ON PARACOMPACT REGULAR SPACES

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A topological space is paracompact if and only if each open cover of the space has an open locally finite refinement. It is well-known that an unusual normality condition is satisfied by each paracompact regular space X [p. 158, 5]: Let α be a locally finite (discrete) family of subsets of X, then there is a neighborhood V of the diagonal $\Delta(X)$ (in $X \times X$), such that $V[x]$ intersects at most a finite number of members (respectively at most one member) of $\{V[A] : A \in \alpha\}$ for each $x \in X$. In this note we will show that a variant of this condition actually characterizes paracompactness. Among other results, an improvement to a recent result of H. H. Corson [2] is given so as to accord with a conjecture of J. L. Kelley [p. 208, 5] more prettily, and we connect paracompactness to metacompactness [1].

1. Definitions. A family $\{A_n : n \in D, \leqq\}$ of subsets of a topological space X is locally non-frequent if $(D, \leqq)$ is a directed system and if to each $x \in X$ there is a neighborhood V_x of x, and an $n(x) \in D$, such that $V_x \cap A_n = 0$ for $n \geqq n(x)$. (In the following we will simply say "V_x disjoint from $\{A_n : n \in D, \leqq\}$ eventually", and we write a locally non-frequent family as $\{A_n : n \in D\}$.)

A uniformity $\mathscr{U}$ is an H-uniformity for the topological space X, if $\mathscr{U}$ is compatible with X and if to each locally non-frequent family $\{A_n : n \in D\}$ of subsets of X, there is a $U \in \mathscr{U}$, such that $U[x]$ disjoint from $\{U[A_n] : n \in D\}$ eventually for each $x \in X$.

Let X be a topological space, $\mathscr{V}$ a family of neighborhoods of the diagonal $\Delta(X)$ (in $X \times X$), m a cardinal number. A net $\{S_n, n \in D\}$ in X is a $\mathscr{V}$-Cauchy-net ($c(\mathscr{V})$-net), if $\{(S_n, S_p), (n, p) \in D \times D\}$ is eventually in each member of $\mathscr{V}$. $\{S_n, D\}$ is a $\mathscr{V}$-Cauchy-m-net ($c(\mathscr{V}, m)$-net), if to each subfamily $\mathscr{W}$ of cardinal not greater than m of $\mathscr{V}$, there is a $c(\mathscr{W})$-subnet.

We make free use of the terminologies, conventions, and notations covered by [5].

2. As is well-known, there is a close parallelism between theorems (and their proofs) stated in terms of filters, and corresponding theorems (and

proofs) stated in terms of directed nets. The following lemma is a translation of one of the results of Corson [2] into our language.

LEMMA. Suppose X is a topological space such that a net in X clusters if each of its continuous images in pseudo-metric spaces clusters, then each open cover of X has a locally finite open refinement.

This lemma can either be deduced from Corson's result, or demonstrated by a proof parallel to his.

We remark also that to every regular space X corresponds a regular Hausdorff space $X^* = \{\{x\}^- : x \in X\}$, and a natural mapping ϕ (continuous, open and closed) of X onto X^*, $\phi(x) = \{x\}^-$. If X is completely regular, so is X^*, and will have a Hausdorff compactification. By the use of the mapping ϕ it is not difficult to extend another of Corson's results to non-Hausdorff spaces, as follows.

LEMMA. Suppose that X is a regular space and that, $X \times \alpha(X^*)$ is normal for a certain Hausdorff compactification $\alpha(X^*)$ of X^*. Then each c $(\mathscr{V}_{\Delta(X)}, 1)$-net in X has a cluster point in X, here $\mathscr{V}_{\Delta(X)}$ is the family of all neighborhoods of the diagonal $\Delta(X)$ in $X \times X$.

3. LEMMA. If X is a paracompact regular space, then the family $\mathscr{V}_{\Delta(X)}$ of all neighborhoods of the diagonal is an H-uniformity for X.

PROOF. That $\mathscr{V}_{\Delta(X)}$ is a uniformity compatible with X is known [p. 157, 5]. Let $\{A_n : n \in D\}$ be a locally non-frequent family of subsets of X, V_x an open neighborhood of x disjoint from $\{A_n : n \in D\}$ eventually. $\{V_x : x \in X\}$ is an open cover of X. Let W be a neighborhood of the diagonal such that $\{W[x] : x \in X\}$ is a refinement of $\{V_x : x \in X\}$ [p. 157, 5]. Let V be a symmetric neighborhood of the diagonal such that $V \circ V \subset W$. It is clear that $V[x] \cap V[A_n] \neq 0$ if and only if $V \circ V[x] \cap A_n \neq 0$, now $V \circ V[x] \subset W[x] \subset V_{x'}$ for some $x' \in X$, thus $V[x]$ disjoint from $\{V[A_n] : n \in D\}$ eventually.

LEMMA. Suppose $\mathscr{U}$ is an H-uniformity for the topological space X, then each $c(\mathscr{U}, 1)$-net in X has a cluster point.

PROOF. If $\{S_r, n \in D\}$ is a $c(\mathscr{U}, 1)$-net not having any cluster point, let $A_n = \{S_p : p \geqq n\}$, then $\{A_n : D\}$ is a locally non-frequent family. Let $U \in \mathscr{U}$, such that $U[x]$ disjoint from $\{U[A_n] : D\}$ eventually. Let $\{S_{n(e)}, e \in E\}$ be a $c(U)$-subnet of $\{S_n, D\}$ such that $(S_{n(e)}, S_{n(e')}) \in U$ for $e, e' \geqq e_0$ for some $e_0 \in E$. It is clear that $U[S_{n(e_0)}]$ intersects A_n for n arbitrarily large; this is impossible.

If γ is an open cover of a topological space X, we denote $\bigcup\{C \times C : C \in \gamma\}$ by V_γ, and by $\mathscr{V}_{pf}$ the family of all V_γ for which γ has an open point-finite refinement. The following is an analogue to compact spaces.

LEMMA. If X is a metacompact space, then each $c(\mathscr{V}_{pf}, 1)$-net has a cluster point. In particular, each $c(\mathscr{V}_{\Delta(X)}, 1)$-net has a cluster point.

PROOF. If $\{S_n, D\}$ is a $c(\mathscr{V}_{pf}, 1)$-net not having any cluster point, let $A_n = \{S_p : p \geqq n\}$, then $\{X - A_n^- : D\}$ is an open cover of X; let γ be an open point-finite refinement, and $\{S_{n(e)}, e \in E\}$ a $c(V_\gamma)$-subnet of $\{S_n, D\}$ such that $(S_{n(e)}, S_{n(e')}) \in V_\gamma$ for $e, e' \geqq e_0$ for some $e_0 \in E$. Thus $\{S_{n(e)}, E\}$ is eventually in $V_\gamma[S_{n(e_0)}]$, and since the latter is a union of a finite number of members of γ, $\{S_{n(e)}, E\}$ and hence $\{S_n, D\}$ must be frequently in some member of γ, say C_0. However, $C_0 \subset X - A_n^-$ for some n; we may choose e such that $S_{n(e)} \in C_0$, and $n(e) \geqq n$. This leads to a contradiction.

Let us agree that ω is the first infinite cardinal. As far as I know, the following is the best positive result so far obtained for the conjecture of Kelley [3].

LEMMA. If X is a topological space and, $\mathscr{U}$ is a uniformity compatible with X such that each $c(\mathscr{U}, \omega)$-net has a cluster point in X, then X is paracompact.

PROOF. Let Q be the gage of $\mathscr{U}$, $\{S_n, D\}$ a net such that its continuous image in any pseudo-metric space has a cluster point, and $\mathscr{V} = \{V_i : i \in \omega\}$ a countable subfamily of $\mathscr{U}$. Then there exists $\mathscr{W} = \{W_i : i \in \omega) \subset \mathscr{U}$, such that $W_i \subset V_i$, $W_{i+1} \circ W_{i+1} \circ W_{i+1} \subset W_i$, and W_i is symmetric for each i. It is well-known that a pseudo-metric d of Q exists, such that $W_{i+1} \subset \{(x, x') : d(x, x') < 2^{-i}\} \subset W_{i-1}$ [p. 185, 5]. Let I be the identity map on (X, Q) to (X, d). Now $\{S_n, D\} = \{I \circ S_n, D\}$ clusters in (X, d); let $\{S_{n(e)}, E\}$ be a convergent subnet of $\{S_n, D\}$ in (X, d). It is clear that $\{S_{n(e)}, E\}$ is a $c(\mathscr{V})$-subnet for $\{S_n, D\}$. Thus $\{S_n, D\}$ is a $c(\mathscr{U}, \omega)$-net in X; by hypothesis, it has a cluster point in X. Apply the first lemma of § 2, X is paracompact.

Corson proved that a Hausdorff space X is paracompact, if there is a uniformity $\mathscr{U}$ compatible with X, such that each $c(\mathscr{U}, 1)$-net has a cluster point in X.

FINAL THEOREM. For a regular space X, the following propositions are equivalent:

(i) X is paracompact.

(ii) There is an H-uniformity compatible with X.

(iii) $\mathscr{V}_{pf}$ (or $\mathscr{V}_{\Delta(X)}$) is a uniformity compatible with X, and X is metacompact.

(iv) There is a uniformity $\mathscr{U}$ compatible with X, such that each $c(\mathscr{U}, \omega)$-net has a cluster point in X.

(v) $\mathscr{V}_{\Delta(X)}$ is a uniformity compatible with X and $X \times \alpha(X^*)$ is normal for some Hausdorff compactification $\alpha(X^*)$ of X^*.

All the implications required to prove these equivalences are either proved above or are already known (cf. Corson [2]).

References

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