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UNIFORM PARTITION AND THE BEST LEAST-SQUARES PIECEWISE POLYNOMIAL APPROXIMATION

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It is shown that the best least-squares piecewise n degree polynomial approximation of x^{n+1} over $[a, b]$ is obtained for a uniform partition. Moreover the approximation is continuous for n odd and discontinuous, with equal stepsizes at the nodes, for n even.

The problem considered here has been introduced by Stone [12] for $n = 1$. In this paper Stone has considered the least-squares continuous piecewise linear approximation of a function $f(\cdot)$ over $[a, b]$. For a quadratic function $f(x) = px^2 + qx + r$, or essentially for $f(x) = x^2$, his result states that the optimal solution is obtained for the uniform partition and the global solution is given by the solution of a least-squares problem on each subinterval. For a general function $f(\cdot)$ he has proposed an iterative method for solving the necessary optimality conditions. The problem for $n = 1$ has also been considered by Ream [10], Bellman [2], Gluss [6], Cantoni [3], Tomeck [13] and others. It is also related to the polygonal approximation of data for computer vision, graphics and image processing (see [7], [8] and [9]).

Let

$$\prod_N = \{\Delta = \{x_i\}_{i=0}^N \mid a = x_0 < \ldots < x_i < \ldots < x_N = b\}$$

be the set of all partitions Δ of $[a, b]$ into exactly N intervals. Let $\mathcal{P}^n[x_{i-1}, x_i]$ be the set of all polynomials of degree at most n defined over $[x_{i-1}, x_i]$ and let

$$\mathcal{P}_N^n = \prod_{i=1}^N \mathcal{P}^n[x_{i-1}, x_i].$$

The object of this note is to show that the minimum of

$$F\left(\Delta, \vec{p}\right) = \sum_{i=1}^N \int_{x_{i-1}}^{x_i} \left(x^{n+1} - p_i(x)\right)^2 dx,$$

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subject to $\Delta \in \Pi_N$ and $\vec{p} = (p_1(\cdot), \ldots, p_N(\cdot)) \in \mathcal{P}_N^n$, is obtained for the uniform partition $\Delta^* = \{x_i^* = a + i(b-a)/N\}_{i=0}^N$. Moreover the optimal $p_i^*(\cdot)$'s form a continuous approximation for n odd or a discontinuous approximation with equal stepsizes at the nodes for n even. The optimal $\vec{p}^* = (p_1^*(\cdot), \ldots, p_N^*(\cdot))$ is called the best least-squares piecewise n degree polynomial approximation of x^{n+1} over $[a, b]$.

The necessary conditions for the optimality of p and Δ are:

(A)
$$\int_{x_{i-1}}^{x_i} (x^{n+1} - p_i(x))q(x)dx = 0 \text{ for all } q(\cdot) \in \mathcal{P}^n[x_{i-1}, x_i] \text{ and } i = 1, \ldots, N;$$
(B)
$$[x_i^{n+1} - p_i(x_i)]^2 - [x_i^{n+1} - p_{i+1}(x_i)]^2 = 0 \text{ for all } i = 1, \ldots, N-1.$$

Before considering (A) and (B), let us recall some properties of the Legendre polynomials $\widehat{\pi}_\ell(\cdot)$ $(\ell = 0, \ldots, k)$ defined on $[-1, 1]$. They form an orthogonal basis of $\mathcal{P}^k[-1, 1]$ with respect to the usual scalar product

$$(\widehat{p}(\cdot), \widehat{q}(\cdot)) = \int_{-1}^{1} \widehat{p}(\xi)\widehat{q}(\xi)d\xi.$$

To obtain an orthogonal basis for $\mathcal{P}^k[x_{i-1}, x_i]$ we consider the transformation

$$x \longrightarrow \xi = T_i(x) : [x_{i-1}, x_i] \longrightarrow [-1, 1]$$

where
$$T_i(x) = \frac{(x - x_{i-1}) - (x_i - x)}{(x_i - x_{i-1})},$$

and the polynomials $\pi_{\ell,i}(\cdot) = \widehat{\pi}_\ell \circ T_i(\cdot)$ $(\ell = 0, \ldots, k)$. It follows that the polynomials $\pi_{\ell,i}(\cdot)$ $(\ell = 0, \ldots, k)$ form an orthogonal basis of $\mathcal{P}^k[x_{i-1}, x_i]$ with respect to the usual scalar product

$$(p(\cdot), q(\cdot)) = \int_{x_{i-1}}^{x_i} p(x)q(x)dx.$$

The main properties of the polynomials $\widehat{\pi}_\ell(\cdot)$ and $\pi_{\ell,i}(\cdot)$ are summarised in the table on the next page (see [4] or [11, pp.126–127]).

Using the orthogonal basis $\{\pi_{\ell,i}(\cdot) \mid i = 0, \ldots, n+1\}$ for $\mathcal{P}^{n+1}[x_{i-1}, x_i]$, we can write $x^{n+1} = \sum_{\ell=0}^{n+1} \alpha_{\ell,i}\pi_{\ell,i}(x)$ where

$$\alpha_{\ell,i} = \int_{x_{i-1}}^{x_i} x^{n+1}\pi_{\ell,i}(x)dx \Big/ \int_{x_{i-1}}^{x_i} \pi_{\ell,i}^2(x)dx$$

	$\widehat{\pi}_\ell(\cdot)$	$\pi_{\ell,i}(\cdot)$
1.	$\widehat{\pi}_\ell(-1)=(-1)^\ell$ and $\widehat{\pi}_\ell(1)=1$	$\pi_{\ell,i}(x_{i-1})=(-1)^\ell$ and $\pi_{\ell,i}(x_i)=1$
2.	$\int_{-1}^1 \xi^\ell \widehat{\pi}_k(\xi)d\xi=0$ $(\ell=0,\dots,k-1)$	$\int_{x_{i-1}}^{x_i} x^\ell \pi_{\ell,i}(x)dx=0$ $(\ell=0,\dots,k-1)$
3.	$\int_{-1}^1 \xi^k \widehat{\pi}_k(\xi)d\xi=\frac{2^{k+1}(k!)^2}{(2k+1)!}$	$\int_{x_{i-1}}^{x_i} x^k \pi_{k,i}(x)dx=\frac{(x_i-x_{i-1})^{k+1}(k!)^2}{(2k+1)!}$
4.	$\int_{-1}^1 \widehat{\pi}_k(\xi)d\xi=\frac{2}{2k+1}$	$\int_{x_{i-1}}^{x_i} \pi_{k,i}^2(x)dx=\frac{(x_i-x_{i-1})}{2k+1}$
5.	$\int_{-1}^1 \widehat{\pi}_k(\xi)\widehat{\pi}_\ell(\xi)d\xi=0$ for $k\neq\ell$	$\int_{x_{i-1}}^{x_i} \pi_{k,i}(x)\pi_{\ell,i}(x)dx=0$ for $k\neq\ell$

TABLE. Basic properties of $\widehat{\pi}_\ell(\cdot)$ and $\pi_{\ell,i}(\cdot)$.

for $\ell=0,\dots,n+1$. In particular, using (3), we have

$$\alpha_{n+1,i}=\frac{(x_i-x_{i-1})^{n+1}[(n+1)!]^2}{(2n+2)!}$$

For a given partition $\triangle=\{x_i\}_{i=0}^N$, the solution of (A) is

$$p_i^*(x)=\sum_{\ell=0}^n \alpha_{\ell,i}\pi_{\ell,i}(x)$$

and

$$x^{n+1}-p_i^*(x)=\alpha_{n+1,i}\pi_{n+1,i}(x).$$

It follows that

$$\int_{x_{i-1}}^{x_i}\left(x^{n+1}-p_i^*(x)\right)^2dx=\int_{x_{i-1}}^{x_i}\left(\alpha_{n+1,i}\pi_{n+1,i}(x)\right)^2dx$$
$$=\frac{(x_i-x_{i-1})^{2n+3}[(n+1)!]^4}{(2n+3)\,[(2n+2)!]^2}$$

and

$$F\left(\triangle,\vec{p}^*\right)=\frac{[(n+1)!]^4}{(2n+3)[(2n+2)!]^2}\sum_{i=1}^N(x_i-x_{i-1})^{2n+3}.$$

But, from an inequality for weighted means (see [5] or [1]), we have

$$\frac{1}{N}\sum_{i=1}^N(x_i-x_{i-1})^{2n+3}\geqslant\left(\frac{1}{N}\sum_{i=1}^N(x_i-x_{i-1})\right)^{2n+3}=\left(\frac{b-a}{N}\right)^{2n+3},$$

with a strict inequality if the (x_i-x_{i-1}) are not all equal. Then

$$F\left(\triangle,\vec{p}^*\right)\geqslant F\left(\triangle^*,\vec{p}^*\right).$$

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Hence $F\left(\triangle, \vec{p}\right)$ is minimised for the uniform partition and the $\vec{p}_i(\cdot)$'s are the best least-squares n degree polynomal approximation of x^{n+1} o over $[x_{i-1}, x_i]$.

Finally, for the uniform partition $\triangle^*$ we have

$$x_i^{n+1} - p_i^*(x_i) = \alpha_{n+1,i}\pi_{n+1}(1) = \left(\frac{b-a}{N}\right)^{n+1} \frac{[(n+1)!]^2}{(2n+2)!}$$

and

$$x_i^{n+1} - p_{i+1}^*(x_i) = \alpha_{n+1,i}\pi_{n+1}(-1) = (-1)^{n+1}\left(\frac{b-a}{N}\right)^{n+1} \frac{[(n+1)!]^2}{(2n+2)!}.$$

Then

(i) the approximation is continuous for n odd, and also $x_i^{n+1} - p_i^*(x_i) = x_{i-1}^{n+1} - p_i^*(x_{i-1})$;

(ii) the approximation is discontinuous for n even with equal stepsizes at the nodes:

$$x_i^{n+1} - p_i^*(x_i) = p_{i+1}^*(x_i) - x_i^{n+1}$$

and also

$$x_i^{n+1} - p_i^*(x_i) = -\left(x_{i-1}^{n+1} - p_i^*(x_{i-1})\right).$$

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