THE COORDINATE CONDITIONS AND THE EQUATIONS OF MOTION

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1. Introduction. The problem of the field equations and the equations of motion in general relativity theory is now sufficiently clarified. The equations of motion can be deduced from pure field equations by treating matter as singularities, [**2**; **3**], or from field equations with the energy momentum tensor [**4**]. Recently two papers appeared in which the problem of the coordinate system was considered [**5**; **8**]. The two papers are in general agreement as far as the role of the coordinate system is concerned. Yet there are some differences which require clarification.

For this purpose it will be useful to analyse a very simple and well-known example that will emphasize some of the essential ideas by which field and motion are connected. The problem of motion is complicated in the case of two bodies but much simpler in the case of one body, when we may use the geodetic principle, which, by the way, can also be deduced from the field equations [**1**; **6**]. Therefore, we shall begin by returning to this rather elementary problem and analyse some of its features, which appear in a changed form in the two-body problem. An interesting by-product of this work is a method of deducing perihelion motion, which seems to me simpler and more instructive than any others I know.

2. The Schwarzschild solution and the change of the coordinate system. It is usual to write the quadratic form of the Schwarzschild solution in the following way:

$$ds = \left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{2.1}$$

What are the characteristic features of this coordinate system? For large r the only difference between the Galilean and the Schwarzschild field consists of additional terms that are essentially the Newtonian potentials. For $r \to \infty$ the field becomes Galilean. As a rule it is in such a coordinate system that we calculate the perihelian motion. We ask: Is it possible to find a coordinate system in which the equations of the path are the ordinary equations of a conic without any perihelion motion? This does not mean that the perihelion motion can be wiped out. But it does mean that we can introduce an artificial observer moving with the angular velocity of the perihelion motion.

To show this, at least for the *mean* path, it is sufficient to introduce a change in one variable, that is ϕ. We write:

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$$\phi = \psi + \omega(\psi). \tag{2.2}$$

We shall assume not that $\omega(\psi)$ is small, but that its derivatives are small, that is we shall consistently omit all expressions of higher order than $d\omega/d\psi$. This is our only assumption.

Thus from (2.2) it follows that:

$$d\phi^2 = d\psi^2\left(1 + 2\frac{d\omega}{d\psi}\right).$$

Therefore, in our new coordinate system, the metrical form (2.1) takes the following form:

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\psi^2\left(1 + 2\frac{d\omega}{d\psi}\right). \tag{2.3}$$

The next step is to calculate the differential equations of the geodetic line. This calculation is absolutely straightforward and little difficulty is added by taking (2.3) instead of (2.1) We begin with the procedure described extensively in any book on general relativity theory and take $\theta = \frac{1}{2}\pi$. We then obtain three equations. The first is simply (2.1) or, in our coordinate system, (2.3) for $\theta = \frac{1}{2}\pi$ and $d\theta = 0$. Then we have two integrals. One of them which represents the conservation of momenta is, because of (2.2),

$$r^2 \frac{d\psi}{ds} + r^2 \frac{d\omega}{ds} = h. \tag{2.4}$$

The third equation, also integrated, is

$$\frac{dt}{ds} = C\left(1 - \frac{2m}{r}\right)^{-1} \tag{2.5}$$

or precisely the same as in the original coordinate system represented by (2.1). Thus (2.3), (2.4), (2.5) represent our equations of motion.

We can write (2.4):

$$r^2 \frac{d\psi}{ds} + r^2 \frac{d\omega}{d\psi}\frac{d\psi}{ds} = h, \tag{2.4a}$$

therefore:

$$\frac{d\psi}{ds} = \frac{h}{r^2} - \frac{d\omega}{d\psi}\frac{d\psi}{ds}.$$

Thus, neglecting higher powers of $d\omega/d\psi$, we can write (2.4):

$$\frac{d\psi}{ds} = \frac{h}{r^2}\left(1 - \frac{d\omega}{d\psi}\right). \tag{2.6}$$

Besides this, we have equations (2.5) and (2.3) for $d\theta = 0$, $\theta = \frac{1}{2}\pi$, which we shall rewrite now:

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$$(2.3\text{a})\quad \left(1-\frac{2m}{r}\right)\left(\frac{dt}{ds}\right)^2-\left(\frac{dr}{ds}\right)^2\left(1-\frac{2m}{r}\right)^{-1}-r^2\left(\frac{d\psi}{ds}\right)^2\left(1+2\frac{d\omega}{d\psi}\right)=1.$$

Here we replace:

$$\frac{dt}{ds}\quad\text{by}\quad C\left(1-\frac{2m}{r}\right)^{-1},\qquad\text{according to (2.5),}$$

$$\frac{dr}{ds}\quad\text{by}\quad \frac{dr}{d\psi}\frac{d\psi}{ds},$$

$$\frac{d\psi}{ds}\quad\text{by}\quad \frac{h}{r^2}\left(1-\frac{d\omega}{d\psi}\right),\qquad\text{according to (2.6).}$$

We obtain

$$(2.7)\qquad C^2-\left(\frac{dr}{d\psi}\right)^2\left(1-2\frac{d\omega}{d\psi}\right)\frac{h^2}{r^4}-\frac{h^2}{r^2}\left(1-\frac{2m}{r}\right)=1-\frac{2m}{r}.$$

We could have done this still more simply by taking the usual equation in the coordinate system (2.1), that is the equation:

$$C^2-\left(\frac{dr}{d\phi}\right)^2\frac{h^2}{r^4}-\frac{h^2}{r^2}\left(1-\frac{2m}{r}\right)=1-\frac{2m}{r}$$

and introducing into this equation the transformation (2.2). Equation (2.7) follows immediately.

Now we write (2.7) in the following form, introducing as usual $u = r^{-1}$:

$$(2.8)\quad C^2-h^2\left(\frac{du}{d\psi}\right)^2-h^2u^2-1+2mu+2h^2\left(\frac{du}{d\psi}\right)^2\left(\frac{d\omega}{d\psi}\right)+2mh^2u^3=0.$$

Our question is whether by a proper choice of the function ω we can rigorously solve the equation (2.8) by

$$(2.9)\qquad u=\frac{1+e\cos\psi}{p}.$$

This is indeed possible and to show it we split the equation (2.8) into two parts which we shall satisfy separately:

$$(2.10)\qquad C'^2-h^2\left(\frac{du}{d\psi}\right)^2-h^2u^2-1+2mu=0,$$

$$(2.11)\qquad C^2-C'^2+2h^2\left(\frac{du}{d\psi}\right)^2\left(\frac{d\omega}{d\psi}\right)+2mh^2u^3=0.$$

Here (2.10) is the Newtonian equation, in which, provisionally, the constant C' may be regarded as unknown. Its solution therefore is (2.9), where e and p are functions of C' and h. Now as to the equation (2.11), we introduce into it (2.9), regarding it as an equation for ω and C'. We obtain from (2.11):

$$(2.11\text{a})\quad e^2\left(\frac{d\omega}{d\psi}\right)\sin^2\psi+\frac{m}{p}(1+3e\cos\psi+3e^2\cos^2\psi+e^3\cos^3\psi)+\frac{(C^2-C'^2)p^2}{2h^2}=0$$

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or, assuming $\sin^2 \psi \neq 0$:

$$\frac{d\omega}{d\psi} + \frac{3m}{pe}\frac{\cos\psi}{\sin^2\psi} - \frac{3m}{p} + \frac{me}{p}\frac{\cos^3\psi}{\sin^2\psi} + \frac{1}{\sin^2\psi}\left(\frac{m}{pe^2} + \frac{3m}{p} + \frac{C^2 - C'^2}{2h^2}p^2\right) = 0. \tag{2.11b}$$

Thus we can solve this equation, by putting:

$$C'^2 = C^2 + \frac{2mh^2}{p^3e^2} + \frac{6mh^2}{p^3} \tag{2.12}$$

and

$$\frac{d\omega}{d\psi} + \frac{3m\cos\psi}{pe\sin^2\psi} - \frac{3m}{p} + \frac{me\cos^3\psi}{p\sin^2\psi} = 0. \tag{2.13}$$

Equation (2.12) says that the ellipse is slightly changed from the one which corresponds to the constant C. Remembering that

$$h^2 \text{ is of the order } r^4\omega^2$$

(r being the "distance" of the planet from the sun) and that

$$p \text{ is of the order of } r,$$

we see that

$$C^2 - C'^2 \text{ is of the order } mv^2/r$$

(v being the "velocity"). Therefore (2.10) in which C^2 was replaced by C'^2 gives the "relativistic" correction to the "energy."

Equation (2.13) is not sufficient to determine ω in the neighbourhood of $2\pi n$. But it is sufficient to determine the perihelion motion. We have:

$$\phi = \psi + \omega(\psi) = f(\psi).$$

Let us write

$$\phi_n = f(2\pi n) = 2\pi n + \omega_n$$

$$\phi_{n+1} = f(2\pi n + 2\pi) = 2\pi(n+1) + \omega_{n+1},$$

then the definition of the perihelion motion is:

$$\phi_{n+1} - \phi_n - 2\pi = \omega_{n+1} - \omega_n.$$

Now integrating (2.13) from $2\pi n$ to $2\pi(n+1)$ we have

$$\omega_{n+1} - \omega_n = 6m\pi/p,$$

which formula expresses the famous perihelion motion.

In order that the perihelion motion describe reality, it must be, in a certain sense at least, independent of the coordinate system. Indeed, the perihelion motion *is* independent of the coordinate system, as long as the coordinate system is Galilean at infinity.

The description of the perihelion motion of, say, Mercury is based on the assumption that the observer is so far away and in such a coordinate system that *his* field is Galilean. Otherwise, nothing else is required from the coordinate system. The one represented by (2.1) would do just as well as any other as long as it is Galilean at infinity.

Imagine an observer (in a Galilean local coordinate system) very far from the "sun," pointing with one rigid rod towards the sun and with another towards Mercury, at a moment when the angle between the two rods is smallest. In principle, this can be done, since, in a Galilean coordinate system, all the concepts—rigid rods, angles, etc., are the familiar Euclidean concepts. Then the rigid rod pointing toward Mercury will describe an ellipse. But after one rotation, the perihelion position (that is, the minimum angle between the two rods) will be slightly shifted. After very many rotations, the original perihelion position from which the idealized observer started will be reached again. We can ask: After how many rotations of Mercury will the perihelion point complete *one* rotation? Such a formulation of the question has nothing to do with any special coordinate system but it does assume that the chosen coordinate system is Galilean at infinity, because only in such a coordinate system can we use freely the concepts of rigid body, angles, etc. Thus we can have either Newtonian motion and a non-Galilean field at infinity, or perihelion motion and a Galilean field at infinity.

3. The two-body problem. The above discussion should help to clarify the problem of the coordinate system for a two-body problem. It was solved in a series of papers, not by assuming the principle of a geodetic line but by deducing the equations of motion from the field equations. One of the essential ideas used then is the "new approximation method." By its use we find the field in the second and third approximations. Then from this knowledge of the field we deduce the equations of motion in the fourth approximation. These are the Newtonian equations of motion. The next step is to find the field in the fourth and fifth approximations. Finally, we deduce the equations of motion up to the sixth approximation. These equations give us the perihelion motion of the two (or many) body problem. It would be extremely difficult, technically, to go beyond the sixth approximation. But it is not necessary to do so. All contributions to the motion in higher than the sixth approximation can be wiped out by a proper choice of a coordinate system [**5**].

Thus having described the procedure in general terms we shall write down the field equations in the second and third approximations [**3**, p. 219]. These are:

$$(3.1)\qquad \underset{2}{\gamma}_{00,ss} = 0$$
$$- \underset{3}{\gamma}_{0m,ss} + \underset{3}{\gamma}_{0s,ms} = \underset{2}{\gamma}_{00,0m}; \qquad m, s = 1, 2, 3,$$

where

$$\gamma_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\sigma\rho}h_{\sigma\rho} \qquad \mu, \nu, \sigma, \rho = 0, 1, 2, 3$$
$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$
$$\eta_{ms} = -\delta_{ms}; \quad \eta_{00} = 1; \quad \eta_{0m} = 0.$$

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Here, and later, Greek indices take the values 0, 1, 2, 3 and Latin indices the values 1, 2, 3. One more remark: the stroke means differentiation:

$$\gamma_{\mu\nu/s} = \frac{\partial g_{\mu\nu}}{\partial x^s} = g_{\mu\nu,s}$$

$$\gamma_{\mu\nu/0} = \frac{\partial g_{\mu\nu}}{\partial x^0} = \frac{\partial g_{\mu\nu}}{\partial \tau}\lambda = \lambda g_{\mu\nu,0} \qquad (\lambda x^0 = \tau).$$

Thus the differentiation with respect to τ raises the order, since λ is the parameter of the development and its order is indicated by a number below the γ's.

The essential point in the use of the "new approximation method" consisted in the fact that we began the development of γ_{00}, γ_{0m}, and γ_{mn} with $\underset{2}{\gamma_{00}}$, $\underset{3}{\gamma_{0m}}$, and $\underset{4}{\gamma_{mn}}$ respectively:

$$(3.2) \qquad \begin{aligned} \gamma_{00} &= \lambda^2 \underset{2}{\gamma_{00}} + \lambda^4 \underset{4}{\gamma_{00}} + \dots \\ \gamma_{0m} &= \lambda^3 \underset{3}{\gamma_{0m}} + \lambda^5 \underset{5}{\gamma_{0m}} + \dots \\ \gamma_{mn} &= \lambda^4 \underset{4}{\gamma_{mn}} + \lambda^6 \underset{6}{\gamma_{mn}} + \dots . \end{aligned}$$

If we look at (3.1) we see that γ_{00} must be a harmonic function. Let us choose an harmonic function that represents two moving singularities. But then γ_{0m} is not uniquely determined. Indeed let us replace

$$\underset{3}{\gamma_{0m}} \quad \text{by} \quad \underset{3}{\gamma_{0m}} + \underset{3}{a_{0,m}}$$

a_0 being an arbitrary function. Then $\underset{3}{\gamma_{0m}} + \underset{3}{a_{0,m}}$ *is* a solution. Thus we could ask whether the arbitrary function $\underset{3}{a_{0,m}}$ can have any influence upon the equations of motion. It certainly does not have any influence on the equations of motion of the fourth order (which are the Newtonian equations of motion). This follows from a theorem proved before [**3**, p. 233]. But a straightforward, though troublesome, calculation [**9**] shows that it has no influence, either, upon the equations of motion up to the sixth order. Thus it is not true [**8**] as Papapetrou claims that we could have changed the equations of motion by changing the coordinate condition involving $\underset{2}{\gamma_{00}}$ and $\underset{3}{\gamma_{0m}}$. Thus nothing can change the equations of motion up to the sixth order, as long as we stick to the following procedures:

1. To introduce solutions of Laplace's equations for $\underset{2}{\gamma_{00}}$ that represent the Newtonian fields.

2. To use the "new approximation method", starting with $\underset{2}{\gamma_{00}}$, $\underset{3}{\gamma_{0m}}$, $\underset{4}{\gamma_{mn}}$ respectively.

Once these procedures are assumed and no singularities later arbitrarily introduced, the equations of motion up to the sixth order are uniquely determined, giving us, in consequence, the perihelion motion of a double star.

But we have seen in the first part, that we can wipe out the perihelion motion by a proper choice of the coordinate system. There seems to be a contradiction between our statement here and the result of the first part of this paper. The solution is simple enough: to introduce a coordinate system that would wipe out the perihelion motion we would have to violate the second of our assumptions just quoted.

Although the corresponding calculation is elementary, we shall show it in detail in order to clarify the problem.

Let us rewrite (2.1) in the form:

$$ds^2 = \left(1 - \frac{2m}{\rho}\right)dt^2 - \left(1 - \frac{2m}{\rho}\right)^{-1} d\rho^2 - \rho^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{3.3}$$

We introduce the transformation

$$\rho = r + m = r + \underset{2}{m}, \tag{3.4}$$

since m is of the second order. Again we rewrite (3.3) keeping only expressions up to the second order:

$$ds^2 = (1 - 2\underset{2}{m}r^{-1})dt^2 - dr^2(1 + 2\underset{2}{m}r^{-1})((dx^1)^2 + (dx^2)^2 + (dx^3)^2). \tag{3.5}$$

(Here the connection between r, θ, ϕ and x^1, x^2, x^3 is obvious.)

Thus we have in (3.5)

$$\underset{2}{h}_{00} = -2\underset{2}{m}r^{-1}; \quad \underset{2}{h}_{mn} = -\delta_{mn}2\underset{2}{m}r^{-1}.$$

Therefore:

$$\underset{2}{\gamma}_{00} = \tfrac{1}{2}h_{00} + \tfrac{1}{2}h_{ss} = -4\underset{2}{m}r^{-1},$$

$$\underset{3}{\gamma}_{0m} = 0,$$

$$\underset{2}{\gamma}_{mn} = h_{mn} - \tfrac{1}{2}\delta_{mn}h_{ss} + \tfrac{1}{2}\delta_{mn}h_{00} = 0.$$

Thus this solution satisfies our conditions, because $\underset{2}{\gamma}_{mn} = 0$ and because $\underset{2}{\gamma}_{00}$ represents a Newtonian field belonging to a particle.

To wipe out the perihelion motion we would have to introduce a rotation in the "plane" of the motion with an angular velocity ω chosen so as to wipe out such a motion. Thus we introduce the transformation:

$$x^1 = \xi^1\cos\omega - \xi^2\sin\omega$$

$$x^2 = \xi^1\sin\omega + \xi^2\cos\omega$$

$$x^3 = \xi^3.$$

In this coordinate system we have the following quadratic form:

$$(3.6) \quad ds^2 = [1 - 2\underset{2}{m}r^{-1} + \dot{\omega}^2((\xi^1)^2 + (\xi^2)^2)]dt^2 + 2\xi^1\dot{\omega}dt\, d\xi^2 - 2\xi^2\dot{\omega}dt\, d\xi^1 - dr^2(1 + 2\underset{2}{m}r^{-1})((d\xi^1)^2 + (d\xi^2)^2 + (d\xi^3)^2).$$

If $\dot{\omega} = \underset{1}{\dot{\omega}}$ is of order one, we see that our quadratic form violates both the conditions on which our approximation procedure is built. First, $\underset{2}{\gamma_{00}}$ does not vanish for $r \to \infty$, and does not represent a Newtonian field of a particle. Secondly, γ_{0m} does not start with $\underset{3}{\gamma_{0m}}$ but with $\underset{1}{\gamma_{0m}}$. Thirdly,

$$\underset{2}{\gamma_{mn}} = \tfrac{1}{2}\dot{\omega}^2((\xi^1)^2 + (\xi^2)^2)\,\delta_{mn},$$

that is, γ_{mn} does not start with $\underset{4}{\gamma_{mn}}$.

Thus, returning to the two-body problem, we see that the conditions of our method are sufficiently stringent to insure uniquely the equations of motion up to the sixth order without any additional conditions concerning the coordinate system.

Therefore the choice of a coordinate system has no influence upon the equations of motion up to the sixth order. A straightforward calculation shows that this is true for $\underset{3}{\gamma_{0m}}$; that the addition of an arbitrary function $a_{0,m}$ does not change the surface integrals that determine the equations of motion. But as has been shown before [**3**, p. 13] the choice of the coordinate system in $\underset{4}{\gamma_{mn}}$, $\underset{4}{\gamma_{00}}$, $\underset{5}{\gamma_{m0}}$, has no influence upon the equations of motion of the sixth order. But it will have influence upon the equations of motion of the eighth order. Therefore it is possible to use such a coordinate system for $\underset{3}{\gamma_{0m}}$, $\underset{4}{\gamma_{mn}}$, $\underset{4}{\gamma_{00}}$, $\underset{5}{\gamma_{0m}}$, so that the contributions to the surface integrals coming from $\underset{8}{\Lambda}$ are zero. We know [**5**], that such a coordinate system exists, though it would not be an easy task to find it explicitly. Similarly we can introduce such coordinate conditions in $\underset{6}{\gamma_{mn}}$, $\underset{6}{\gamma_{00}}$, $\underset{7}{\gamma_{m0}}$ that the contributions to the surface integral $\underset{10}{\Lambda}$ will be zero. Thus it is possible to regard the equations of motion that we obtained before [**2**] as the *exact* equations of motion in a properly chosen coordinate system. We can get rid of all contributions to the surface integral beginning with $\underset{8}{\Lambda}$, but we can not get rid of $\underset{4}{\Lambda}$ and $\underset{6}{\Lambda}$; these expressions, and also the equations of motion up to the sixth order, are determined not by the choice of our coordinate system, but essentially by the procedure concerning the use of our new approximation method.

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