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RINGS IN WHICH ELEMENTS ARE UNIQUELY THE SUM OF AN IDEMPOTENT AND A UNIT

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Abstract. An associative ring with unity is called *clean* if every element is the sum of an idempotent and a unit; if this representation is unique for every element, we call the ring *uniquely clean*. These rings represent a natural generalization of the Boolean rings in that a ring is uniquely clean if and only if it is Boolean modulo the Jacobson radical and idempotents lift uniquely modulo the radical. We also show that every image of a uniquely clean ring is uniquely clean, and construct several noncommutative examples.

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1. Introduction. An element a in a ring R is called *clean* if a is the sum of an idempotent and a unit in R, and R is called a *clean ring* if every element is clean. It is known 4, Proposition 1.8 that clean rings are exchange rings (see Warfield 6) and that the two concepts are equivalent for rings with all idempotents central. Moreover Camillo and Yu have shown that every unit regular ring is clean 2, Theorem 5, and that the clean rings with no infinite orthogonal families of idempotents are precisely the semiperfect rings 2, Theorem 9.

In this paper we investigate the *uniquely clean* rings in which each element has a unique representation as the sum of an idempotent and a unit, and find that these rings are closely related to the Boolean rings. In fact, we prove the following results.

THEOREM. *A local ring R is uniquely clean if and only if $R/J(R) \cong \mathbb{Z}_2$.*

THEOREM. *A ring R is uniquely clean if and only if $R/J(R)$ is Boolean and idempotents lift uniquely modulo $J(R)$. In particular R is Boolean if and only if R is uniquely clean and $J(R) = 0$.*

THEOREM. *Every image of a uniquely clean ring is again uniquely clean.*

We also use ideal extensions to construct several examples of uniquely clean rings, some of which are not commutative.

Throughout this paper all rings are associative with unity (unless otherwise noted) and all modules are unitary. We denote the group of units of the ring R by $U = U(R)$, the center by $C(R)$, and the Jacobson radical by $J = J(R)$, and we write $I \lhd R$ to indicate that I is an ideal (right and left) of R. The ring of integers is denoted by $\mathbb{Z}$, and we write $M_n(R)$ and $T_n(R)$ for the rings of all (respectively, all upper triangular) $n \times n$ matrices over the ring R.

2. Examples and basic properties. An element a in a ring R is called *uniquely clean* if $a = e + u$ where $e^2 = e$ and $u \in U$, and this representation is unique. A ring R is called a *uniquely clean ring* if every element is uniquely clean.

EXAMPLE 1. Central idempotents and central nilpotents are uniquely clean in any ring.

Proof. If $e^2 = e$ we have $e = (1 - e) + (2e - 1)$. Suppose that $e = f + u$, $f^2 = f$, $u \in U$. If $eu = ue$ we obtain $f + u = (f + u)^2 = f + 2fu + u^2$, so $u = 1 - 2f$. Hence $f = 1 - e$, as required.

If a is nilpotent we have $a = 1 + (a - 1)$. Suppose $a^n = 0$ and $a = e + u$, $e^2 = e$, $u \in U$. If $au = ua$, the binomial theorem gives $0 = (e + u)^n = e + \binom{n}{1}eu + \binom{n}{2}eu^2 + \cdots + \binom{n}{n-1}eu^{n-1} + u^n$. Hence $u^n \in eR$, so $e = 1$ as required. □

COROLLARY 2. *Every Boolean ring is uniquely clean.*

Note that in the proof of Example 1 it is only required that e and a commute with the unit u. On the other hand, *something* is needed because

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

A routine elementary argument establishes the following result.

EXAMPLE 3. A direct product $\Pi_i R_i$ of rings is uniquely clean if and only if each R_i is uniquely clean.

We are going to give examples of noncommutative uniquely clean rings, and the following result will be needed.

LEMMA 4. *Every idempotent in a uniquely clean ring is central.*

Proof. Let $e^2 = e \in R$. If $r \in R$, then $e + (er - ere)$ is an idempotent. Hence $1 + (er - ere)$ is a unit, so the fact that $[e + (er - ere)] + 1 = e + [1 + (er - ere)]$ implies that $e + (er - ere) = e$ because R is uniquely clean. It follows that $er = ere$, and similarly $re = ere$. □

In particular, no matrix ring $M_n(R)$, and no triangular matrix ring $T_n(R)$, is uniquely clean if $n \geq 2$.

Example 3 and Lemma 4 give immediately the following result.

COROLLARY 5. *If R is a uniquely clean ring and $e^2 = e \in R$, then eRe is uniquely clean.*

COROLLARY 6. *Every uniquely clean ring R is directly finite* ($ab = 1$ *implies* $ba = 1$).

Proof. If $ab = 1$, then ba is a (central) idempotent, so $ba = ba(ab) = a(ba)b = 1$. □

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To exhibit noncommutative examples of uniquely clean rings, we need the following construction. Let R be a ring and let ${}_RV_R$ be an R-R-bimodule which is a general ring (possibly with no unity) in which $(vw)r = v(wr)$, $(vr)w = v(rw)$ and $(rv)w = r(vw)$ hold for all $v, w \in V$ and $r \in R$. Then the *ideal-extension* $I(R; V)$ of R by V is defined to be the additive abelian group $I(R; V) = R \oplus V$ with multiplication $(r, v)(s, w) = (rs, rw + vs + vw)$. Note that if S is a ring and $S = R \oplus A$, where R is a subring and $A \lhd S$, then $S \cong I(R; A)$.

PROPOSITION 7. *An ideal-extension $S = I(R; V)$ is uniquely clean if the following conditions are satisfied:*

(a) *R is uniquely clean*;

(b) *if $e^2 = e \in R$ then $ev = ve$ for all $v \in V$*;

(c) *if $v \in V$ then $v + w + vw = 0$ for some $w \in V$.*

Furthermore, conditions (*a*), (*b*) *and* (*c*) *are necessary if S is uniquely clean and V contains no nonzero idempotents.*

Proof. Assume that (a), (b) and (c) are satisfied. Let $s = (r, v) \in S$ and (by (a)) write $r = e + u$, $e^2 = e \in R$, $u \in U(R)$. Then $s = (e, 0) + (u, v)$ so S is clean because $(u, v) \in U(S)$ [in fact $(u, v) = (u, 0)(1, u^{-1}v)$, and $(1, u^{-1}v) = (1, 0) + (0, u^{-1}v) \in U(S)$ because $(0, V) \subseteq J(S)$ by (c)]. Now suppose that $(e, x) + (u, v) = (e', x') + (u', v')$ in S where (e, x) and (e', x') are idempotents and (u, v) and (u', v') are units. Then $e + u = e' + u'$ in R where e and e' are idempotents and u and u' are units, so $(e, x) = (e', x')$ by the following result.

Claim. If $(e, x)^2 = (e, x) \in S$ then $e^2 = e$ and $x = 0$.

Proof. $(e, x)^2 = (e, x)$ gives $e^2 = e$ and $x = 2ex + x^2$ using (b). Then multiplying by e gives $ex + ex^2 = 0$, and multiplying by x gives $x^2 = 2ex^2 + x^3$. Hence adding this latter equation to $x = 2ex + x^2$ yields $x = x^3$, and so x^2 is an idempotent in V. By (c) $-x^2 + y + (-x^2)y = 0$, for some $y \in V$, so that $x^2 + w = x^2w$ where $w = -y$. Multiplying by x^2 yields $x^2 = 0$, whence $x = x^3 = 0$, proving the Claim.

On the other hand, suppose that S is uniquely clean and V contains no nonzero idempotents. It is routine to see that (a) holds. If $e^2 = e \in R$ then $(e, 0)$ is an idempotent in S and so is central (Lemma 4). In particular $(e, 0)$ commutes with $(0, v)$ for every $v \in V$, and (b) follows. Finally, given $v \in V$ write $(1, v) = (e, x) + (u, z)$ where (e, x) is an idempotent and (u, z) is a unit. Then $1 = e + u$ in R so $e = 0$ by (a). This implies that $x^2 = x \in V$, so $x = 0$ by hypothesis. Hence $(1, v)$ is a unit in S and (c) follows where $(1, v)^{-1} = (a, w)$, $a \in R$, $w \in V$. □

It is worth noting that it is not necessary for V to contain no nonzero idempotent for $I(R; V)$ to be uniquely clean in Proposition 7. In fact, the ring $I(\mathbb{Z}_2; \mathbb{Z}_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is uniquely clean.

We can now give some noncommutative examples of uniquely clean rings.

EXAMPLE 8. Let R be uniquely clean and let $S = \{[a_{ij}] \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$. Then S is uniquely clean and is noncommutative if $n \geq 3$.

Proof. Take $V = \{[a_{ij}] \in T_n(R) \mid a_{11} = \cdots = a_{nn} = 0\}$. Then $S \cong I(R; V)$. Apply Proposition 7. (a) is clear; (b) holds because idempotents in R are central and the idempotents in S are diagonal matrices (by a routine, diagonal by diagonal computation), and (c) follows because $V \subseteq J(S)$. □

If R is a ring and $\alpha : R \to R$ is a ring endomorphism, let $R[[x, \alpha]]$ denote the ring of *skew formal power series* over R; that is all formal power series in x with coefficients from R with multiplication defined by $xr = \alpha(r)x$ for all $r \in R$. In particular, $R[[x]] = R[[x, 1_R]]$ is the ring of formal power series over R.

EXAMPLE 9. If R is a ring and $\alpha : R \to R$ is a ring endomorphism, then $R[[x, \alpha]]$ is uniquely clean if and only if R is uniquely clean and $e = \alpha(e)$ for all $e^2 = e \in R$.

Proof. We have $R[[x, \alpha]] \cong I(R; (x))$ where (x) is the ideal generated by x. If $I(R; (x))$ is uniquely clean then R is uniquely clean by Proposition 7, and $e = \alpha(e)$ because $ex = xe = \alpha(e)x$. Conversely, condition (a) in Proposition 7 clearly holds, and (c) holds because $(x) \subseteq J(R[[x, \alpha]])$. To prove (b), observe that $\alpha^k(e)$ is an idempotent for each $k \geq 1$, so our hypothesis gives $e = \alpha(e) = \alpha^2(e)$. Continuing in this way we find that $e = \alpha^k(e)$ for each $k \geq 1$. Since e is central in R, it follows that $e(ax^k) = (ax^k)e$ for all $a \in R$ and all $k \geq 1$, so $ev = ve$ for all $v \in (x)$. $\square$

Taking $\alpha = 1_R$ in Example 9 gives our next result.

COROLLARY 10. *If R is a ring, then the ring $R[[x]]$ of formal power series is uniquely clean if and only if R is uniquely clean.*

EXAMPLE 11. Let $R = \mathbb{Z}_4 \times \mathbb{Z}_4$ and define $\alpha : R \to R$ by $\alpha(a, b) = (b, a)$. Then R is uniquely clean but $R[[x, \alpha]]$ is not uniquely clean.

It is clear that the center $C(R)$ of every uniquely clean ring R is again uniquely clean. However, in contrast to the Boolean rings, subrings of uniquely clean rings need not inherit the property.

EXAMPLE 12. If R is any uniquely clean ring, the power series ring $R[[x]]$ is uniquely clean but its subring $R[x]$ is not uniquely clean.

Proof. $R[[x]]$ is uniquely clean by Corollary 10; the rest follows from the following result. $\square$

PROPOSITION 13. *The polynomial ring $R[x]$ is never clean if $R \neq 0$.*

Proof. We show that x is not clean in $R[x]$. Suppose that $x = e + u$ where e is an idempotent and $u \in U(R[x])$. If $e = e_0 + e_1x + \cdots$ and $u = u_0 + u_1x + \cdots$ then $e_0 = -u_0$ is both a unit and an idempotent in R, so $e_0 = 1$. If $e \neq 1$ then e has the form $e = 1 + x^m g$ where $m \geq 1$ and $g = a + bx + \cdots$ where $a \neq 0$. Comparing coefficients of x^m in $e^2 = e$ gives $2a = a$, a contradiction. Hence $e = 1$, so $1 - x = -u$ is a unit in $R[x]$. But if $(1 - x)^{-1} = a_0 + a_1x + \cdots + a_nx^n$ then $a_0 = 1$, $a_1 - a_0 = 0, \cdots, a_n - a_{n-1} = 0$, $a_n = 0$, a contradiction. $\square$

3. Structure theorems. We begin with a characterization of the local uniquely clean rings. The following fact about clean rings will be needed and has some interest in itself.

LEMMA 14. *A ring $R \neq 0$ is local if and only if it is clean and 0 and 1 are the only idempotents in R.*

Proof. If R is local and $a \in R$ then a is clean. If $a \in J$ we have $a = 1 + (1 - a)$, while if $a \notin J$ then $a = 0 + a$ since a is a unit. Hence R is clean. Conversely, given the conditions, let $a \notin J$. Then $1 - ar$ is a nonunit for some $r \in R$, so $1 - ar = 0 + u$

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is impossible for any unit u. Hence $1-ar=1+u$ by hypothesis, and it follows that $av=1$ for some $v\in R$. Similarly $wa=1$ for $w\in R$, so a is a unit. This proves that R is local. □

THEOREM 15. *The following are equivalent for a ring* $R\neq 0$:
(1) *R is local and uniquely clean*;
(2) *R is uniquely clean and the only idempotents in R are* 0 *and* 1;
(3) $R/J\cong\mathbb{Z}_2$.

Proof. (1)⇒(2) is clear.
(2)⇒(3). If $\bar{a}\neq\bar{0}$ in $\bar{R}=R/J$, we show that $\bar{a}=\bar{1}$. If not then both a and $1-a$ are units because R is local by Lemma 14. Hence $0+(1-a)=1+(-a)$, which implies that $0=1$ because R is uniquely clean, a contradiction.
(3)⇒(1). R is local by (3), and so is clean by Lemma 14. Suppose that $e+u=f+v$ where $e^2=e, f^2=f, u^{-1}\in R$ and $v^{-1}\in R$. If $e\neq f$ then, as 0 and 1 are the only idempotents in R, we may assume $e=0$ and $f=1$. It follows that both u and $1-u$ are units in R. But $R/J\cong\mathbb{Z}_2$ so this means $\bar{u}=\bar{1}=\bar{1}+\bar{u}$ in R/J, a contradiction. □

REMARK. *The proof of* (2)⇒(3) *in Theorem* 15 *shows that if R is a uniquely clean ring and $a\in R$, then a and $1-a$ cannot both be units.*

COROLLARY 16. *A ring R is uniquely clean and contains no infinite orthogonal set of idempotents if and only if $R\cong R_1\times\cdots\times R_n$ for some $n\geq 1$ where $R_i/J(R_i)\cong\mathbb{Z}_2$ for each i.*

Proof. If R is uniquely clean with no infinite orthogonal set of idempotents then $R\cong R_1\times\cdots\times R_n$ where each R_i is indecomposable and uniquely clean. Since idempotents in R_i are central, $R_i/J(R_i)\cong\mathbb{Z}_2$ by Theorem 15. The converse follows from Example 3 and Theorem 15. □

The following observation will be needed several times. It follows from 4, Propositions 1.8 and 1.9, but we include a compact, direct proof for completeness. We say that *idempotents lift* modulo an ideal I of a ring R if whenever $a^2-a\in I$ there exists $e^2=e\in R$ such that $e-a\in R$. In this case we say that e *lifts* a.

LEMMA 17. *Let R be a clean ring.*
(1) *Idempotents lift modulo every ideal I of R.*
(2) *If $T\nsubseteq J$ is a right* (*or left*) *ideal of R there exists* $0\neq e^2=e\in T$.

Proof. Let $a\in R$, and write $a=e+u$, where $e^2=e$ and $u\in U$. Then

$$[a-u(1-e)u^{-1}]u=eu+ue+u^2-u=a^2-a. \qquad (*)$$

(1) If $a^2-a\in I$ then $(*)$ shows that a lifts to $u(1-e)u^{-1}$ and so idempotents lift modulo I.
(2) Suppose $T\nsubseteq J$ is a right ideal containing no nonzero idempotent. If $a\in T$ and $a=e+u$ as above, then $(*)$ gives $u(1-e)u^{-1}=a-(a^2-a)u^{-1}\in T$. Hence $e=1$ so $1-a=-u$ is a unit. Thus $T\subseteq J$, a contradiction. A similar argument works if T is a left ideal. □

Returning to uniquely clean rings, we prove the following result.

LEMMA 18. *If R is a uniquely clean ring then R/J has characteristic* 2.

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Proof. We must show that $2 = 1 + 1$ is in J. If $2 \notin J$ there exists $0 \neq e^2 = e \in 2R$ by Lemma 17. Hence $e = 2b$, where $b \in R$. We may assume that $eb = b = be$. Then $u = (1 - e) - 2e$ is a unit with inverse $(1 - e) - b$. Hence

$$(1 - e) + (1 - 2e) = 1 + u$$

shows that $1 - e = 1$ by hypothesis, a contradiction. □

THEOREM 19. *The following are equivalent for a ring R:*

(1) *R is uniquely clean and $J = 0$;*
(2) *R is clean, $char(R) = 2$ and 1 is the only unit in R;*
(3) *R is Boolean;*
(4) *R is (von Neumann) regular and uniquely clean.*

Proof. (3)⇒(4) by Corollary 2; (4)⇒(1) is clear; and (2)⇒(3) because if $a = e + u$ in R where $e^2 = e$ and $u \in U$, then $a = 1 + e = 1 - e$ is an idempotent.

(1)⇒(2). Given (1), $char(R) = 2$ by Lemma 18. Let $u \in U$ and suppose that $u \neq 1$. By Lemma 17 let $0 \neq e^2 = e \in (1 - u)R$, say $e = (1 - u)r$, $r \in R$. Since e is central by Lemma 4, it follows that $e(1 - u)e$ is a unit in eRe by Corollaries 5 and 6. But then $0 + e(1 - u)e = e - eue$ where eue is a unit in eRe, so $0 = e$, again by Corollary 5, a contradiction. Hence $u = 1$, proving (2). □

If $I \lhd R$ we say that *idempotents lift uniquely* modulo I if whenever $a^2 - a \in I$, there exists a unique idempotent $e \in R$ such that $e - a \in I$. Note that this condition implies that if $e - f \in I$, $e^2 = e$, $f^2 = f$, then $e = f$ (since e and f both lift f); in particular 0 is the only idempotent in I. Hence, in a clean ring the unique lifting condition implies that $I \subseteq J$ by Lemma 17.

THEOREM 20. *The following are equivalent for a ring R:*

(1) *R is uniquely clean;*
(2) *R/J is Boolean and idempotents lift uniquely modulo J;*
(3) *R/J is Boolean, idempotents lift modulo J and idempotents in R are central;*
(4) *for all $a \in R$ there exists a unique idempotent $e \in R$ such that $e - a \in J$.*

Proof. (1)⇒(2). Given (1), idempotents lift modulo J by Lemma 17. Suppose that $a^2 - a \in J$, $e - a \in J$, $f - a \in J$, where $e^2 = e, f^2 = f$. Then $(1 - e) - [1 - (e - a)] = -a = (1 - f) - [1 - (f - a)]$ so $e = f$ by the uniqueness. Hence idempotents lift uniquely modulo J, and it remains (by Theorem 19) to show that $\bar{R} = R/J$ is uniquely clean. Certainly $\bar{R}$ is clean. Suppose $\bar{a} = a + J$ has two representations $\bar{a} = \bar{e} + \bar{u} = \bar{f} + \bar{v}$ where $\bar{e}^2 = \bar{e}$, $\bar{f}^2 = \bar{f}$, and $\bar{u}, \bar{v} \in U(\bar{R})$. We may assume that $e^2 = e$ and $f^2 = f$ by Lemma 17, and that u and v are units. Write $x = a - e - u$ and $y = a - f - v$. Then $x, y \in J$ and $e + (u + x) = a = f + (v + y)$ so, as $u + x$ and $v + y$ are units, we obtain $e = f$. Hence $\bar{e} = \bar{f}$, as required.

(2)⇒(3). If $e^2 = e \in R$ and $r \in R$ then e and $e + (er - ere)$ are both idempotents in R lifting e, so $er - ere = 0$ by (2). Hence $er = ere$; a similar argument shows that $re = ere$. This proves (3).

(3)⇒(4). Given $a \in R$ an idempotent e with $e - a \in J$ exists by (3); we must prove uniqueness. If $f - a \in J$, $f^2 = f$, then $e - f \in J$, so $e(1 - f) = (e - f)(1 - f) \in J$. But $e(1 - f)$ is an idempotent (since $ef = fe$) and it follows that $e = ef$. Considering $(1 - e)f$ shows that $f = ef$ too, proving (4).

(4)⇒(1). If $a \in R$ apply (4) to $-a$ to get $e^2 = e$ such that $e + a \in J$. Hence the fact that $a = (1 - e) + [-1 + (e + a)]$ shows that R is clean. Finally, suppose that $a = f + u$, $f^2 = f$, $u \in U(R)$. Then $(1 - f) - a = (1 - 2f) - u \in J$ because $1 + J$ is the only unit in R/J by (4). Hence the uniqueness in (4) shows that $1 - f$ (and hence f) is uniquely determined by a. This proves (1). □

In Theorem 20, the hypotheses that idempotents lift uniquely in (2), and that idempotents commute in (3), cannot be dropped.

EXAMPLE 21. Let $R = \left[\begin{smallmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{smallmatrix}\right]$. Then $R/J \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is Boolean and idempotents lift modulo J, but R is not uniquely clean by Lemma 4.

Let R be a commutative uniquely clean ring. In 1 the authors ask whether $R/M \cong \mathbb{Z}_2$ for every maximal ideal M of R. The answer is affirmative by the following theorem and the fact that $\mathbb{Z}_2$ is the only uniquely clean division ring (Theorem 15).

THEOREM 22. *Every factor ring of a uniquely clean ring is again uniquely clean.*

Proof. If R is uniquely clean and $I \lhd R$, write $\bar{R} = R/I$. We have $(J + I)/I \subseteq J(\bar{R})$, so $\bar{R}/J(\bar{R})$ is an image of R/J, and so is Boolean by Theorem 20. Since $\bar{R}$ is clean, idempotents lift modulo $J(\bar{R})$ by Lemma 17, so it remains to show that they lift uniquely. Hence let $\bar{a}^2 - \bar{a} \in J(\bar{R})$ and suppose that $\bar{e}^2 = \bar{e}$ and $\bar{f}^2 = \bar{f}$ are such that $\bar{e} - \bar{a}$ and $\bar{f} - \bar{a}$ are in $J(\bar{R})$. As R is clean, idempotents lift modulo I by Lemma 17 so we may assume that $e^2 = e$ and $f^2 = f$. Hence e and f are central in R, whence $\bar{e}$ and $\bar{f}$ are central in $\bar{R}$. But then $\bar{e}(\bar{1} - \bar{f})$ is an idempotent and $\bar{e}(\bar{1} - \bar{f}) = (\bar{e} - \bar{f})(\bar{1} - \bar{f}) \in J(\bar{R})$, so $\bar{e} = \bar{e}\bar{f}$. Similarly $\bar{f} = \bar{f}\bar{e}$ and it follows that $\bar{e} = \bar{f}$. This completes the proof. □

In fact we have the following result for maximal one-sided ideals. A ring R is called *left quasi-duo* (respectively *right quasi-duo*) if every maximal left (right) ideal of R is an ideal.

PROPOSITION 23. *Every uniquely clean ring is left and right quasi-duo.*

Proof. Let M be a maximal left ideal in a uniquely clean ring R. Since $\bar{R} = R/J$ is Boolean we have $\bar{R}/\bar{M} \cong \mathbb{Z}_2$. But then the R-module $R/M \cong \bar{R}/\bar{M}$ has two elements, so $R = M \cup (1 + M)$. Now let $m \in M$ and $r \in R$; we must show that $mr \in M$. This is clear if $r \in M$; otherwise $r = 1 + m_1$, $m_1 \in M$, so $mr = m + mm_1 \in M$, as required. □

Note that a clean, left and right quasi-duo ring need not be uniquely clean. Indeed, if F is a field and $R = \{\left[\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right] \mid a, b \in F\}$ then R is a clean, commutative (and local) ring, but it is not uniquely clean if $F \ncong \mathbb{Z}_2$.

If R is a ring and G is a group, let RG denote the group ring.

PROPOSITION 24. *Let C_n denote the cyclic group of order n.*

(1) *If R is a commutative uniquely clean ring, then RC_{2^k} is uniquely clean for all $k \geq 0$.*

(2) *If $n \geq 3$ is odd and R is a Boolean ring, then RC_n is clean, but not uniquely clean.*

Proof. (1). It is routine to verify that $RC_{2k} \cong (RC_k)C_2$, so it suffices to show that if RC_2 is uniquely clean. We show first that RC_2 is clean. Write $C_2 = \{1, g\}$, let $x = a + bg \in RC_2$, and write $a^2 - b^2 = e + u$, $e^2 = e$, $u^{-1} \in R$. Then $x = e + ((a - e) + bg)$ and, since $v = (a - e)^2 - b^2 = u + 2e(1 - a)$ is a unit in R (by Lemma 18), we have $((a - e) + bg)^{-1} = v^{-1}((a - e) - bg)$. Hence RC_2 is clean.

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To check uniqueness, let $x = f + z$ where $f^2 = f \in RC_2$ and $z^{-1} \in RC_2$. If $f = r + sg$ then $r^2 + s^2 = r$ and $2rs = s$, so $s = 0$ (as $2 \in J(R)$). Hence $f^2 = f = r \in R$. Now let $z = p + qg$, so that $a = f + p$ and $b = q$. Then $a + b = f + (p + q)$, and $p + q$ is a unit in R because $p + qg \mapsto p + q$ is a ring homomorphism $RC_2 \to R$. Since R is uniquely clean, this shows that f (and hence z) is uniquely determined by x.

(2). Write $C_n = \{1, g, g^2, \cdots, g^{n-1}\}$ where $g^n = 1$, choose $b \in R$, and put $x = g + g^2 + \cdots + g^{n-1}$. Then $g^{n-1} = bx + (bx + g^{n-1})$ so, to see that RC_n is not uniquely clean, it suffices to show that $x^2 = x$ and $y = bx + g^{n-1}$ is invertible for any $b \in R$. To see that $x^2 = x$, observe that $gx = x$, and so $g^k x = x$ for each $k \geq 1$. But then $x^2 = \Sigma_{k=1}^{n-1} g^k x = (n-1)x = x$ because n is odd.

To see that y is invertible, view it as the linear transformation $RC_n \to RC_n$ given by $t \mapsto ty$. Since $y = 0 + bg + bg^2 + \cdots + bg^{n-2} + (1 + b)g^{n-1}$, the matrix of y (with respect to the basis $\{1, g, g^2, \cdots, g^{n-1}\}$) is the $n \times n$ matrix A_n below. Hence it suffices to show that $det(A_n) = 1$ for all choices of b, we have

$$A_n = \begin{bmatrix} 0 & b & b & \cdots & b & b & b+1 \\ b+1 & 0 & b & \cdots & b & b & b \\ b & b+1 & 0 & \cdots & b & b & b \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ b & b & b+1 & \cdots & 0 & b & b \\ b & b & b & \cdots & b+1 & 0 & b \\ b & b & b & \cdots & b & b+1 & 0 \end{bmatrix},$$

$$B_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & b & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & b & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & b & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & b & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & b \end{bmatrix}.$$

If rows 2 through n of A_n are added to row 1, and then b times row 1 is added to each of the other rows, the result (since n is odd) is that $det(A_n) = det(B_n)$. Now expand $det(B_n)$ by column 1 to obtain $det(B_n) = b + det(B_{n-1})$. Repeating this for B_{n-1}, we obtain $det(B_n) = det(B_{n-2})$. It follows that $det(B_n) = det(B_3) = 1$, so $det(A_n) = 1$, as required.

Finally, to see that RC_n is clean, let $w = \Sigma a_i g^i \in RC_n$. Then $w \in R_0 C_n$ where R_0 is the subring of R generated by the coefficients a_i. But R_0 is finite (R is Boolean), so $R_0 G_0$ is a finite ring, and hence is clean (it is semiperfect—see 2). Thus w is clean in $R_0 C_n$, and hence in RC_n. □

PROPOSITION 25. *The following conditions are equivalent for a ring R.*

(1) *R is uniquely clean*;

(2) *$R = C(R) + J(R)$ where $C(R)$ is uniquely clean and all idempotents of R are in $C(R)$*;

(3) *$R = C + V$ where C is a subring and $V \lhd R$ such that*

(a) *C is uniquely clean, and if $c \in C$ is a unit in R then $c^{-1} \in C$*;

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(b) $V \subseteq J(R)$;

(c) *every idempotent in* R *has the form* $e + v$ *where* $e^2 = e \in C$, $v \in V$ *and* $ev = ve$.

Proof. (1)⇒(2). This is by Theorem 20, Lemma 4, and the remark after Example 11.
(2)⇒(3). Take $C = C(R)$ and $V = J(R)$.
(3)⇒(1). Assume the conditions in (3).

Claim. If $a^2 = a \in R$ then $a \in C$.

Proof. By (c) write $a = e + v$ where $e^2 = e \in C$, $v \in V$, and $ev = ve$. Since $a^2 = a$ we obtain $v = 2ev + v^2$; then multiplying by e gives $ev + ev^2 = 0$, and multiplying by v gives $v^2 = 2ev^2 + v^3$. Hence $v^3 = v^2 - 2ev^2 = v^2 + 2ev = v$. Thus v^2 is an idempotent in $V \subseteq J(R)$, that is $v^2 = 0$. Hence $v = v^3 = 0$ and $a = e \in C$. This proves the Claim.

Given $r \in R$, say $r = c_0 + v$, $c_0 \in C$, $v \in V$, write $c_0 = e + c$, where $e^2 = e \in C$ and $c \in C$ is a unit in C and so in R. Hence $r = e + (c + v)$ and $c + v$ is a unit in R because $V \subseteq J(R)$. So R is clean; to see that R is uniquely clean, let $e + a = f + b$ in R where $e^2 = e$, $f^2 = f$, and $a^{-1}, b^{-1} \in R$. Then write $a = c + v$ and $b = d + w$ where $c, d \in C$ and $v, w \in V \subseteq J(R)$. Hence c and d are units in R, and $e, f \in C$ by the Claim. Then $e + a = f + b$ becomes

$$e + c + v = f + d + w.$$

Hence $v - w \in C$ and so $e + (c + v - w) = f + d$ is a decomposition in C where d and $c + v - w$ are units of R, and hence in C by (a). Since C is uniquely clean, we obtain $e = f$. This completes the proof of (1). □

Lemma 26. *Let* $I(R; V)$ *be an ideal extension.*

(1) *If* V *has a unity* f *then* $rf = fr$ *for all* $r \in R$ *and* $I(R; V) \cong R \times V$.

(2) *If* $V = V_1 \oplus V_2$ *where each* V_i *is an* R-R-*bimodule, then* V_2 *is an* $I(R; V_1)$-$I(R; V_1)$-*bimodule via* $(r, v_1)v_2 = rv_2 + v_1v_2$ *and* $v_2(r, v_1) = v_2r + v_2v_1$, *and we have*

$$I(R; V_1 \oplus V_2) \cong I(I(R; V_1); V_2).$$

(3) *If* $W \lhd V$ *and* $W = {}_RW_R$ *then* $I(R; V/W)$ *is an image of* $I(R; V)$.

Proof. (1). We have $rf - fr = f(rf - fr)f = 0$ because $rf - fr \in V$. In particular, $e = (0, f)$ is a central idempotent in $S = I(R; V)$, so $S = eS \oplus (1 - e)S$ is a direct product of rings. Since $V \cong eS$ via $v \mapsto (0, v)$ and $R \cong (1 - e)S$ via $r \mapsto (r, -fr)$, this proves (1).

(2). The map $(r, v_1 + v_2) \mapsto ((r, v_1), v_2)$ is a ring isomorphism $I(R; V_1 \oplus V_2) \to I(I(R; V_1); V_2)$.

(3). The map $(r, v) \mapsto (r, v + W)$ is an onto ring morphism $I(R; V) \to I(R; V/W)$. □

Proposition 27. *Let* $V = V_1 \oplus V_2$ *as rings and* R-R-*bimodules where* V_2 *contains no nonzero idempotents and* $V_1 = \Sigma_{\alpha \in \Lambda} e_\alpha V$ *with* $e_\alpha^2 = e_\alpha$ *for each* α. *Then* $S = I(R; V)$ *is uniquely clean if and only if the following conditions are satisfied.*

(1) $e_\alpha \in C(V)$ *and* $re_\alpha = e_\alpha r$ *for all* $r \in R$ *and* $\alpha \in \Lambda$.

(2) R *is uniquely clean.*

(3) *Each* $e_\alpha V$ *is a uniquely clean ring.*

(4) *For each* $v \in V_2$ *there exists* $w \in V_2$ *such that* $v + w + vw = 0$.

(5) *If* $a^2 = a \in R$ *then* $av = va$ *for all* $v \in V_2$.

Proof. Assume that S is uniquely clean, so that $(0, e_\alpha)$ is a central idempotent in S for each α. This proves (1), and (2) follows by Theorem 22. As to (3), $V = e_\alpha V \oplus V'$ where $V' = \{v - e_\alpha v \mid v \in V\}$, so Lemma 26 shows $R \times e_\alpha V \cong I(R; e_\alpha V) \cong I(R; V/V')$ is an image of $I(R; V)$. Hence $R \times e_\alpha V$ and $e_\alpha V$ are uniquely clean, proving (3). Next, $I(R; V_2) \cong I(R; V/V_1)$ is an image of $I(R; V)$ by Lemma 26. Since V_2 has no nonzero idempotents, (4) follows from Proposition 7. Finally, if $a^2 = a \in R$ then $(a, 0)$ is a (central) idempotent in S, and (5) follows.

Conversely, assume these conditions hold. We have $I(R; V_1 \oplus V_2) \cong I(I(R; V_1); V_2)$ by Lemma 26, so by Proposition 7 it suffices to prove that $I(R; V_1)$ is uniquely clean.

Claim. For any finite subset $\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ *of* Λ, $I(R; \Sigma_{i=1}^k e_{\alpha_i} V)$ *is uniquely clean.*

Proof. Since each $e_{\alpha_i} \in C(V)$ a standard argument shows that $\Sigma_{i=1}^k e_{\alpha_i} V = fV$ where $f^2 = f \in C(V)$. We claim that fV is uniquely clean. Indeed, $\Sigma_{i=1}^{k-1} e_{\alpha_i} V = gV$ for some $g^2 = g \in C(V)$ so $fV = gV \oplus V'$ and $V' \cong fV/gV \cong e_{\alpha_k} V/(gV \cap e_{\alpha_k} V)$ is uniquely clean since $e_{\alpha_k} V$ is uniquely clean. It follows by induction on k that fV is uniquely clean. By Lemma 26 it follows that $I(R; fV) \cong R \times fV$ is uniquely clean.

If $(a, x) \in I(R; V_1)$ then $x \in \Sigma_{i=1}^k e_{\alpha_i} V$ for some k, so $(a, x) \in I(R; \Sigma_{i=1}^k e_{\alpha_i} V)$ and hence (a, x) is clean in $I(R; V_1)$ by the Claim. So $I(R; V_1)$ is clean. Now suppose that $(e_1, z_1) + (u_1, w_1) = (e_2, z_2) + (u_2, w_2)$ where each (e_i, z_i) is an idempotent in $I(R; V_1)$ and each (u_i, w_i) is a unit in $I(R; V_1)$ with $(u_i, w_i)^{-1} = (u_i', w_i') \in I(R; V_1)$. Then there exists $n > 0$ such that $w_i, z_i, w_i' \in \Sigma_{i=1}^n e_{\alpha_i} V$ for $i = 1, 2$, so (e_i, z_i), (u_i, w_i) and (u_i', w_i') are in $I(R; \Sigma_{i=1}^n e_{\alpha_i} V)$. Since $I(R; \Sigma_{i=1}^n e_{\alpha_i} V)$ is uniquely clean by the Claim, it follows that $(e_1, z_1) = (e_2, z_2)$ and $(u_1, w_1) = (u_2, w_2)$. This shows that $I(R; V_1)$ is uniquely clean. □

As a corollary, we obtain two examples of uniquely clean rings.

Example 28. Let $I_\alpha \lhd R$ for each $\alpha \in \Lambda$ where R is a uniquely clean ring. Then $I(R; \oplus_{\alpha\in\Lambda} R/I_a)$ is a uniquely clean ring.

Example 29. For $k \geq 2$ let $R = \mathbb{Z}_{2^k}$ and $F = \{2^i \mathbb{Z}_{2^k} \mid i = 0, 1, \cdots, k-1\}$. For any family $\{R_\alpha \mid \alpha \in \Lambda\} \subseteq F$, $I(R; \oplus_{\alpha\in\Lambda} R_\alpha)$ is a uniquely clean ring.

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REFERENCES

1. D. D. Anderson and V. P. Camillo, Commutative rings whose elements are a sum of a unit and idempotent, *Comm. Algebra* **30** (2002), 3327–3336.

2. V. P. Camillo and H.-P. Yu, Exchange rings, units and idempotents, *Comm. Algebra* **22** (1994), 4737–4749.

3. T. Y. Lam, *Lectures on modules and rings*, Graduate Texts in Mathematics, Vol. **189** (Springer-Verlag, 1999).

4. W. K. Nicholson, Lifting idempotents and exchange rings, *Trans. Amer. Math. Soc.* **229** (1977), 269–278.

5. W. K. Nicholson, On exchange rings, *Comm. Algebra* **25** (1997), 1917–1918.

6. R. B. Warfield, Exchange rings and decompositions of modules, *Math. Ann.* **199** (1972), 31–36.

7. H.-P. Yu, Stable range one for exchange rings, *J. Pure Appl. Algebra* **98** (1995), 105–109.