience. Here we show that these functions together with Excel's strong graphical capabilities are well suited to illustrate simple complex mappings. The complex functions in Excel could of course also be used in other areas of complex analysis, for example to illustrate the convergence of sequences

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graphically. Complex functions can be handled equally well in the spreadsheet Calc in the open source suite OpenOffice [6].

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