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THE EXACT ORDER OF GENERALIZED DIAPHONY AND MULTIDIMENSIONAL NUMERICAL INTEGRATION

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Abstract

For a point set in the multidimensional unit torus we introduce an L^κ-measure of uniformity of distribution, which for $\kappa = 2$ reduces to diaphony (and thus in this case essentially coincides with Weyl L^2-discrepancy). For $\kappa \in [1, 2]$ we establish a sharp asymptotic for this new measure as the number of points of the set tends to infinity. Upper and lower-bound estimates are given also for $\kappa > 2$.

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1. Introduction

Let S be a finite weighted point set in the s-dimensional unit torus $\mathbb{I}^s$. How uniformly are the points of S distributed? This problem goes back to Weyl [18]. Since then, a number of measures for the uniformity of distribution have been proposed, the most widely studied are various versions of discrepancy. The most common kind of discrepancy is the L^κ-average of the local discrepancies on all aligned rectangular boxes with the 'left bottom corner' at the origin:

$$D_\kappa(S) = \left(\int_{\mathbb{I}^s} \left| \sum_{x^{(k)} < x} \rho_k - V(x) \right|^\kappa dx \right)^{1/\kappa},$$

where $x^{(k)}$ are the points of the set S, ρ_k are the corresponding weights, $V(x)$ is the volume of the aligned rectangular box with the 'minimum' at the origin and the 'maximum' at x, and the sum is extended over all points of S in this box. An

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obvious drawback of this discrepancy is that it is biased in giving more attention to the irregularities of S which are closer to the origin than to those far from it. This is harmless for the metric of L^∞, as the supremum of local discrepancies on all the boxes with the 'left bottom corner' at x can easily be seen to be of the same order of magnitude for all $x \in \mathbb{I}^s$. However, the situation changes drastically for $\kappa < \infty$, see [7].

A natural way to improve the definition is to average local discrepancies on all aligned rectangular boxes with 'floating left bottom corner'. In fact, it is this kind of discrepancy which was originally considered in the pioneering paper of Weyl, and we call it *Weyl discrepancy* below. Unlike regular discrepancy, that of Weyl is invariant under translates and symmetry reflections of S, and is easier to handle due to its averaging nature; clearly, it is an L^κ-average of regular discrepancies of the translates $S + x$ of S by all the vectors $x \in \mathbb{I}^s$:

$$\widetilde{D}_\kappa(S) = \left(\int_{\mathbb{I}^s} D_\kappa^\kappa(S + x)\, dx \right)^{1/\kappa},$$

where D_κ stands for the regular discrepancy, and $\widetilde{D}_\kappa$ for Weyl discrepancy.

Another (and undeservedly less known) measure of uniformity of distribution of S is diaphony, which we denote below $F(S)$ and define formally in Section 3. It was introduced in 1976 by Zinterhof (see [20]), who used an analytical definition via absolutely convergent series and to the best of our knowledge, did not know that the analytic construction he brought into consideration also has a transparent geometric interpretation. This geometric face of diaphony was discovered almost 20 years later in [13], where it is shown that diaphony essentially (up to a bounded multiplicative factor) coincides with Weyl L^2-discrepancy. This allows one to use Weyl L^2-discrepancy and diaphony interchangeably, applying either a geometrical or analytical definition, whichever is more convenient. It is worth mentioning that diaphony, like Weyl discrepancy, is invariant with respect to translates and symmetry reflections of S. To summarize, compared to the 'usual' discrepancy, Weyl discrepancy and diaphony (which are two faces of the same coin)

- are more natural, giving non-biased treatment to all irregularities of S;
- are easier to handle due to the existence of equivalent definitions of distinct types and the averaging character of the geometric definition;
- have a number of nice invariance properties, which one expects a 'good' measure of irregularity of distribution to possess and which the usual discrepancy lacks.

Since 1976, diaphony has been considered in numerous papers (see, for instance, the citations below and the references given there). Many authors use diaphony implicitly, without introducing any special notation, to study distribution problems. Most of the papers that explicitly investigate diaphony concentrate on the one-dimensional case or on estimates of diaphony of particular sequences (as, for instance, [2, 3, 14–16, 19]). A remarkable exception is the paper of Bykovsky [1], where the asymptotic behavior of $F(S)$ is studied in general case, as $N = |S|$ approaches infinity. Results of Bykovsky imply that

$$\inf_S F(S) \gg \frac{(\ln N)^{(s-1)/2}}{N}$$

for any dimension s (with a constant depending only on s). However, the question of sharpness of this estimate remained open for $s > 1$. In this paper we answer it in the affirmative.

Actually, we do not consider diaphony by itself, but instead introduce a parametric family of measures for the uniformity of distribution of S, diaphony $F(S)$ being obtained for some particular values of the parameters. This is motivated primarily by the requirements of numerical integration applications. (One possible application is given in Theorem 1 below.) We extend the lower bound of Bykovsky to our generalized diaphony, and give a construction of S for which this lower bound is attained. In fact, this construction is borrowed from [4] where it was used to obtain lower-bound estimates of discrepancy; it can be characterized as a weighted version of Frolov's algebraic nets (see [5]).

All our results are obtained in the general setting of weighted S.

2. Notation

Fix $s \in \mathbb{Z}$, $s \geq 1$. We denote vectors of $\mathbb{R}^s$ by lower italic or Greek letters, with or without superscripts; subscripts are reserved for vector coordinates (say, $x_j^{(k)}$ is the j-th coordinate of the vector $x^{(k)}$). To distinguish them from scalars, the following two constant vectors will be written in boldface: $\mathbf{0} = (0, \ldots, 0)$ and $\mathbf{1} = (1, \ldots, 1)$. The s-dimensional unit cube is

$$\mathbb{I}^s = \{x \in \mathbb{R}^s \mid \mathbf{0} \leq x < \mathbf{1}\};$$

here $x \geq \mathbf{0}$ means $x_j \geq 0$ ($j = 1, \ldots, s$) and $x < \mathbf{1}$ means $x_j < 1$ ($j = 1, \ldots, s$). Other vector inequalities below should be interpreted similarly.

By a *net* we mean a pair $S = (X, \rho)$, where X is a finite set of points in $\mathbb{I}^s$:

$$X = \{x^{(k)} \in \mathbb{I}^s \mid k = 1, \ldots, N\},$$

and ρ is a set of non-negative real weights, corresponding to these points:

$$\rho = \{\rho_k \geq 0 \mid k = 1, \ldots, N\}.$$

The pair $(x^{(k)}, \rho_k)$ is called the k-th node of S. Numeration of nodes is not essential: we will not distinguish nets which differ only in the order of their nodes.

We put $\rho_0 = \rho_1 + \cdots + \rho_N$. The case $\rho_0 = 0$ is of no interest and we assume below that at least one of the weights is strictly positive. If $\rho_0 = 1$ we say that S is *normalized.*

The *exponential sum* of S is the complex-valued function on $\mathbb{Z}^s$ defined by

$$T(m) = \sum_{k=1}^{N} \rho_k e^{2\pi i \langle m, x^{(k)} \rangle},$$

angular brackets standing for the standard inner product. Clearly, $T(\mathbf{0}) = \rho_0$, and $|T(m)| \le \rho_0$ for $m \neq \mathbf{0}$.

For $x \in \mathbb{R}$, we write $\overline{x} = \max\{|x|, 1\}$. For $x, \alpha \in \mathbb{R}^s$ we write

$$\overline{x} = \overline{x}_1 \cdots \overline{x}_s, \qquad \overline{x}^{\alpha} = \overline{x}_1^{\alpha_1} \cdots \overline{x}_s^{\alpha_s}.$$

3. Preliminaries

The original diaphony of Zinterhof was defined with a normalized net S in mind by

$$F(S) = \left(\sum_{m \in \mathbb{Z}^s}{}' \frac{|T(m)|^2}{\overline{m}^2} \right)^{1/2}$$

(here and below, primed sums extend over all non-zero vectors). Generalizing this definition, for $\alpha > \mathbf{0}$, $\kappa \ge 1$ and *arbitrary* (not necessarily normalized) S we consider

$$F_{\kappa,\alpha}(S) = \left(\sum_{m \in \mathbb{Z}^s}{}' \frac{|T(m)|^{\kappa}}{\overline{m}^{\kappa\alpha}} + |T(\mathbf{0}) - 1|^{\kappa} \right)^{1/\kappa}.$$

The second summand may look somewhat odd at first, but deeper examination shows that it arises very naturally (see Theorem 1 below or [13, Theorem 1]). Qualitatively, it allows one to exclude the situation where a poorly distributed net has small diaphony only because all its weights are very small. However, the principal part of the definition is, of course, the first summand: if it is small, we can make the entire diaphony small by normalizing the weights (save for the above mentioned exotic case of 'very small weights').

Without loss of generality, we assume that the coordinates of α are arranged in ascending order, and write $r \in [1, s]$ for the number of appearances of the minimal coordinate:

$$\alpha_1 = \cdots = \alpha_r < \alpha_{r+1} \le \cdots \le \alpha_s.$$

Throughout the paper, we assume $\kappa\alpha_1 > 1$; since $T(m)$ is bounded, this guarantees the absolute convergence of the series for $F_{\kappa,\alpha}$.

Our primary objective is to prove the following theorem, in which the constants depend on α and κ only (possible dependence of the constants on s is always assumed by default).

MAIN THEOREM.

(i) *Assume $\kappa \in [1,2]$, and let S be a net with N nodes. Then*

$$F_{\kappa,\alpha}(S) \gg \frac{(\ln N)^{(r-1)/\kappa}}{N^{\alpha_1}}.$$

(ii) *Assume $\kappa \geq 2$, and let S be a net with N nodes. Then*

$$F_{\kappa,\alpha}(S) \gg \frac{(\ln N)^{(r-1)/\kappa}}{N^{\alpha_1+(1/2-1/\kappa)}}.$$

(iii) *For each $\kappa \geq 1$, there exists a net S with N nodes (and even a normalized one) such that*

$$F_{\kappa,\alpha}(S) \ll \frac{(\ln N)^{(r-1)/\kappa}}{N^{\alpha_1}}.$$

For $\kappa \leq 2$ this theorem establishes the exact order of $\inf_S F_{\kappa,\alpha}(S)$, the infimum being taken over all normalized net with N nodes. In particular, for the 'usual' diaphony $F(S) = F_{2,1}(S)$, it immediately implies

$$\inf_S F(S) \asymp \frac{(\ln N)^{(s-1)/2}}{N}. \tag{1}$$

For $\kappa > 2$ we have a gap between the estimates. It is not clear what exactly in our argument gives rise to this gap.

The proof of Main Theorem will consist of two independent counterparts: the lower-bound estimate of $F_{\kappa,\alpha}$ (Section 4) and an explicit construction of S for which this estimate is attained (Section 5).

For the important special case $\kappa = 2$, $\alpha_1 = \cdots = \alpha_s$, the lower-bound estimate was obtained in [1] by Bykovsky. We show that the general case $\kappa \leq 2$ reduces to this one (while the case $\kappa > 2$ requires certain additional considerations). To keep the paper self-contained, we then give a proof of this particular case. Our proof is inspired by that of Bykovsky, though it differs somewhat from the original; the idea behind both proofs comes from the classical paper of Roth [17].

As to the lower-bound estimate in (1), it was observed in [10] that there is another and immediate way to obtain it using Roth's well-known lower-bound estimate for

L^2-discrepancy. Namely, if $S+x$ is the translate of S modulo $\mathbb{I}^s$ by a vector $x \in \mathbb{I}^s$, and if $D_2(S+x)$ is the L^2-discrepancy of $S+x$, then as shown in [10, 13]

$$F(S) \asymp \left(\int_{\mathbb{I}^s} D_2^2(S+x)\,dx\right)^{1/2},$$

and the estimate in question now follows from Roth's result:

$$D_2(S+x) \gg \frac{(\ln N)^{(s-1)/2}}{N} \quad \text{uniformly in } x.$$

We turn to the numerical integration problem. For $M > 0$ and $\tau \in [1, \infty]$, denote by $E_{\tau,\alpha}(M)$ the class of all continuous $\mathbb{I}^s$-periodic functions $f : \mathbb{R}^s \to \mathbb{R}$ with Fourier coefficients $\hat{f}$ satisfying

$$\left(\sum_{m\in\mathbb{Z}^s} \left|\hat{f}(m)\overline{m}^{\alpha}\right|^{\tau}\right)^{1/\tau} \le M$$

(with the standard interpretation for $\tau = \infty$). This classifies $\mathbb{I}^s$-periodic functions according to the rate of decrease of their Fourier coefficients.

It can easily be shown that the error of numerical integration on $E_{\tau,\alpha}(M)$ using a net S is intimately related to the generalized diaphony $F_{\kappa,\alpha}(S)$, where κ is the conjugate to τ: $1/\kappa + 1/\tau = 1$.

THEOREM 1. *Assume $\kappa\alpha_1 > 1$. Then*

$$\sup_{f\in E_{\tau,\alpha}(M)} \left|\int_{\mathbb{I}^s} f(x)\,dx - \sum_{k=1}^{N} \rho_k f(x^{(k)})\right| = M F_{\kappa,\alpha}(S).$$

The quantity on the left is the maximal possible error which can occur when we approximate the integral of a function from $E_{\tau,\alpha}(M)$ by a finite sum over the points of $S = (X, \rho)$. The proof will be given in Section 6.

Theorem 1 along with our Main Theorem solve the problem of numerical integration on $E_{\tau,\alpha}(M)$ as follows.

COROLLARY 1. *Let $\tau \in [2, \infty]$ and assume $\alpha_1 > 1 - 1/\tau$. Then*

$$\inf_S \sup_{f\in E_{\tau,\alpha}(M)} \left|\int_{\mathbb{I}^s} f(x)\,dx - \sum_{k=1}^{N} \rho_k f(x^{(k)})\right| \asymp M \frac{(\ln N)^{(r-1)(1-1/\tau)}}{N^{\alpha_1}},$$

the infimum being taken over all normalized nets with N nodes.

For $\kappa \le 2$, the results of this paper were obtained in [8] (using a somewhat more complicated argument) and announced without proof in [12]. The case $\kappa > 2$ has never been considered before.

4. Lower-bound estimate of $F_{\kappa,\alpha}(S)$

Let S be a net with N nodes. If $T(\mathbf{0}) < 1/2$ then $F_{\kappa,\alpha}(S) > 1/2$ by the definition and so the lower-bound estimate of Main Theorem is trivial. Otherwise, dividing all the weights of S by $\rho_0 = T(\mathbf{0})$ (which may increase the diaphony at most twice) we get into the situation when S is normalized.

We first consider the case $\kappa \in [1, 2]$ and prove that

$$\text{(2)} \qquad \sum_{m\in\mathbb{Z}^s}{}' \frac{|T(m)|^\kappa}{\overline{m}^{\kappa\alpha}} \gg \frac{(\ln N)^{r-1}}{N^{\kappa\alpha_1}}.$$

We can assume $r = s$. For, if S' is the net in $\mathbb{I}^r$ obtained by the projection of the points of S onto the first r coordinates, if T' is the exponential sum of S', and if $\alpha' = (\alpha_1, \dots, \alpha_r)$, then the sum $\sum'_{m\in\mathbb{Z}^r} |T'(m)|^\kappa/\overline{m}^{\kappa\alpha'}$ is included in the sum on the left-hand side of (2) when $m_{r+1} = \cdots = m_s = 0$, and hence (2) will follow from the analogous estimate for S'.

Next, we can replace $|T(m)|^\kappa$ on the left-hand side of (2) by $|T(m)|^2$, since $|T(m)| \le T(\mathbf{0}) = 1$ (as S is normalized). Therefore, it suffices to prove

$$\text{(3)} \qquad \sum_{m\in\mathbb{Z}^s}{}' \frac{|T(m)|^2}{\overline{m}^{\kappa\alpha}} \gg \frac{(\ln N)^{s-1}}{N^{\kappa\alpha_1}}$$

under the assumptions $\alpha_1 = \cdots = \alpha_s,\ T(\mathbf{0}) = 1$.

Fix a non-negative function $\varphi\colon \mathbb{R} \to \mathbb{R}$ with finite support, having derivatives of any order and satisfying $\varphi(0) = 2$, $\int_{-\infty}^{\infty} \varphi(x)\,dx = 1$ (the values of $\varphi(0)$ and of the integral have no special meaning; those indicated above are chosen to somewhat simplify the calculations below). For $t \in \mathbb{R}^s$, $\mathbf{0} < t < \mathbf{1}$ and $x \in \mathbb{R}^s$ we define

$$\varphi_t(x) = \sum_{n\in\mathbb{Z}^s} \varphi\left(\frac{x_1+n_1}{t_1}\right)\cdots\varphi\left(\frac{x_s+n_s}{t_s}\right),$$

$$\Phi_t(x) = \sum_{k=1}^{N} \rho_k \varphi_t(x - x^{(k)}).$$

(Note that for any x the series for $\varphi_t(x)$ actually has only a finite number of non-zero terms.) Obviously, both φ_t and Φ_t are $\mathbb{I}^s$-periodic.

Consider the Fourier expansion of Φ_t:

$$\Phi_t(x) = \sum_{m\in\mathbb{Z}^s} \widehat{\Phi}_t(m) e^{2\pi i\langle m,x\rangle}.$$

Since $\Phi_t(x)$ is smooth, the series on the right-hand side converges absolutely, and therefore

$$\text{(4)} \qquad \sum_{k=1}^{N} \rho_k \Phi_t(x^{(k)}) = \sum_{m\in\mathbb{Z}^s} \widehat{\Phi}_t(m) \sum_{k=1}^{N} \rho_k e^{2\pi i\langle m,x^{(k)}\rangle} = \sum_{m\in\mathbb{Z}^s} \widehat{\Phi}_t(m) T(m).$$

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The Fourier coefficients $\widehat{\Phi}_t(m)$ can be evaluated as follows:

$$
\begin{aligned}
\widehat{\Phi}_t(m) &= \sum_{k=1}^{N} \rho_k \int_{\mathbb{I}^s} \varphi_t(x - x^{(k)}) e^{-2\pi i \langle m, x\rangle}\, dx \\
&= \sum_{k=1}^{N} \rho_k e^{-2\pi i \langle m, x^{(k)}\rangle} \int_{\mathbb{I}^s} \varphi_t(x - x^{(k)}) e^{-2\pi i \langle m, x - x^{(k)}\rangle}\, dx \\
&= T(-m) \int_{\mathbb{I}^s} \varphi_t(x) e^{-2\pi i \langle m, x\rangle}\, dx \\
&= T(-m) \sum_{n\in\mathbb{Z}^s} \prod_{j=1}^{s} \int_0^1 \varphi\Big(\frac{x_j + n_j}{t_j}\Big) e^{-2\pi i m_j x_j}\, dx_j \\
&= T(-m) \prod_{j=1}^{s} \sum_{n_j\in\mathbb{Z}} \int_0^1 \varphi\Big(\frac{x_j + n_j}{t_j}\Big) e^{-2\pi i m_j x_j}\, dx_j \\
&= T(-m) \prod_{j=1}^{s} \int_{-\infty}^{\infty} \varphi(x_j/t_j) e^{-2\pi i m_j x_j}\, dx_j \\
&= T(-m) t_1 \cdots t_s \prod_{j=1}^{s} \int_{-\infty}^{\infty} \varphi(x_j) e^{-2\pi i m_j t_j x_j}\, dx_j . \qquad (5)
\end{aligned}
$$

In particular,

$$\widehat{\Phi}_t(\mathbf{0}) = t_1 \cdots t_s. \qquad (6)$$

We now return to (4) and estimate the sum on the left-hand side from below:

$$\sum_{k=1}^{N} \rho_k \Phi_t(x^{(k)}) \ge \sum_{k=1}^{N} \rho_k^2 \varphi_t(\mathbf{0}) \ge 2^s \sum_{k=1}^{N} \rho_k^2 \ge \frac{2^s}{N} \left(\sum_{k=1}^{N} \rho_k \right)^2 = \frac{2^s}{N}. \qquad (7)$$

Then, for $t_1 \cdots t_s \le 1/N$ from (4), (6) and (7) we obtain

$$\sum_{m\in\mathbb{Z}^s}{}' \, T(m) \widehat{\Phi}(m) \ge \frac{2^s - 1}{N}. \qquad (8)$$

To estimate $\widehat{\Phi}(m)$, we use (5). If $|m_j t_j| > 1$, then integrating by parts $c+1$ times on the right-hand side of (5) (with a sufficiently large integer c, say $c = \lceil \kappa \alpha_s \rceil$) we obtain

$$\int_{-\infty}^{\infty} \varphi(x_j) e^{-2\pi i m_j t_j x_j}\, dx_j \ll \frac{1}{\overline{m_j t_j}^{\,c+1}}$$

(with a constant depending only on φ and c), and of course, this holds also if $|m_j t_j| \le 1$. Therefore,

$$|\widehat{\Phi}_t(m)| \ll t_1 \cdots t_s \frac{|T(m)|}{(\overline{t_1 m_1} \cdots \overline{t_s m_s})^{c+1}},$$

and substitution into (8) gives

$$t_1 \cdots t_s \sum_{m \in \mathbb{Z}^s}{}' \frac{|T(m)|^2}{\left(\overline{t_1 m_1} \cdots \overline{t_s m_s}\right)^{c+1}} \gg \frac{1}{N}.$$

Recalling that $t_1 \cdots t_s \le 1/N$, $c \ge \kappa\alpha_1$ we obtain

$$\sum_{m \in \mathbb{Z}^s}{}' \frac{|T(m)|^2}{\left(\overline{t_1 m_1} \cdots \overline{t_s m_s}\right)^{\kappa\alpha_1+1}} \gg 1.$$

To prove (3) it remains to integrate this last inequality over the region

$$\Omega_s(N) = \{t \in \mathbb{I}^s \mid \tfrac{1}{2N} \le t_1 \cdots t_s \le \tfrac{1}{N}\}$$

using the following technical lemma:

LEMMA 1. *Let $K \in \mathbb{R}$ and $c \in \mathbb{R}$, $c > 1$. Then*

(i)
$$\int_{\Omega_s(K)} dt \gg \frac{(\ln K)^{s-1}}{K} \qquad (K \ge 1),$$

(ii)
$$\int_{\Omega_s(K)} \frac{dt}{\left(\overline{t_1 m_1} \cdots \overline{t_s m_s}\right)^{c+1}} \ll \frac{K^{c-1}}{(\overline{m}_1 \cdots \overline{m}_s)^c} \qquad (K > 0),$$

where the constants depend only on s and c.

PROOF. (i) We denote the integral by $I_s(K)$ and use induction on s. The case $s = 1$ is trivial. Let $s \ge 2$. Then

$$\begin{aligned}
I_s(K) &\ge \int_{1/K}^1 \left(\int_{1/(2Kt_s) \le t_1 \cdots t_{s-1} \le 1/(Kt_s)} dt_1 \cdots dt_{s-1} \right) dt_s \\
&= \int_{1/K}^1 I_{s-1}(Kt_s)\, dt_s \\
&\gg \frac{1}{K} \int_{1/K}^1 \frac{(\ln Kt_s)^{s-2}}{t_s}\, dt_s \\
&\ge \frac{1}{K} \int_{1/\sqrt{K}}^1 \frac{(\ln \sqrt{K})^{s-2}}{t_s}\, dt_s \\
&\gg \frac{(\ln K)^{s-2}}{K} \int_{1/\sqrt{K}}^1 \frac{dt_s}{t_s} \\
&\gg \frac{(\ln K)^{s-1}}{K}.
\end{aligned}$$

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(ii) We denote the integral by $J_s(K)$ and use induction on s. Observe, that $\overline{t_j m_j} = \max\{1, |t_j m_j|\} \geq \max\{t_j, t_j|m_j|\} = t_j \overline{m}_j$, and hence

$$J_1(K) \leq \int_{1/2K}^{1/K} \frac{dt_1}{\overline{t_1 m_1}^c} \leq \frac{1}{\overline{m}_1^c} \int_{1/2K}^{1/K} \frac{dt_1}{t_1^c} \ll \frac{K^{c-1}}{\overline{m}_1^c}.$$

Now let $s \geq 2$. If $K < 1/2$, then $\Omega_s(K)$ is empty and the proof is complete. If $1/2 \leq K \leq 1$ and $t \in \Omega_s(K)$, then $t_j \geq 1/(2K) \geq 1/2$ $(j = 1, \ldots, s)$ and thus $\overline{t_j m_j} \geq t_j \overline{m}_j \geq \overline{m}_j/2$, hence

$$J_s(K) \ll \frac{1}{(\overline{m}_1 \cdots \overline{m}_s)^{c+1}} \ll \frac{K^{c-1}}{(\overline{m}_1 \cdots \overline{m}_s)^c}.$$

Finally, if $K > 1$, then by the induction hypothesis

$$\begin{aligned} J_s(K) &= \int_{1/2K}^{1} \frac{1}{\overline{t_s m_s}^{c+1}} \left(\int_{1/(2Kt_s) \leq t_1 \cdots t_{s-1} \leq 1/(Kt_s)} \frac{dt_1 \cdots dt_{s-1}}{\left(\overline{m_1 t_1} \cdots \overline{m_{s-1} t_{s-1}}\right)^{c+1}} \right) dt_s \\ &\ll \frac{K^{c-1}}{(\overline{m}_1 \cdots \overline{m}_{s-1})^c} \int_{1/2K}^{1} \frac{t_s^{c-1}}{\overline{t_s m_s}^{c+1}} \, dt_s, \end{aligned}$$

and it suffices to observe that

$$\int_{2K}^{1} \frac{t_s^{c-1}}{\overline{t_s m_s}^{c+1}} \, dt_s \leq \int_0^{1/\overline{m}_s} + \int_{1/\overline{m}_s}^{1} = \int_0^{1/\overline{m}_s} t_s^{c-1} \, dt_s + \frac{1}{\overline{m}_s^{c+1}} \int_{1/\overline{m}_s}^{1} \frac{dt_s}{t_s^2} \ll \frac{1}{\overline{m}_s^c}.$$

□

We now return to the proof of the Main Theorem and consider the case $\kappa > 2$. As above, we may assume $r = s$, and using the same argument we get

$$\sum_{m \in \mathbb{Z}^s}{}' \frac{|T(m)|^2}{\left(\overline{t_1 m_1} \cdots \overline{t_s m_s}\right)^{c+2}} \gg 1$$

for any positive integer c. The problem is that the sum we wish to estimate incorporates $|T(m)|^\kappa \leq |T(m)|^2$. To overcome this difficulty, we define τ by $1/(\kappa/2) + 1/\tau = 1$ and use Hölder's inequality to obtain

$$\left(\sum_{m \in \mathbb{Z}^s}{}' \frac{|T(m)|^\kappa}{\left(\overline{t_1 m_1} \cdots \overline{t_s m_s}\right)^{c\kappa/2}} \right)^{2/\kappa} \left(\sum_{m \in \mathbb{Z}^s}{}' \frac{1}{\left(\overline{t_1 m_1} \cdots \overline{t_s m_s}\right)^{2\tau}} \right)^{1/\tau} \gg 1.$$

But

$$\sum_{m \in \mathbb{Z}^s}{}' \frac{1}{\left(\overline{t_1 m_1} \cdots \overline{t_s m_s}\right)^{2\tau}} \leq \sum_{m \in \mathbb{Z}^s} \frac{1}{\left(\overline{t_1 m_1} \cdots \overline{t_s m_s}\right)^2} = \prod_{j=1}^{s} \sum_{m_j \in \mathbb{Z}} \frac{1}{\overline{t_j m_j}^2} \ll \frac{1}{t_1 \cdots t_s} \leq 2N$$

for $t_1 \cdots t_s \geq 1/(2N)$; hence,

$$\sum_{m\in\mathbb{Z}^s}{}' \frac{|T(m)|^\kappa}{(\overline{t_1 m_1} \cdots \overline{t_s m_s})^{c\kappa/2}} \gg N^{-\kappa/(2\tau)} = N^{1-\kappa/2}.$$

Since c is arbitrary, it follows that

$$\sum_{m\in\mathbb{Z}^s}{}' \frac{|T(m)|^\kappa}{(\overline{t_1 m_1} \cdots \overline{t_s m_s})^{\kappa\alpha_1+1}} \gg N^{1-\kappa/2},$$

and Lemma 1 now gives

$$\sum_{m\in\mathbb{Z}^s}{}' \frac{|T(m)|^\kappa}{\overline{m}^{\kappa\alpha}} \gg \frac{(\ln N)^{s-1}}{N^{\kappa\alpha_1+(\kappa/2-1)}},$$

which was to be proved.

5. Nets with small order of $F_{\kappa,\alpha}(S)$

First, note that constructing a net with small $F_{\kappa,\alpha}(S)$ we may ignore the issue of normality: if a net S satisfies

$$F_{\kappa,\alpha}(S) \ll \frac{(\ln N)^{(r-1)/\kappa}}{N^{\alpha_1}} \tag{9}$$

then automatically $T(\mathbf{0}) \geq 1/2$ for sufficiently large N, and so the net obtained by a normalization of S will also satisfy (9).

We now describe a construction of nets, proposed by Dobrovolsky in [4]. Denote by $\mathbb{K}^s$ the doubled unit cube:

$$\mathbb{K}^s = \{x \in \mathbb{R}^s \mid -\mathbf{1} \leq x < \mathbf{1}\}.$$

For a real non-singular matrix $A \in M_{s\times s}[\mathbb{R}]$ let $\Lambda = \{A^t n \mid n \in \mathbb{Z}^s\}$ be the lattice, generated by A, and let $\Lambda^* = \{A^{-1} n \mid n \in \mathbb{Z}^s\}$ be the dual lattice. The set of points of our net is the set of fractional parts of all those points of Λ^* which fall into $\mathbb{K}^s$:

$$X = \{\{A^{-1}n\} \mid n \in A\mathbb{K}^s \cap \mathbb{Z}^s\}.$$

To define the weights, we fix a non-negative function $\psi\colon \mathbb{R} \to \mathbb{R}$, having derivatives of any order, vanishing outside $[-1, 1]$ and satisfying $\psi(x) + \psi(x-1) = 1$ for every real $x \in [0, 1]$ (the existence of such functions can be easily verified). Next, we define $\Psi\colon \mathbb{R}^s \to \mathbb{R}$ by $\Psi(x) = \psi(x_1)\cdots\psi(x_s)$. Now, with each point $x = \{A^{-1}n\} \in X$ we associate the weight $\rho(x) = \Psi(A^{-1}n)/\det\Lambda$.

Let $S(\Lambda)$ be the net obtained in this way. Denote

$$\delta_m = \begin{cases} 1 & \text{if } m = \mathbf{0}, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 2. *For $m \in \mathbb{Z}^s$ the exponential sum of $S(\Lambda)$ is*

$$T(m) = \delta_m + \sum_{z \in \Lambda}{}' \int_{\mathbb{K}^s} \Psi(x) e^{2\pi i \langle m - z, x \rangle} dx.$$

PROOF. Using the multidimensional Poisson summation formula and then changing the variable of integration, we get:

$$\begin{aligned} T(m) &= \frac{1}{\det \Lambda} \sum_{n \in A\mathbb{K}^s \cap \mathbb{Z}^s} \Psi(A^{-1}n) e^{2\pi i \langle m, A^{-1}n \rangle} \\ &= \frac{1}{\det \Lambda} \sum_{n \in \mathbb{Z}^s} \int_{A\mathbb{K}^s} \Psi(A^{-1}x) e^{2\pi i \langle m, A^{-1}x \rangle} e^{-2\pi i \langle n, x \rangle}\, dx \\ &= \sum_{n \in \mathbb{Z}^s} \int_{\mathbb{K}^s} \Psi(x) e^{2\pi i \langle m, x \rangle} e^{-2\pi i \langle n, Ax \rangle}\, dx \\ &= \sum_{n \in \mathbb{Z}^s} \int_{\mathbb{K}^s} \Psi(x) e^{2\pi i \langle m - A^t n, x \rangle}\, dx \\ &= \sum_{z \in \Lambda} \int_{\mathbb{K}^s} \Psi(x) e^{2\pi i \langle m - z, x \rangle}\, dx, \end{aligned}$$

since $A^t n$ varies over all the points of Λ. It remains to evaluate the term with $z = \mathbf{0}$:

$$\begin{aligned} \int_{\mathbb{K}^s} \Psi(x) e^{2\pi i \langle m, x \rangle}\, dx &= \prod_{j=1}^{s} \int_{-1}^{1} \psi(x_j) e^{2\pi i m_j x_j}\, dx_j \\ &= \prod_{j=1}^{s} \left(\int_0^1 \psi(x_j - 1) e^{2\pi i m_j x_j}\, dx_j + \int_0^1 \psi(x_j) e^{2\pi i m_j x_j}\, dx_j \right) \\ &= \prod_{j=1}^{s} \int_0^1 e^{2\pi i m_j x_j}\, dx_j = \delta_m. \end{aligned}$$

□

COROLLARY 2. *For $m \in \mathbb{Z}^s$ we have:*

$$T(m) = \delta_m + O\left(\sum_{z \in \Lambda}{}' \frac{1}{\overline{z - m}^{\kappa\alpha}} \right)$$

where the constant depends only on α and κ.

PROOF. We fix a sufficiently large integer c (say, $c = \lceil \kappa\alpha_s \rceil$) and use Lemma 2, integrating by parts c times over x_j for all j satisfying $|m_j - z_j| > 1$. $\square$

Denote

$$\zeta_\Lambda(\alpha) = \sum_{z\in\Lambda}{}' \frac{1}{\overline{z}^\alpha}.$$

PROPOSITION 1. *Assume $\zeta_\Lambda(\kappa\alpha) \le C$ for a positive constant C. Then*

$$F^\kappa_{\kappa,\alpha}(S(\Lambda)) \ll \zeta_\Lambda(\kappa\alpha)$$

with the constant depending on C, α, κ.

PROOF. By Corollary 2,

$$|T(\mathbf{0}) - 1|^\kappa \ll (\zeta_\Lambda(\kappa\alpha))^\kappa \ll \zeta_\Lambda(\kappa\alpha)$$

and hence also $T(\mathbf{0}) = O(1)$. Thus applying Corollary (2) once again we obtain

$$\begin{aligned}\sum_{m\in\mathbb{Z}^s}{}' \frac{|T(m)|^\kappa}{\overline{m}^{\kappa\alpha}} &\le (T(\mathbf{0}))^{\kappa-1} \sum_{m\in\mathbb{Z}^s}{}' \frac{|T(m)|}{\overline{m}^{\kappa\alpha}} \\ &\ll \sum_{m\in\mathbb{Z}^s}{}' \sum_{z\in\Lambda}{}' \frac{1}{\overline{m}^{\kappa\alpha}\,\overline{z-m}^{\kappa\alpha}} \\ &\le \sum_{z\in\Lambda}{}' \prod_{j=1}^{s} \sum_{m_j\in\mathbb{Z}} \frac{1}{(\overline{m}_j \overline{z_j - m_j})^{\kappa\alpha_j}},\end{aligned}$$

and it can be easily seen (see, for instance, [6, Chapter I, Section 2]) that the inner sum is bounded by $1/\overline{z}_j^{\kappa\alpha_j}$. $\square$

An important characteristic of Λ which determines the quality of $S(\Lambda)$ is

$$q(\Lambda) = \min_{z\in\Lambda}{}' \overline{z}$$

(here z assumes non-zero points of Λ). The following lemma was proved in the particular case $\alpha_1 = \cdots = \alpha_s$ in [4], and in the general case in [9, 11].

LEMMA 3. *Assume $q(\Lambda) \ge 2$ and $\alpha_1 > 1$. Then*

$$\zeta_\Lambda(\alpha) \ll \frac{(\ln q(\Lambda))^{r-1}}{(q(\Lambda))^{\alpha_1}}.$$

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COROLLARY 3. *Assume* $q(\Lambda) \geq 2$. *Then*

$$F^{\kappa}_{\kappa,\alpha}(S(\Lambda)) \ll \frac{(\ln q(\Lambda))^{r-1}}{(q(\Lambda))^{\kappa\alpha_1}}.$$

We are now in a position to choose the matrix A. Let $A = \mu A_0$, where μ is a real factor, tending to infinity, and A_0 is a fixed matrix of special form, described below. To emphasize the dependence on μ, we will write S_μ, q_μ instead of $S(\Lambda)$, $q(\Lambda)$, and let N_μ be the number of nodes of S_μ. Since N_μ is the number of integer points in $\mu A_0 \mathbb{K}^s$, we have

$$N_\mu = 2^s \mu^s |\det A_0| (1 + o(1)). \tag{10}$$

Following the remarkable idea of Frolov [5], we choose

$$A_0 = \begin{pmatrix} \omega_1^{(1)} & \dots & \omega_1^{(s)} \\ \vdots & & \vdots \\ \omega_s^{(1)} & \dots & \omega_s^{(s)} \end{pmatrix},$$

where $\omega_1^{(1)}, \dots, \omega_s^{(1)}$ is an integer algebraic basis of some fixed real extension of the field of rational numbers, and $\omega_1^{(j)}, \dots, \omega_s^{(j)}$ $(j = 2, \dots, s)$ are the conjugate bases. Then the coordinates of any non-zero point $z \in \Lambda$ are of the form

$$z_j = \mu(n_1 \omega_1^{(j)} + \dots + n_s \omega_s^{(j)}) \qquad (j = 1, \dots, s)$$

(for some non-zero $n \in \mathbb{Z}^s$), and hence

$$\overline{z} \geq |z_1 \cdots z_s| = \mu^s \left| \prod_{j=1}^{s} (n_1 \omega_1^{(j)} + \dots + n_s \omega_s^{(j)}) \right|.$$

But the product on the right-hand side is the norm of a non-zero algebraic integer, hence $\overline{z} \geq \mu^s$ and $q_\mu \geq \mu^s \gg N_\mu$. Therefore by Corollary 3

$$F^{\kappa}_{\kappa,\alpha}(S_\mu) \ll \frac{(\ln N_\mu)^{r-1}}{N_\mu^{\kappa\alpha_1}}.$$

The only problem remaining is that we cannot guarantee that for any given N there exists μ such that $N_\mu = N$. This can be easily dealt with as follows. By (10), we can always find μ in such a way that

$$N_\mu \leq N < N_\mu (1 + o(1)). \tag{11}$$

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Now, based on the net $S_\mu = (X, \rho)$, we build another net $\widetilde{S} = (\widetilde{X}, \widetilde{\rho})$ with precisely N nodes defining

$$(\widetilde{x}^{(k)}, \widetilde{\rho}_k) = \begin{cases} (x^{(k)}, \rho_k) & k = 1, \ldots, N_\mu, \\ (\mathbf{0}, 0) & k = N_\mu + 1, \ldots, N. \end{cases}$$

Obviously, the exponential sum of $\widetilde{S}$ coincides with that of S, and therefore

$$F_{\kappa,\alpha}(\widetilde{S}) = F_{\kappa,\alpha}(S_\mu) \ll \frac{(\ln N_\mu)^{(r-1)/\kappa}}{N_\mu^{\alpha_1}} \ll \frac{(\ln N)^{(r-1)/\kappa}}{N^{\alpha_1}},$$

completing the proof.

6. Numerical integration of functions with constrained Fourier coefficients

Recall that for $M > 0$, $\tau \in [1, \infty]$ we defined $E_{\tau,\alpha}(M)$ as the class of all those continuous $\mathbb{I}^s$-periodic functions $f : \mathbb{R}^s \to \mathbb{R}$ with Fourier coefficients $\hat{f}$ satisfying

$$\left(\sum_{m \in \mathbb{Z}^s} \left| \hat{f}(m) \overline{m}^\alpha \right|^\tau \right)^{1/\tau} \le M.$$

For $f \in E_{\tau,\alpha}(M)$ denote

$$R(f; S) = \sum_{k=1}^{N} \rho_k f(x^{(k)}) - \int_{\mathbb{I}^s} f(x)\, dx$$

(the error of numerical integration of f on the net S) and let

$$R(E_{\tau,\alpha}(M); S) = \sup_{f \in E_{\tau,\alpha}(M)} |R(f; S)|$$

(the error of numerical integration of $E_{\tau,\alpha}(M)$ on S).

Proof of Theorem 1. We consider only the case $\tau < \infty$; the modifications to be made for $\tau = \infty$ are obvious. For brevity, we write $t(m) = T(m)$, if $m \neq \mathbf{0}$, and $t(\mathbf{0}) = T(\mathbf{0}) - 1$. Since

$$\sum_{m \in \mathbb{Z}^s} \left| \hat{f}(m) \right| = \sum_{m \in \mathbb{Z}^s} \left| \hat{f}(m) \overline{m}^\alpha \right| \frac{1}{\overline{m}^\alpha} \le \left(\sum_{m \in \mathbb{Z}^s} \left| \hat{f}(m) \overline{m}^\alpha \right|^\tau \right)^{1/\tau} \left(\sum_{m \in \mathbb{Z}^s} \frac{1}{\overline{m}^{\kappa\alpha}} \right)^{1/\kappa},$$

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the assumption $\kappa\alpha_1 > 1$ ensures the absolute convergence of the Fourier series for f, and therefore (as in (4))

$$\sum_{k=1}^{N} \rho_k f(x^{(k)}) = \sum_{m\in\mathbb{Z}^s} \hat{f}(m)T(m) = \sum_{m\in\mathbb{Z}^s} \hat{f}(m)t(m) + \hat{f}(\mathbf{0}),$$

$$R(f;S) = \sum_{m\in\mathbb{Z}^s} \hat{f}(m)t(m) = \sum_{m\in\mathbb{Z}^s} \hat{f}(m)\overline{m}^{\alpha}\,\frac{t(m)}{\overline{m}^{\alpha}}.$$

The estimate

$$|R(f;S)| \le M F_{\kappa,\alpha}(S)$$

now follows by the inequality of Hölder. The equality is attained for the function f defined by

$$\hat{f}(m) = \begin{cases} C\,\overline{t(m)}\,|t(m)|^{\kappa-2}/\overline{m}^{\kappa\alpha} & \text{if } t(m)\neq 0,\\ 0 & \text{otherwise,}\end{cases}$$

where C is a positive real normalizing factor (which, of course, could be written out explicitly). Note, that f is real since $\hat{f}(-m) = \overline{\hat{f}(m)}$.

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