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ON THE SEMIGROUP OF PROBABILITY MEASURES OF A LOCALLY COMPACT SEMIGROUP II

BY

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ABSTRACT. This is a sequel to the author's paper "On the semigroup of probability measures of a locally compact semigroup." We continue to investigate the relationship between amenability of spaces of functions and functionals associated with a locally compact semigroups S and its convolution semigroup $M_0(S)$ of probability measures and fixed point properties of actions of S and $M_0(S)$ on compact convex sets.

1. **Introduction**. Let S be a locally compact semigroup with convolution measure algebra $M(S)$ and probability measures $M_0(S)$. In Wong [21], it is shown that $M(S)^*$ has a topological left invariant means iff the space of all "uniformly" left uniformly continuous functions on the convolution semigroup $S_1 = M_0(S)$ has a left invariant mean or equivalently, iff S_1 has the fixed point property for uniformly continuous affine actions of S_1 on compact convex sets. (S_1 having topology $\sigma(M(S), M(S)^*)$). In this paper, we consider certain subspaces X of $M(S)^*$ and obtain some interesting results. In particular, we show that if $X = AF(S)$, the space of all almost periodic functionals in $M(S)^*$, then X has a topological left invariant mean iff S_1 has the fixed point property for equi-uniformly continuous affine actions of S_1 on compact convex sets. (S_1 having topology $\sigma(M(S), X)$). A similar result is obtained for the space $X = AP(S)$, the almost periodic functions on S (considered as a subspace of $M(S)^*$). These are analogues of a fixed point theorem of Lau in [11]. Other subspaces of $M(S)^*$ are considered together with the corresponding fixed point properties. Moreover, we also investigate the relationship between certain affine actions of the semigroups S and $S_1 = M_0(S)$, unifying their corresponding fixed point theorems.

2. **Main Results**. We shall follow Wong [21] for definitions and terminologies not explained here. Let S be a locally compact semi-topological semigroup with convolution measure algebra $M(S)$ and convolution semigroup $S_1 = M_0(S)$ of probability measures. A functional $F \in M(S)^*$ is called almost periodic if the set $\{l_\mu F : \mu \in S_1\}$ is relatively compact in the norm topology of $M(S)^*$, where $l_\mu F$ is defined by $l_\mu F(\nu) = F(\mu * \nu)$, $\nu \in M(S)$. Let $AF(S)$ denote the space of all such functionals. Define a map $\tau: M(S)^* \to m(S_1)$ (the bounded functions on S_1 with supremum norm) by $\tau(F)(\mu)$

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$= F(\mu)$, $F \in M(S)^*$, $\mu \in S_1$. Let S_1 have the topology $\sigma(M(S), AF(S))$ and let $CB(S_1)$ be the space of all continuous bounded functions on S_1. Clearly $\tau(AF(S)) \subset CB(S_1)$. We first present the following Lemma whose (straightforward) proof is omitted.

LEMMA 2.1: *(1) $AF(S)$ is a topological left introverted topological left and right invariant linear subspace of $M(S)^*$ containing the constant functionals.*

(2) S_1 is a semi-topological semigroup under the topology $\sigma(M(S), AF(S))$.

(3) $\tau : M(S)^* \to m(S_1)$ is one to one, bounded linear, $\tau \geq 0$, $\tau(1) = 1$ and $\tau(l_\mu F) = l_\mu \tau(F)$, $\mu \in S_1$, $F \in M(S)^*$. Moreover, $\tau(AF(S)) \subset AP(S_1)$, the almost periodic functions on S_1. As a result:

(4) $F \in AF(S)$ iff $\{r_\mu F : \mu \in S_1\}$ is relatively compact in the norm topology of $M(S)^*$.

As in Wong [21], let $\mathcal{LUC}(S_1)$ be the space of all functions $f \in CB(S_1)$ such that the map $\mu \to l_\mu f$ of S_1 into $CB(S_1)$ is uniformly continuous with respect to the uniformity on S_1 induced by $\sigma(M(S), AF(S))$ and that on $CB(S_1)$ induced by the supremum norm. Also an affine action $T : S_1 \times K \to K$ (where $(\mu, x) \to T_\mu(x)$) on the compact convex set K is $\sigma(M(S), AF(S))$ equi-uniformly continuous if for each $U \in \mathcal{U}$, the unique uniformity on K, there is some *nbhd* N in $\sigma(M(S), AF(S))$ such that $\mu, \nu \in S_1$ and $\mu - \nu \in N$ implies $(T_\mu(x), T_\nu(x)) \in U$ for all $x \in K$. It is called equicontinuous on K if for each $x \in K$ and each $U \in \mathcal{U}$, there is some $V \in \mathcal{U}$ such that $(T_\mu x, T_\mu y) \in U \; \forall \, (x, y) \in V$ and $\mu \in S_1$. Note that continuity of each T_μ is part of the definition of an action (Wong [21, §2] and Lau [11, §3]).

THEOREM 2.2. *The following statements are all equivalent.*

(1) $AF(S)$ has a topological left invariant mean (TLIM).

(2) $\mathcal{LUC}(S_1)$ has a left invariant mean (LIM).

(3) S_1 has the fixed point property for $\sigma(M(S), AF(S))$ equi-uniformly continuous affine actions of S_1 on compact convex sets.

(4) S_1 has the fixed point property for $\sigma(M(S), AF(S))$ equi-uniformly continuous and equicontinuous affine actions of S_1 on compact convex sets.

PROOF.

(1) implies (2).

Suppose $AF(S)$ has a *TLIM*. Then by Wong [21, Theorem 2.1] applied to $X = AF(S)$, $\mathcal{WLUC}(S_1)$ has a *LIM* where $\mathcal{WLUC}(S_1)$ is the space of $f \in CB(S_1)$ such that the map $\mu \to l_\mu f$ is uniformly continuous when $CB(S_1)$ has the uniformity induced by the weak topology $\sigma(CB(S_1), CB(S_1)^*)$. Hence $\mathcal{LUC}(S_1) \subset \mathcal{WLUC}(S_1)$ also has a *LIM*.

(2) implies (3).

We first show that right multiplications in $M_0(S)$ are equi-uniformly continuous under the topology $\sigma(M(S), AF(S))$. Define $R_\theta : M_0(S) \to M_0(S)$ by $R_\theta(\mu) = \mu * \theta$, $\mu, \theta \in S_1 = M_0(S)$. Given $N = \{\nu \in M(S) : |F(\nu)| < \epsilon\}$ where $\epsilon > 0$ and $F \in AF(S)$.

By Lemma 2.1 (4), $\{r_\theta F: \theta \in S_1\}$ is totally bounded. Hence there exist $\theta_1, \theta_2, \ldots, \theta_n \in S_1$ such that $\forall\, \theta \in S_1$, $\|r_\theta F - r_{\theta_j} F\| < \epsilon/4$ for some $1 \leq j \leq n$. Define $M = \bigcap_{i=1}^{n} \{\nu \in M(S): |r_{\theta_i}(F)(\nu)| < \epsilon/2\}$, a *nbhd* in $(\sigma M(S), AF(S))$. $\forall\, \theta \in S_1$, $\mu, \nu \in M_0(S)$, $\mu - \nu \in M$, we have $|F(R_\theta(\mu) - R_\theta(\nu))| = |r_\theta F(\mu - \nu)| \leq |r_\theta F(\mu - \nu) - r_{\theta_j} F(\mu - \nu)| + |r_{\theta_j} F(\mu - \nu)| \leq \|r_\theta F - r_{\theta_j} F\| \cdot \|\mu - \nu\| + |r_{\theta_j} F(\mu - \nu)| \leq (\epsilon/4) \cdot 2 + \epsilon/2 = \epsilon$ or $R_\theta(\mu) - R_\theta(\nu) \in N$. That is, $\{R_\theta: \theta \in S_1\}$ are equi-uniformly continuous. This together with equi-uniform continuity of the affine action $T: S_1 \times K \to K$ implies that T is an A-representation of the pair S_1, $\mathcal{LUC}(S_1)$ in the sense of Argabright [1]. Hence by [1, Theorem 1], T has a common fixed point (cf. Wong [21, Theorem 2.2 (2) implies (3)]).

(3) implies (4)
obvious

(4) implies (1)
Let $W = AP(S_1) \cap \mathcal{LUC}(S_1)$. Then W is a left introverted, left invariant linear subspace of $CB(S_1)$ containing the constants. Let K be the set of all means on W with weak* topology $\sigma(W^*, W)$ and consider the action $S_1 \times K \to K$ defined by $(\mu, m) \to l_\mu^* m$, $\mu \in S_1$, $m \in K$. Since $W \subset \mathcal{LUC}(S_1)$, the definition of $\mathcal{LUC}(S_1)$ shows that this action is $\sigma(M(S), AF(S))$ equi-uniformly continuous. Since $W \subset AP(S_1)$, the arguments used in Lau [11, Theorem 3.2] applied to S_1 show that the same action is equicontinuous. By assumption, this action has a fixed point m which is a left invariant mean on W. We now show that $\tau: AF(S) \to W$. Clearly $f = \tau(F) \in AP(S_1)$ $\forall\, F \in AF(S)$, by Lemma 2.1 (3). To show that $f \in \mathcal{LUC}(S_1)$, let $\epsilon > 0$ be given. As before, total boundedness of $\{r_\theta F: \theta \in S_1\}$ implies that there is some *nbhd* M in $\sigma(M(S), AF(S))$ such that $\mu, \nu \in S_1$, $\mu - \nu \in M$ implies $\|l_\mu f - l_\nu f\| = \sup\{|f(m * \theta) - f(\nu * \theta)|: \theta \in S_1\} = \sup\{|r_\theta F(\mu - \nu)|: \theta \in S_1\} < \epsilon$ or $f \in \mathcal{LUC}(S_1)$. Therefore $\tau^*: W^* \to AF(S)^*$ and $\tau^* m$ is the a topological left invariant mean on $AF(S)$. This completes the proof.

REMARKS

(a) Theorem 2.2 (1) $\Leftrightarrow$ (3) is an analogue of a fixed point theorem in Lau [11, Theorem 3.2, p. 71] where equicontinuity of the action is on K while the equi (uniform) continuity of the action in Theorem 2.2 (3) is on $S_1 = M_0(S)$. Also Theorem 2.2. ((2) implies (1)) sharpens the result in Wong [21, Theorem 2.1, (2) implies (1)] for the case $X = AF(S)$.

(b) There are versions of Theorem 2.2 for actions of S_1 on compact Hansdorff spaces and multiplicative left invariant means on $\mathcal{LUC}(S_1)$. We omit the details. Note that $\mathcal{LUC}(S_1)$ is an algebra under pointwise operations but in general neither $\mathcal{WLUC}(S_1)$ nor $M(S)^*$ is an algebra under pointwise operations.

(c) We can also consider $X = WAF(S)$, the weakly almost periodic functionals (obvious definition). Theorem 2.2 remains valid if we replace "$AF(S)$" by "$WAF(S)$", "$\mathcal{LUC}(S_1)$" by $\mathcal{WLUC}(S_1)$", "$\sigma(M(S), AF(S))$" by $\sigma(M(S), WAF(S))$", "equi-uniformly continuous" by "uniformly continuous" and "equicontinuous" by "quasi-

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equicontinuous" throughout. See Berglund, Junghenn and Milnes [2] for definition of quasi-equicontinuous actions). The difference is that unlike the case for $AF(S)$, we can only show $\tau(WAF(S)) \subset \mathcal{WLUC}(S_1) \cap WAP(S_1)$.

In order to obtain similar fixed point theorems for subspaces $X \subset M(S)^*$ arising from continuous bounded functions in $CB(S)$, and unify the fixed point properties and theorems obtained in Mitchell [13], Lau [11] and Wong [21], we first embed $CB(S)$ in $M(S)^*$ by integration. Define a map $\phi : CB(S) \to M(S)^*$ by $\phi(f)(\mu) = \int f d\mu$, $f \in CB(S)$, $\mu \in M(S)$. Also let $WLUC(S)$ and $WAP(S)$ be respectively the space of all weakly left uniformly continuous and weakly almost periodic functions in $CB(S)$ as defined in Mitchell [13] and Lau [11]. If $f \in CB(S)$ and $\mu \in M(S)$, we define $l_\mu f \in CB(S)$ by $l_\mu f(s) = \int r_s f \, d\mu = \int f(ts) \, d\mu(t)$. Similarly for $r_\mu f$ (see Glicksberg [7] and Williamson [15]). We have

LEMMA 2.3. *Φ is a linear isometry of $CB(S)$ into $M(S)^*$ such that $\Phi \geq 0$, $\Phi(1) = 1$ and Φ commutes with left and right convolutions, i.e. $\Phi(l_\mu f) = l_\mu \Phi(f)$, $\Phi(r_\mu f) = r_\mu(\Phi f)$, $f \in CB(S)$, $\mu \in M(S)$.*

PROOF: Straightforward verification.

LEMMA 2.4. *(1) $AP(S) \subset WAP(S) \subset WLUC(S)$ are all left introverted, left and right invariant linear subspaces of $CB(S)$ containing the constants. (2) If $m \in CB(S)^*$, then $\int m_l(f) \, d\mu = m(l_\mu f)$ for any $f \in WLUC(S)$ where $m_l(f)(s) = m(l_s f)$, $s \in S$. (3) $AP(S)$, $WAP(S)$ and $WLUC(S)$ (or more precisely, their images under Φ) are topological left introverted, topological left and right invariant linear subspaces of $M(S)^*$ containing the constant functionals.*

PROOF: (1) is well known, see for example Berglund, Junghenn and Milnes [2]. For (2) and (3), see Kharaghani [10, § 2.2 and 2.3]. Note that semigroups in [10] are jointly continuous. But the corresponding results there are valid for semi-topological semigroups because of the convolution formula in Wong [19] (which also implies the identity $\Phi(l_\mu f) = l_\mu \Phi(f)$ of Lemma 2.3).

We shall often identify $AP(S)$, $WAP(S)$ and $WLUC(S)$ with their images under Φ.

Next, we investigate the relationship between certain actions of S and $S_1 = M_0(S)$ on compact convex sets and actions of the algebra $M(S)$ on separated convex spaces. An action $T : M_0(S) \times K \to K$ is called bi-affine if it is affine and if $T_{\alpha\mu + \beta\nu}(x) = \alpha T_\mu(x) + \beta T_\nu(x)$ for all $x \in K$, μ, $\nu \in M_0(S)$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. An action of the algebra $M(S)$ on a separated convex space E is a bilinear map $T : M(S) \times E \to E$ such that $T_{\mu * \nu} = T_\mu \circ T_\nu$ for any μ, $\nu \in M(S)$ where $T_\mu : E \to E$ is continuous for each $\mu \in M(S)$. If K is a compact convex $M_0(S)$-invariant subset of E (i.e. $T_\mu(K) \subset K$ for all $\mu \in M_0(S)$), then T induces a bi-affine action $T : M_0(S) \times K \to K$ of $M_0(S)$ on K but not every bi-affine action of $M_0(S)$ comes from action of the algebra $M(S)$. Let $T : S \times K \to K$ be any separately continuous affine action of S on a compact convex subset K of a separated convex space E. We shall show that T can be extended to a $\sigma(M(S), WLUC(S))$ uniformly continuous bi-affine action of S_1 on K and that every such action of S_1 arises in this manner. But first, we have

THEOREM 2.5. *Each* $\sigma(M(S), WLUC(S))$ *separately continuous bi-affine action of* $S_1 = M_0(S)$ *is* $\sigma(M(S), WLUC(S))$ *uniformly continuous.*

PROOF. For brevity, let $X = WLUC(S)$. Let $T : S_1 \times K \to K$ be a $\sigma(M(S), X)$ separately continuous bi-affine action of S_1. Consider the restriction $T : S \times K \to K$ to S (where $T_s(x) = T_{\delta(s)}(x)$) which is a separately continuous affine action of S (because functions in X are continuous). By Mitchell [13, Theorem 4] for each $h \in A(K)$, $x \in K$, the function $T_x h$ is in $X = WLUC(S)$ where $T_x h(s) = hT_s(x)$, $s \in S$. We claim that $h(T_\mu(x)) = \int T_x h \, d\mu = \int h(T_s(x)) \, d\mu(s)$. If $\mu = \delta(s)$, this is clear. In general, $\mu \in M_0(S)$ can be regarded as a mean on X. Hence there is a net of finite convex combinations of Dirac measures μ_α such that $\mu_\alpha \to \mu$ in $\sigma(M(S), X)$. Since T is bi-affine and h is affine, $h(T_{\mu_\alpha}(x)) = \int T_x h \, d\mu_\alpha$. So $h(T_\mu(x)) = \lim_\alpha h(T_{\mu_\alpha}(x)) = \lim_\alpha \int T_x h \, d\mu_\alpha = \int T_x h \, d\mu = \int h(T_s(x) \, d\mu(s)$. (Since T is $\sigma(M(S), X)$ separately continuous, h is continuous and $T_x h \in X$.) To show that $T : S_1 \times K \to K$ is $\sigma(M(S), X)$ uniformly continuous, fix $x \in K$ and let $U = \{y \in E : |x^*(y)| < \varepsilon\}$ where $x^* \in E^*$ and $\varepsilon > 0$. Define $h = x^*|_K \in A(K)$ and $N = \{\theta \in M(S) : |\theta(T_x h)| < \varepsilon\}$. Then N is a *nbhd* in $\sigma(M(S), X)$. Moreover, for all $\mu, \nu \in S_1$, $\mu - \nu \in N$, we have $|x^*(T_\mu(x) - T_\nu(x))| = |x^*(T_\mu(x)) - x^*(T_\nu(x))| = |h(T_\mu(x)) - h(T_\nu(x))| = |\int T_x h \, d\mu - \int T_x h \, d\nu| = |T_x h(\mu - \nu)| < \varepsilon$. That is, the action $T : S_1 \times K \to K$ is $\sigma(M(S), X)$ uniformly continuous. (Note that since K is compact, the weak topology $\sigma(E, E^*)$ of E induces the unique uniformity on K.)

REMARKS. (a) A $\sigma(M(S), X)$ separately continuous affine action may not be $\sigma(M(S), X)$ uniformly continuous if it is not bi-affine.

(b) If $X \subset M(S)^*$ contains functionals not coming from functions in $CB(S)$, then the restriction of a $\sigma(M(S), X)$ separately continuous affine action of S_1 to S may not be separately continuous.

THEOREM 2.6. *Let* $S_1 = M_0(S)$ *have the topology* $\sigma(M(S), X)$ *where* $X = WLUC(S)$. *Each separately continuous affine action* $T : S \times K \to K$ *of* S *extends uniquely to a* $\sigma(M(S), X)$ *separately (hence uniformly) continuous bi-affine action* $T : S_1 \times K \to K$ *of* S_1 *such that* $h(T_\mu(x)) = \int T_x h \, d\mu$ *for any* $x \in K$, $h \in A(K)$ *and* $\mu \in S_1$. *Conversely, every such action of* S_1 *is the extension of a separately continuous action of* S. *Moreover, the two actions have the same fixed points (if any).*

PROOF. Since $h \to \int T_x h \, d\mu$ is a mean on $A(K)$ and $A(K)$ separates points, by Argabright [1, Lemma 2, p. 127], the formula $h(T_\mu(x)) = \int T_x h \, d\mu$ defines $T_\mu(x) \in K$. Also, each T_μ is affine. So is $\mu \to T_\mu(x)$, $x \in K$. By the convolution formula in Wong [19], $T_{\mu * \nu} = T_\mu \circ T_\nu$. To show that each T_μ is continuous, suppose $x_\alpha \to x$ in K. Then $T_{x_\alpha} h \to T_x h$ pointwise in $CB(S)$. But $\{T_x h : x \in K\} \subset CB(S)$ is norm bounded (in fact, $\|T_x h\| \leq \|h\|$) and is pointwise compact (since the map $x \to T_x h$ of K into $CB(S)$ is pointwise continuous). By Glicksberg [7, Theorem 1.1] $\{T_x h : x \in K\}$ is weak* compact in $M(S)^*$ and the two topologies coincide on $\{T_x h : x \in K\}$. Hence $T_{x_\alpha} h \to T_x h$ weak* in $M(S)^*$. That is $\int T_{x_\alpha} h \, d\mu \to \int T_x h \, d\mu$ or $h(T_\mu(x_\alpha)) \to h(T_\mu(x))$ for each $h \in A(K)$, $\mu \in S_1$ or T_μ is continuous. Finally, to show that $T : S_1 \times K \to K$ is $\sigma(M(S),$

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X) uniformly continuous (necessarily separately continuous), we can repeat the arguments used in the proof of the preceeding theorem, which depend only on the formula $h(T_\mu(x)) = \int T_x h \, d\mu$. Uniqueness follows from the fact that a $\sigma(M(S), X)$ separately continuous bi-affine action of S_1 is completely determined by its restriction to S. The converse is now established by restriction and the last part of the theorem follows from the fact that the extension $T : S_1 \times K \to K$ is bi-affine and $\sigma(M(S), X)$ continuous in μ for fixed x.

We now unify all fixed point properties which characterise left amenability of $WLUC(S)$ in the following theorem.

THEOREM 2.7. *The following statements are all equivalent.*

(1) $WLUC(S)$ has a LIM.

(2) S has the fixed point property for separately continuous affine actions of S on compact convex sets.

(3) $WLUC(S)$ has a TLIM.

(4) $\mathcal{WLUC}(S_1)$ has a LIM (S_1 has the topology $\sigma(M(S), WLUC(S))$.

(5) S_1 has the fixed point property for $\sigma(M(S), WLUC(S))$ uniformly continuous affine actions of S_1 on compact convex sets.

(6) S_1 has the fixed point property for $\sigma(M(S), WLUC(S))$ separately continuous bi-affine actions of S_1 on compact convex sets.

(7) S has the fixed point property (on compacta) for $\sigma(M(S), WLUC(S))$ separately continuous (bilinear) action of $M(S)$ on separated convex spaces. (See Wong [21, Theorem 2.3] for fixed point property of actions of the algebra $M(S)$).

PROOF. By Mitchell [13, Theorem 4], (1) and (2) are equivalent. So are (3), (4) and (5) by Wong [21, Theorems 2.1 and 2.2]. Also (1) and (3) are equivalent by Lemma 2.5 (2) (each *LIM* on $WLUC(S)$ is also a *TLIM*). Finally, (5) is formally stronger than (6) (Theorem 2.6) which in turn is formally stronger than (7). By Wong [21, Theorem 2.3], (7) implies (3). This completes the proof.

REMARK. (a) One can also use Theorem 2.6 to establish the equivalence (2) $\Leftrightarrow$ (6). Indeed because of Theorem 2.6 and the fact that all left invariant means on $WLUC(S)$ are topological left invariant, the equivalence (3) $\Leftrightarrow$ (6) is the most natural analogue of Mitchell's fixed point theorem [13, Theorem 4] (i.e. (1) $\Leftrightarrow$ (2) for the case of locally compact semigroups.

(b) It is interesting to note that by Mitchell's fixed point theorem [13, Theorem 4] applied to the semi-topological semigroup $S_1 = M_0(S)$ under the topology $\sigma(M(S), WLUC(S))$, $WLUC(S_1)$ has a *LIM* iff S_1 has the fixed point property for $\sigma(M(S), WLUC(S))$ separately continuous affine actions of S_1 on compact convex sets while the equivalence (4) $\Leftrightarrow$ (6) of Theorem 2.7 characterises left amenability of $\mathcal{WLUC}(S_1) \subset WLUC(S_1)$ in terms of fixed point property for bi-affine actions of S_1 of the same sort.

(c) Suppose S is a locally compact group, then $WLUC(S) = LUC(S)$ and every separately continuous affine action is jointly continuous (see Mitchell [13, §5]). Therefore each of the seven conditions in Theorem 2.8 is equivalent to the amenability of the locally compact group S (see Greenleaf [8]).

(d) If S is a compact semi-topological semigroup then $WLUC(S) = CB(S) = C_0(S)$ (Berglund, Junghenn and Milnes [2, p. 126]) and $S_1 = M_0(S)$ is also a compact semitopological semigroup in the topology $\sigma(M(S), WLUC(S)) = \sigma(M(S), C_0(S))$ (Glicksberg [7]). Moreover, $\mathcal{WLUC}(S_1) = WLUC(S_1) = CB(S_1)$. Hence each of the seven conditions in Theorem 2.7 is equivalent to $CB(S)$ or $CB(S_1)$ having a *LIM*.

For the space $AP(S)$ of almost periodic functions on S, it is known that, $AP(S)$ has a *LIM* iff S has the fixed point property for separately continuous and equicontinuous affine actions of S on compact convex sets (Lau [11, Theorem]). Now let $T : S \times K \to K$ be such an action. As in Theorem 2.6, T has an extension $T : S_1 \times K \to K$ to $S_1 = M_0(S)$ which is a $\sigma(M(S), AP(S))$ uniformly continuous bi-affine action of S_1. However, in this case, we shall show (Theorem 2.9 below) that the extension is actually $\sigma(M(S), AP(S)$ equi-uniformly continuous and equicontinuous.

THEOREM 2.8. *Each $\sigma(M(S), AP(S))$ separately continuous and equicontinuous bi-affine action of $S_1 = M_0(S)$ is $\sigma(M(S), AP(S))$ equi-uniformly continuous.*

PROOF: The first part of the proof is similar to that of Theorem 2.6. Put $X = AP(S)$ and as before consider the restriction $T : S \times K \to K$ of a $\sigma(M(S), AP(S))$ separately continuous and equicontinuous bi-affine action of S_1 where $T_s(x) = T_{\delta(s)}(x)$, $s \in S$, $x \in K$. This affine action of S is clearly separately continuous and equicontinuous. Hence by Lau [11, Lemma 3.1], $T_x h \in AP(S)$ for any $x \in K$, $h \in A(K)$. The same arguments used in the proof of Theorem 2.6 show that $hT_\mu(x) = \int T_x h \, d\mu \ \forall \ \mu \in M_0(S)$, $x \in K$ and $h \in A(K)$. To show that $T : S_1 \times K \to K$ is $\sigma(M(S), AP(S))$ equi-uniformly continuous, let $U = \{y \in E : |x^*(y)| < \varepsilon\}$ where $x^* \in E^*$ and $\varepsilon > 0$. Put $h = x^*|_K \in A(K)$. By Lau [11, Lemma 3.1], the map $x \to T_x h$ of K into $CB(S)$ is norm continuous. Hence $\{T_x h : x \in K\}$ is compact hence totally bounded in norm. Therefore there exist $x_1, x_2, \ldots, x_n \in K$ such that for any $x \in K$, $\|T_x h - T_{x_j} h\| < \varepsilon/4$ for some $1 \leq j \leq n$. Define $N = \{\theta \in M(S) : \sup_{1 \leq i \leq n} |T_{x_i} h(\theta)| < \varepsilon/2\}$ which is a *nbhd* in $\sigma(M(S), AP(S))$. Moreover, for all $\mu, \nu \in S_1$, $\mu - \nu \in N$ and $x \in K$, we have $|x^*(T_\mu(x) - T_\nu(x))| = |x^* T_\mu(x) - x^* T_\nu(x)| = |h(T_\mu(x)) - h(T_\nu(x))| = |\int T_x h \, d(\mu - \nu)| = |T_x h(\mu - \nu)| \leq |(T_x h - T_{x_j} h)(\mu - \nu)| + |T_{x_j} h(\mu - \nu)| \leq (\varepsilon/4) \cdot 2 + \varepsilon/2 = \varepsilon$. That is, the action $T : S_1 \times K \to K$ is $\sigma(M(S), AP(S))$ equi-uniformly continuous.

THEOREM 2.9. *Let $S_1 = M_0(S)$ have the topology $\sigma(M(S), X)$ where $X = AP(S)$. Each separately continuous and equicontinuous affine action $T : S \times K \to K$ of S extends uniquely to a $\sigma(M(S), X)$ separately continuous and equicontinuous (hence necessarily $\sigma(M(S), X)$ equi-uniformly continuous) bi-affine action $T : S_1 \times K \to K$ of S_1 such that $h(T_\mu(x)) = \int T_x h \, d\mu$ for $h \in A(K)$, $x \in K$, $\mu \in S_1$. Conversely, every such action of S_1 is the extension of a separately continuous and equicontinuous action of S. Moreover, the two actions have the same fixed points (if any).*

PROOF. As in the proof of Theorem 2.7, it is clear that the formula $h(T_\mu(x)) =$

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$\int T_x h\,d\mu$, $h \in A(K)$, $x \in K$, $\mu \in S_1$ defines a bi-affine action of S_1 which extends $T : S \times K \to K$. We first show that this extension is also equicontinous. Let $U = \{y \in E : |x^*(y)| < \varepsilon\}$ where $x^* \in E^*$ and $\varepsilon > 0$. Put $h = x^*|_K \in A(K)$. Then h is uniformly continuous on K. Hence there exists $W \in \mathcal{U}$, the unique uniformity on K such that $(p, q) \in W$ implies $|h(p) - h(q)| < \varepsilon/2$. By equicontinuity of the action $T : S \times K \to K$ on S at $x_0 \in K$, there is some $V \in \mathcal{U}$ such that $(T_s(x), T_s(x_0)) \in W$ whenever $(x, x_0) \in V$ and $s \in S$. Hence $(x, x_0) \in V$ implies $|h(T_s(x) - h(T_s(x_0))| < \varepsilon/2\ \forall\ s \in S$ or $|h(T_\mu(x) - h(T_\mu(x_0)| = |\int hT_s(x)\,d\mu(s) - \int hT_s(x_0)\,d\mu(s)| < \varepsilon\ \forall\ \mu \in M_0(S)$. That is, $(T_\mu(x), T_\mu(x_0)) \in U$ for any $\mu \in M_0(S)$, and $T : S_1 \times K \to K$ is equicontinuous at x_0. To show that the extension is $\sigma(M(S), AP(S))$ equi-uniformly continuous, we follow the arguments used in the proof of Theorem 2.8 which depend only on the separate and equicontinuity of $T : S \times K \to K$ and the formula $h(T_\mu(x)) = \int T_x h\,d\mu$. The rest of the proof is now trivial.

THEOREM 2.10. *The following statements are all equivalent.*

(1) $AP(S)$ has LIM.

(2) S has the fixed point property for separately continuous and equicontinuous affine action of S on compact convex sets.

(3) $AP(S)$ has TLIM.

(4) $\mathcal{LUC}(S_1)$ has a LIM (S_1 with the topology $\sigma(M(S), AP(S))$.

(5) S_1 has the fixed point property for $\sigma(M(S), AP(S))$ equi-uniformly continuous affine actions of S_1 on compact convex sets.

(6) S_1 has the fixed point property for $\sigma(M(S), AP(S))$ equi-uniformly continuous and equicontinuous affine actions of S_1 on compact convex sets.

(7) S has the fixed point property (on compacta) for $\sigma(M(S), AP(S))$ separately continuous (bilinear) actions of the measure algebra $M(S)$ on separated convex spaces.

PROOF. (1) and (2) are equivalent by Lau [11, Theorem 3.2]. The equivalence of (3), (4), (5) and (6) can be established by using the same arguments in theorem 2.2 (equivalence of (1), (2), (3) and (4)) and replacing $AF(S)$ there by $\Phi(AP(S))$ where Φ is the natural embedding $\Phi : CB(S) \to M(S)^*$ (See Lemma 2.3). Note that $f \in AP(S)$ iff $\{r_\mu f : \mu \in M_0(S)\}$ is relatively compact in the sup norm topology of $CB(S)$. Clearly (1) and (3) are equivalent. So are (3) and (7) by Wong [21, Theorem 2.3, (1) $\Leftrightarrow$ (4) with $X = AP(S)$]. This completes the proof.

REMARKS. (a) Theorems 2.8, 2.9 and 2.10 have their analogues for $WAP(S)$. Indeed, if we replace "$AP(S)$" by "$WAP(S)$", "$\mathcal{LUC}(S_1)$" by "$\mathcal{WLUC}(S_1)$", "equicontinuous" by "quasi-equicontinuous" and "equi-uniformly continuous" by "uniformly continuous" throughout, then theorems 2.8, 2.9 and 2.10 remain valid.

(b) We are unable to obtain analogues of Theorems 2.5, 2.6 and 2.7 for $LUC(S)$, the left uniformly continuous functions. More precisely, it is not known if every jointly continuous affine action $T : S \times K \to K$ has a "jointly continuous" extension $T : M_0(S) \times K \to K$.

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