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φ-CONTRACTIBILITY AND φ-CONNES AMENABILITY COINCIDE WITH SOME OLDER NOTIONS

AMIN MAHMOODI

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Abstract

It is shown that various definitions of φ-Connes amenability and φ-contractibility are equivalent to older and simpler concepts.

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1. Introduction

The fertile notion of amenability was introduced by Johnson in [5]. A generalisation of amenability depending on homomorphisms was introduced and studied by Kaniuth *et al.* [6] and independently by Monfared [8]. For a Banach algebra $\mathfrak{A}$, we write $\Delta(\mathfrak{A})$ for the set of all homomorphisms from $\mathfrak{A}$ onto $\mathbb{C}$. Let $\varphi \in \Delta(\mathfrak{A})$. An element $m \in \mathfrak{A}^{**}$ is called a *right* [*left*] *φ-mean* if $m(\varphi) = 1$ and $m(f \cdot a) = \varphi(a)m(f)$ $[m(a \cdot f) = \varphi(a)m(f)]$ for $a \in \mathfrak{A}$ and $f \in \mathfrak{A}^*$. A Banach algebra is *right* [*left*] *φ-amenable* if it has a right [left] φ-mean [6, 8]. We say that $\mathfrak{A}$ is *φ-amenable* if it is both left and right φ-amenable.

The concept of φ-contractibility was introduced by Hu *et al.* in [4]. Let $\mathfrak{A}$ be a Banach algebra and E be a Banach $\mathfrak{A}$-bimodule. A continuous linear operator $D : \mathfrak{A} \to E$ is a *derivation* if it satisfies $D(ab) = D(a) \cdot b + a \cdot D(b)$ for all $a, b \in \mathfrak{A}$. Given $x \in E$, the *inner* derivation $ad_x : \mathfrak{A} \to E$ is defined by $ad_x(a) = a \cdot x - x \cdot a$. Let $\varphi \in \Delta(\mathfrak{A})$. We write $\mathbb{M}_\varphi$ $[{}_\varphi\mathbb{M}]$ for the set of all Banach $\mathfrak{A}$-bimodules E such that the right [left] module action of $\mathfrak{A}$ on E is given by $x \cdot a := \varphi(a)x$ $[a \cdot x := \varphi(a)x]$ for $a \in \mathfrak{A}$, $x \in E$. We say that $\mathfrak{A}$ is *right* [*left*] *φ-contractible* if for each Banach $\mathfrak{A}$-bimodule $E \in \mathbb{M}_\varphi$ $[E \in {}_\varphi\mathbb{M}]$, every derivation $D : \mathfrak{A} \to E$ is inner. We say that $\mathfrak{A}$ is φ-contractible if it is both left and right φ-contractible.

Recently and motivated by these notions, several authors have defined and studied the concept of φ-Connes amenability of dual Banach algebras, where φ is a w^*-continuous homomorphism onto $\mathbb{C}$ (see [2, 7, 10]). However, these three variations on φ-Connes amenability are not actually new. We show that they coincide with φ-amenability and φ-contractibility (in the particular contexts considered in [2, 7, 10]).

We also show that φ-contractibility is equivalent to an existing notion. In fact, a Banach algebra $\mathfrak{A}$ is φ-contractible if and only if the one-dimensional Banach $\mathfrak{A}$-bimodule $\mathbb{C}_\varphi$ is projective. This concept goes back to Helemskii in the 1970s (see his book [3] and the paper of White [11]).

2. φ-Connes amenability

Let $\mathfrak{A}$ be a Banach algebra. A Banach $\mathfrak{A}$-bimodule E is *dual* if there is a closed submodule E_* of E^* such that $E = (E_*)^*$. We call E_* the *predual* of E. A Banach algebra is *dual* if it is dual as a Banach $\mathfrak{A}$-bimodule. We write $\mathfrak{A} = (\mathfrak{A}_*)^*$ if we wish to stress that $\mathfrak{A}$ is a dual Banach algebra with predual $\mathfrak{A}_*$. For a dual Banach algebra $\mathfrak{A}$, let $\Delta_{w^*}(\mathfrak{A})$ denote the set of all w^*-continuous homomorphisms from $\mathfrak{A}$ onto $\mathbb{C}$.

We start with the definition of φ-Connes amenability in the sense of [2]. Let $\mathfrak{A} = (\mathfrak{A}_*)^*$ be a dual Banach algebra and let $\varphi \in \Delta_{w^*}(\mathfrak{A})$. A dual Banach $\mathfrak{A}$-bimodule $E \in {}_\varphi\mathbb{M}$ is *normal* if the module action $a \mapsto x \cdot a$ of $\mathfrak{A}$ on E is w^*-continuous. A dual Banach algebra $\mathfrak{A} = (\mathfrak{A}_*)^*$ is *left φ-Connes amenable* if for every normal dual Banach $\mathfrak{A}$-bimodule $E \in {}_\varphi\mathbb{M}$, every w^*-continuous derivation $D : \mathfrak{A} \to E$ is inner. We can make similar definitions for right φ-Connes amenable and φ-Connes amenable Banach algebras. It is shown in [2, Theorem 2.3] that (left) φ-Connes amenability of $\mathfrak{A}$ is equivalent to the existence of a (left) φ-mean.

Theorem 2.1. *Suppose that $\mathfrak{A} = (\mathfrak{A}_*)^*$ is a dual Banach algebra and $\varphi \in \Delta_{w^*}(\mathfrak{A})$. Then the following statements are equivalent:*

(i) *$\mathfrak{A}$ is φ-Connes amenable (in the sense of [2]);*
(ii) *$\mathfrak{A}$ is φ-contractible;*
(iii) *$\mathfrak{A}$ is φ-amenable.*

Proof. (i) $\Rightarrow$ (ii) By [2, Theorem 2.3], we can take a φ-mean $m \in \mathfrak{A}^{**}$. Consider the $\mathfrak{A}$-bimodule inclusion map $\iota : \mathfrak{A}_* \to \mathfrak{A}^*$. Taking adjoints, we obtain a w^*-w^*-continuous $\mathfrak{A}$-bimodule map $\xi : \mathfrak{A}^{**} \to \mathfrak{A}$. Now put $u = \xi(m) \in \mathfrak{A}$. It is easily checked that $\varphi(u) = 1$ and $ua = au = \varphi(a)u$ for all $a \in \mathfrak{A}$. Therefore, by Theorem 3.1 below, $\mathfrak{A}$ is φ-contractible.

(ii) $\Rightarrow$ (iii) Choose $u \in \mathfrak{A}$ as in Theorem 3.1. Then it is readily seen that u is a φ-mean and hence $\mathfrak{A}$ is φ-amenable.

(iii) $\Rightarrow$ (i) By [2, Theorem 2.3] again, every φ-amenable Banach algebra is φ-Connes amenable. This is noted in [2] (see the remark following Definition 2.2). Alternatively, we can appeal to [6, Theorem 1.1]. Let $E = (E_*)^*$ be a normal dual Banach $\mathfrak{A}$-bimodule with $E \in {}_\varphi\mathbb{M}$ and let $D : \mathfrak{A} \to E$ be a w^*-continuous derivation. Since $\mathfrak{A}$ is φ-amenable, D is inner by [6, Theorem 1.1]. Hence, $\mathfrak{A}$ is φ-Connes amenable. □

Now we consider the definition of φ-Connes amenability in the sense of [10]. Let $\mathfrak{A}$ be a dual Banach algebra and let $\varphi \in \Delta_{w^*}(\mathfrak{A})$. According to [10, Definition 2.1], $\mathfrak{A}$ is (*left*) *φ-Connes amenable* if there exists $m \in \mathfrak{A}$ such that $m(\varphi) = 1$ and $am = \varphi(a)m$

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for every $a \in \mathfrak{A}$. By Theorem 3.1, left φ-Connes amenability is nothing but right φ-contractibility. Hence, the following theorem is straightforward.

THEOREM 2.2. *Suppose that $\mathfrak{A}$ is a dual Banach algebra and $\varphi \in \Delta_{w^*}(\mathfrak{A})$. Then the following statements are equivalent:*

(i) *$\mathfrak{A}$ is φ-Connes amenable (in the sense of* [10]*);*
(ii) *$\mathfrak{A}$ is φ-contractible;*
(iii) *$\mathfrak{A}$ is φ-amenable.*

Let $\mathfrak{A}$ be a dual Banach algebra and let E be a Banach $\mathfrak{A}$-bimodule. As in [9], we write $\sigma wc(E)$ for the set of all elements $x \in E$ such that the maps

$$\mathfrak{A} \to E, \quad a \mapsto \begin{cases} a \,.\, x, \\ x \,.\, a \end{cases}$$

are w^*-weak continuous.

The following lemma is immediate.

LEMMA 2.3. *Suppose that $\mathfrak{A}$ is a dual Banach algebra. Then $\Delta_{w^*}(\mathfrak{A}) = \Delta(\mathfrak{A}) \cap \sigma wc(\mathfrak{A}^*)$.*

The notion of φ-Connes amenability as defined in [7] is based on Lemma 2.3. Suppose that $\mathfrak{A}$ is a dual Banach algebra and $\varphi \in \Delta_{w^*}(\mathfrak{A})$. We call $\mathfrak{A}$ *(right) φ-Connes amenable* if $\mathfrak{A}$ admits a *(right) φ-Connes mean m*, that is, there exists a bounded linear functional m on $\sigma wc(\mathfrak{A}^*)$ satisfying $m(\varphi) = 1$ and $m(f \,.\, a) = \varphi(a)m(f)$ for all $a \in \mathfrak{A}$ and $f \in \sigma wc(\mathfrak{A}^*)$. Similarly, we may consider *left* φ-Connes amenability. We say that $\mathfrak{A}$ is *φ-Connes amenable* if it is both left and right φ-Connes amenable.

THEOREM 2.4. *Suppose that $\mathfrak{A} = (\mathfrak{A}_*)^*$ is a dual Banach algebra and $\varphi \in \Delta_{w^*}(\mathfrak{A})$. Then the following statements are equivalent:*

(i) *$\mathfrak{A}$ is φ-Connes amenable (in the sense of* [7]*);*
(ii) *$\mathfrak{A}$ is φ-contractible;*
(iii) *$\mathfrak{A}$ is φ-amenable.*

PROOF. Only (i) $\Leftrightarrow$ (ii) needs proof. Let $\iota : \mathfrak{A} \to \sigma wc(\mathfrak{A}^*)^*$ be the $\mathfrak{A}$-bimodule map obtained by composing the canonical inclusion $\mathfrak{A} \to \mathfrak{A}^{**}$ with the quotient map $\mathfrak{A}^{**} \to \sigma wc(\mathfrak{A}^*)^*$, so that $\langle \iota(a), \psi \rangle = \psi(a)$ for all $a \in \mathfrak{A}$ and $\psi \in \sigma wc(\mathfrak{A}^*)$.

(i) $\Rightarrow$ (ii) Since $\mathfrak{A}$ is a dual Banach algebra, $\mathfrak{A}_*$ is an $\mathfrak{A}$-bimodule and $\mathfrak{A}_* \subseteq \sigma wc(\mathfrak{A}^*)$ (see [9, Corollary 4.6]). Taking adjoints gives a w^*–w^*-continuous $\mathfrak{A}$-bimodule map $\xi : \sigma wc(\mathfrak{A}^*)^* \longrightarrow \mathfrak{A}$. Notice that $\xi \circ \iota(a) = a$ for all $a \in \mathfrak{A}$. By the assumption, there exists a φ-Connes mean $m \in \sigma wc(\mathfrak{A}^*)^*$. Set $u = \xi(m) \in \mathfrak{A}$. Then $\mathfrak{A}$ is φ-contractible by Theorem 3.1.

(ii) $\Rightarrow$ (i) Take $u \in \mathfrak{A}$ satisfying $\varphi(u) = 1$ and $ua = au = \varphi(a)u$ for all $a \in \mathfrak{A}$. Then $\iota(u)$ is a φ-Connes mean on $\sigma wc(\mathfrak{A}^*)$. □

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For a given dual Banach algebra $\mathfrak{A}$, it would be interesting to determine the set $\Delta_{w^*}(\mathfrak{A})$. We give some illustrative examples. It is known that $\mathcal{L}(\mathfrak{H})$, the algebra of all bounded linear operators on a Hilbert space $\mathfrak{H}$, is a dual Banach algebra. Let $\mathfrak{A}_V$ be the dual Banach algebra generated by the *Volterra operator* $V \in \mathcal{L}(L^2([0,1]))$ defined by $Vf(x) = \int_0^x f(t)\,dt$. In fact, $\mathfrak{A}_V$ is the w^*-closure of the algebra of all polynomials $p(V)$ in $\mathcal{L}(L^2([0,1]))$. In [1, Corollary 3.10], it was shown that $\Delta_{w^*}(\mathfrak{A}_V) = \emptyset$. Next, for the discrete convolution algebra $\ell^1(\mathbb{Z}^+)$, it is well known that $\Delta(\ell^1(\mathbb{Z}^+)) \cong \bar{\mathbb{D}}$, where $\mathbb{D}$ denotes the open unit disc in $\mathbb{C}$ centred at 0 (see [6, Example 2.5(2)]). Indeed, $\Delta(\ell^1(\mathbb{Z}^+))$ consists precisely of the point evaluations $\varphi_z(f) = f(z)$ for $f \in \ell^1(\mathbb{Z}^+)$ and $z \in \bar{\mathbb{D}}$. It is easy to check that φ_z is w^*-continuous when $z \in \mathbb{D}$. Therefore, $\Delta_{w^*}(\ell^1(\mathbb{Z}^+))$ is a dense subset of $\Delta(\ell^1(\mathbb{Z}^+))$.

3. φ-contractibility

It is known that right [left] φ-contractibility of $\mathfrak{A}$ is equivalent to the existence of a *right* [*left*] *φ-diagonal* for $\mathfrak{A}$, that is, an element $m \in \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ such that $\varphi(\pi(m)) = 1$ and $a \cdot m = \varphi(a)m$ [$m \cdot a = \varphi(a)m$] for $a \in \mathfrak{A}$, where $\pi : \mathfrak{A}\widehat{\otimes}\mathfrak{A} \to \mathfrak{A}$ is the bounded linear map determined by $\pi(a \otimes b) = ab$. If m is both a left and a right φ-diagonal, it is called a *φ-diagonal*.

The following theorem is likely to be well known, but since we could not locate a reference, we include a proof.

THEOREM 3.1. *Suppose that $\mathfrak{A}$ is a Banach algebra and $\varphi \in \Delta(\mathfrak{A})$. Then $\mathfrak{A}$ is φ-contractible if and only if there exists an element $u \in \mathfrak{A}$ satisfying*

$$\varphi(u) = 1 \quad \textit{and} \quad ua = au = \varphi(a)u \quad \textit{for } a \in \mathfrak{A}. \tag{$*$}$$

PROOF. Suppose that $u \in \mathfrak{A}$ satisfies the conditions in ($*$). Let $D : \mathfrak{A} \to E$ be a derivation for a Banach $\mathfrak{A}$-bimodule $E \in \mathbb{M}_\varphi$. Then $D^*(f \cdot a) = D^*(f) \cdot a - \langle Da, f\rangle\varphi$ for each $f \in E^*$ and $a \in \mathfrak{A}$. Put $t := Du \in E$. For $f \in E^*$ and $a \in \mathfrak{A}$,

$$\begin{aligned}\langle f, a \cdot t\rangle &= \langle D^*(f \cdot a), u\rangle = \langle D^*(f), au\rangle - \langle Da, f\rangle \\ &= \varphi(a)\langle D^*(f), u\rangle - \langle Da, f\rangle = \varphi(a)\langle f, t\rangle - \langle Da, f\rangle.\end{aligned}$$

Therefore, $a \cdot t = \varphi(a)t - Da$ for $a \in \mathfrak{A}$. Thus, $D = ad_{-t}$ and $\mathfrak{A}$ is right φ-contractible. Notice that to get right φ-contractibility, we use only $au = \varphi(a)u$. A similar argument shows that $\mathfrak{A}$ is also left φ-contractible.

Conversely, suppose that $m \in \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ is a φ-diagonal for $\mathfrak{A}$. Put $u := \pi(m) \in \mathfrak{A}$. It is easily checked that u has the properties in ($*$). □

Let $\mathfrak{A}$ be a Banach algebra with the *unitisation* $\mathfrak{A}^\sharp$ and let P be a left Banach $\mathfrak{A}$-module. We recall that P is a *projective* left $\mathfrak{A}$-module if the multiplication map

$$\pi : \mathfrak{A}^\sharp \widehat{\otimes} P \to P, \quad a \otimes x \mapsto a \cdot x \quad \text{for } a \in \mathfrak{A}^\sharp, x \in P,$$

has a right inverse which is also a left $\mathfrak{A}$-module homomorphism. Similar definitions hold for projective right $\mathfrak{A}$-modules and projective $\mathfrak{A}$-bimodules.

For $\varphi \in \Delta(\mathfrak{A})$, the space $\mathbb{C}_\varphi = \{\alpha\varphi \;:\; \alpha \in \mathbb{C}\}$ is a Banach $\mathfrak{A}$-bimodule with module actions $a \cdot \varphi = \varphi \cdot a := \varphi(a)\varphi$ for $a \in \mathfrak{A}$.

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Theorem 3.2. *Suppose that $\mathfrak{A}$ is a Banach algebra and $\varphi \in \Delta(\mathfrak{A})$. Then $\mathfrak{A}$ is φ-contractible if and only if $\mathbb{C}_\varphi$ is a projective Banach $\mathfrak{A}$-bimodule.*

Proof. Without loss of generality, we may assume that $\mathfrak{A}$ is unital. Let $\mathbb{C}_\varphi$ be projective as a left $\mathfrak{A}$-module. Then there exists a bounded linear map $\rho : \mathbb{C}_\varphi \longrightarrow \mathfrak{A}\widehat{\otimes}\mathbb{C}_\varphi$ satisfying $\pi\rho(\varphi) = \varphi$ and $a \cdot \rho(\varphi) = \varphi(a)\rho(\varphi)$ for each $a \in \mathfrak{A}$. We have $\rho(\varphi) = \sum_{n=1}^{\infty} a_n \otimes \varphi$, where $a_n \in \mathfrak{A}$ for $n = 1, 2, \ldots$ and $\sum_{n=1}^{\infty} \|a_n\| < \infty$. Put $u := \sum_{n=1}^{\infty} a_n \in \mathfrak{A}$. Then $\varphi(u) = 1$ and $au = \varphi(a)u$ for $a \in \mathfrak{A}$. By Theorem 3.1, $\mathfrak{A}$ is right φ-contractible.

Conversely, let $\mathfrak{A}$ be right φ-contractible. Take $u \in \mathfrak{A}$ with $\varphi(u) = 1$ and $au = \varphi(a)u$ for all $a \in \mathfrak{A}$. Then it is easy to verify that the map $\rho : \mathbb{C}_\varphi \longrightarrow \mathfrak{A}\widehat{\otimes}\mathbb{C}_\varphi$ defined by $\rho(\varphi) := u \otimes \varphi$ is a left $\mathfrak{A}$-module homomorphism which is a right inverse of π. It follows that $\mathbb{C}_\varphi$ is a projective left $\mathfrak{A}$-module.

Similarly, one can see that $\mathbb{C}_\varphi$ is a projective right $\mathfrak{A}$-module if and only if $\mathfrak{A}$ is left φ-contractible. □

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AMIN MAHMOODI, Department of Mathematics,
Central Tehran Branch, Islamic Azad University, Tehran, Iran
e-mail: a_mahmoodi@iauctb.ac.ir