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FINITE CONJUGACY IN ALGEBRAS AND ORDERS

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Abstract Herstein showed that the conjugacy class of a non-central element in the multiplicative group of a division ring is infinite. We prove similar results for units in algebras and orders and give applications to group rings.

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1. Introduction

For a given group G, we denote by $\Delta(G)$ the *FC-centre* (finite-conjugacy centre) of G, that is:

$$\Delta(G) = \{g \in G \mid [G : \mathcal{C}_G(g)] < \infty\}.$$

Also, for a ring R we shall denote by $\mathcal{U}R$ the group of units of R, i.e. the set of invertible elements of R. Herstein showed in [**6**] that if D is a division ring then $\Delta(\mathcal{U}D)$ coincides with $\mathcal{Z}(\mathcal{U}D)$, the centre of $\mathcal{U}D$.

The study of the FC-centre of groups of units of group rings started with papers by Sehgal and Zassenhaus [**14**], Polcino Milies [**8**] and Cliff and Sehgal [**2**]. Also, Williamson [**16**], studied elements of a periodic group G that has finite conjugacy class in the group of units of its integral group ring. A more general approach was given by Sehgal and Zassenhaus in [**15**]. This work was followed by several papers studying group rings over fields [**3**, **9**].

Theorem 2.2 below shows that a result similar to that of Herstein holds for finite-dimensional algebras over infinite fields and this fact is extended to algebraic algebras in Corollary 2.3. However, it follows from [**3**, Example 1] that it cannot be extended to all infinite-dimensional algebras. In § 3 we consider orders in finite-dimensional algebras where the situation is more complicated. In particular, a theorem of Williamson for integral group rings [**16**] shows that an analogue of Herstein's result does not hold for orders. However, we are able to obtain general positive results for large classes of orders and we give a partial extension of a theorem of Sehgal and Zassenhaus [**15**, Theorem 1]. In § 4 we consider some applications to group rings and, in particular, we obtain a short proof of the theorem of Williamson.

2. Algebras

The following fact should be known; however, we include an argument for the sake of completeness.

Proposition 2.1. *Let G be a connected algebraic group over an infinite field. Then every FC-element of G is central.*

Proof. Let G be a connected algebraic group and $x \in G$ an FC-element. Then, the centralizer $\mathcal{C}_G(x)$ is a closed subgroup of finite index G. For a fixed $y \in G$ the map $z \to yz \in G$ is polynomial and, therefore, continuous. Since the same holds for its inverse, it is a homeomorphism. Hence, each coset $y\mathcal{C}_G(x)$ is closed in G and G is a union of a finite number of them. Since G is connected (even irreducible), it follows that $x \in \mathcal{Z}(G)$. □

For a group G let $\boldsymbol{T}G$ denote the torsion part of G.

Theorem 2.2. *Let A be an algebra with unity over an infinite field K.*

(i) *If A is finite dimensional, then $\mathcal{U}A$ is a connected linear algebraic group and, consequently,*

$$\Delta(\mathcal{U}A) = \mathcal{Z}(\mathcal{U}A).$$

Moreover, A is generated by its units, as a vector space over K and, therefore, $\mathcal{U}A$ is FC if and only if A is commutative.

(ii) *Every torsion unit of $\Delta(\mathcal{U}A)$ commutes with each algebraic unit of A and, consequently, $\Delta(\mathcal{U}A)$ is solvable of length at most 2.*

(iii) *Every element of $\Delta(\mathcal{U}A)$ commutes with each nilpotent element of A.*

Proof. (i) Let $n = \dim_K A$ and $\Gamma : A \to M_n(K)$ be the regular representation of A. Then $x \in A$ is invertible if and only if $\det(\Gamma(x)) \neq 0$. Indeed, if $\det(\Gamma(x)) \neq 0$, then $\Gamma(x)$ is invertible in $M_n(K)$, thus it cannot be a zero divisor in $\Gamma(A)$. Since an element in a finite-dimensional algebra is either a zero divisor or invertible, the statement follows.

Taking a basis in A, $\det(\Gamma(x))$ can be considered as a polynomial f in coordinates $x_1, \dots, x_n$ of x. Hence

$$\mathcal{U}A = \{(x_1, \dots, x_n) \in A \mid f(x_1, \dots, x_n) \neq 0\}.$$

So, with respect to the Zariski topology on A, we have that $\mathcal{U}A$ is a principal open subset in $A = K^n$. Therefore, $\mathcal{U}A$ is an irreducible (i.e. connected) algebraic group. Therefore, by Proposition 2.1, we have that $\Delta(\mathcal{U}A) = \mathcal{Z}(\mathcal{U}A)$.

Note that $\mathcal{U}A$ is a linear algebraic group as both $\Gamma : A \to \Gamma(A)$ and its inverse are polynomial maps. Since $\mathcal{U}A$ is an open subset of A, we have that $\overline{\mathcal{U}A} = A$. Let A_1 be the linear span of $\mathcal{U}A$. Since every linear subspace is closed under Zariski's topology, we have that $\overline{\mathcal{U}A} \subset A_1$; hence $A_1 = A$.

Item (ii) is an easy consequence of (i), indeed, let $x \in \boldsymbol{T}\Delta(\mathcal{U}A)$, $y \in \mathcal{U}A$ be algebraic and n be the dimension of the subalgebra generated by y. Then, by Dietzmann's Lemma (see [**11**, 15.1.11]), the conjugates of x in $H = \langle x, y\rangle$ generate a finite normal subgroup N, and every element of H can be written as a K-linear combination of elements of the form hy^i with $h \in N$ and $0 \leqslant i \leqslant n-1$. Hence the K-linear span of H is a finite-dimensional algebra and, by item (i), $xy = yx$, as desired. In particular, $\boldsymbol{T}\Delta(\mathcal{U}A)$ is abelian and, since $\Delta(\mathcal{U}A)' \subset \boldsymbol{T}\Delta(\mathcal{U}A)$, by Neuman's Theorem [**11**, 15.1.7], $\Delta(\mathcal{U}A)$ is solvable of length at most 2.

(iii) Let $y \in A$ be a nilpotent element and let n be a positive integer such that $y^n \neq 0$ and $y^{n+1} = 0$. For each $\alpha \in K$ we consider $z_\alpha = 1 + \alpha y$, which is a unit whose inverse is $z_\alpha^{-1} = \sum_{i=0}^{n} (-1)^i (\alpha y)^i$ and we have that

$$z_\alpha^{-1} x z_\alpha = x + \sum_{i=1}^{n} (v_i \alpha^i),$$

where $v_i = (-1)^i (y^i x - y^{i-1} x y)$, $1 \leqslant i \leqslant n$.

Since $x \in \Delta(\mathcal{U}A)$, there exists an infinite set S of K such that the set $\{z_\alpha^{-1} x z_\alpha \mid \alpha \in S\}$ consists of a single element.

Let $\mathcal{B}$ be a K-basis of A and set $\alpha, \beta \in S$, with β fixed. Write

$$v_i = \sum_{b \in \mathcal{B}} v_i(b) b$$

and

$$z_\beta^{-1} x z_\beta - x = \sum_{b \in \mathcal{B}} w(b) b.$$

As $z_\alpha^{-1} x z_\alpha = z_\beta^{-1} x z_\beta$, we get

$$\sum_{b \in \mathcal{B}} w(b) b = \alpha \left(\sum_{b \in \mathcal{B}} v_1(b) b \right) + \alpha^2 \left(\sum_{b \in \mathcal{B}} v_2(b) b \right) + \cdots .$$

Assume that $v_1 = xy - yx \neq 0$. Then, there exists an element $b_0 \in \mathcal{B}$ such that $v_1(b_0) \neq 0$. Consequently, the polynomial $-w(b_0) + \alpha v_1(b_0) + \alpha^2 v_2(b_0) + \cdots$ is non zero and has infinitely many roots, since it is zero for every $\alpha \in S$, a contradiction. $\square$

We note that the proof of (iii) also works in the case of orders, as we show in the beginning of the next section.

Notice that if K is a finite field, results similar to those of the previous theorem need not hold. In fact, let A be a direct sum of infinitely many copies of a full matrix ring $M_n(K)$, $n > 2$. Then every unit in A is FC so

$$\mathcal{U}A = \Delta(\mathcal{U}A) \neq \mathcal{Z}(\mathcal{U}A).$$

Moreover, $\mathcal{U}A$ is not solvable and clearly units need not commute with nilpotent elements so none of the statements of the theorem above hold.

Throughout this section we shall always assume that the algebras considered are taken over an infinite field K.

Corollary 2.3. *If $\mathcal{U}A$ and $\Delta(\mathcal{U}A)$ are generated by algebraic units, then*

$$\Delta(\mathcal{U}A) = \mathcal{Z}(\mathcal{U}A).$$

In particular, this happens if A is an algebraic algebra. In this case A is generated by units as a vector space, and $\mathcal{U}A$ is FC if and only if A is commutative.

Proof. Let $x \in \Delta(\mathcal{U}A)$, $y \in \mathcal{U}A$ be algebraic, $H = \langle x, y\rangle$ and $x_1, \dots, x_s$ be the conjugates of x in H. Each commutator of $\langle x_1, \dots, x_s\rangle$ is torsion and, therefore, central by (ii) of Theorem 2.2. Hence, each $h \in H$ can be written as

$$h = y^{\alpha} x_1^{\beta_1} \dots x_s^{\beta_s} \prod [x_i, x_j]^{\gamma_{ij}},$$

with $\alpha, \beta_1, \dots, \beta_s, \gamma_{ij} \in \mathbb{Z}$. Now, let n_i (respectively n) be the dimension of the subalgebra generated by x_i (respectively y). Then h is a K-linear combination of elements of the form

$$y^{\delta} x_1^{\varepsilon_1} \dots x_s^{\varepsilon_s} \prod [x_i, x_j]^{\omega_{ij}}$$

with $0 \leqslant \delta \leqslant n$, $0 \leqslant \varepsilon_i \leqslant n_i$, $0 \leqslant \omega_{ij} \leqslant o([x_i, x_j])$. We have finitely many such elements and, consequently, the K-linear span of H is a finite-dimensional algebra. It follows from (i) of Theorem 2.2 that x and y commute, as desired. The last statement also follows from part (i) of that theorem. $\square$

Now, we wish to consider algebras with many units; more precisely, algebras that are generated, as a vector space, by their units. These include large classes of algebras, such as group rings, crossed products, finite-dimensional algebras, algebraic algebras and algebras unitally generated by nilpotent elements such as those considered in [**1**].

The following lemma is an extension of [**3**, Lemma 2.1] to the general case.

Lemma 2.4. *Let $x \in A$ be an element such that $x^2 = bx$ for some $b \in K$. Then $xy = yx$ for all $y \in \Delta(\mathcal{U}A)$.*

Proof. Let k be an arbitrary element in K. If $b \neq 0$ we set $u_k = 1 - b^{-1}x + b^{-1}kx$. Then u_k is a unit of A whose inverse is $u_k^{-1} = 1 - b^{-1}x + b^{-1}k^{-1}x$. Given an element $y \in \mathcal{U}A_1$, we compute

$$\begin{aligned} y_k = u_k y u_k^{-1} &= (1 - b^{-1}x + b^{-1}kx)y(1 - b^{-1}x + b^{-1}k^{-1}x) \\ &= y - b^{-1}xy - b^{-1}yx + b^{-1}kxy + b^{-1}k^{-1}yx + (2b^{-2} - b^{-2}k - b^{-2}k^{-1})xyx. \end{aligned}$$

If we denote $c = yxy^{-1}$, we have that $yx = cy$ and we can write

$$y_k = (1 + b^{-1}kx - b^{-1}x - b^{-1}c + b^{-1}k^{-1}c + 2b^{-2}xc - b^{-2}kxc - b^{-2}k^{-1}xc)y.$$

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Hence

$$xy_k = x(k + b^{-1}c - b^{-1}kc)y = k(x - b^{-1}xc)y + b^{-1}xcy.$$

Thus, if $x - b^{-1}xc \neq 0$, as K is infinite, we would have infinitely many conjugates for y. So we must have that $x = b^{-1}xc$ and, back in the expression of y_k, we obtain

$$y_k = (1 - b^{-1}c + b^{-1}x + b^{-1}k^{-1}(c - x))y.$$

Once again, if $c \neq x$ we would have infinitely many conjugates for y, a contradiction. Hence, $x = y^{-1}xy$, as desired.

The case where $b = 0$ can be obtained by a similar argument, considering the unit $u_a = 1 + ax$. It also follows immediately from Theorem 2.2. □

Let A_1 denote the linear span of $\Delta(\mathcal{U}A)$ in A. Since $\Delta(\mathcal{U}A)$ is a group, it follows immediately that A_1 is a subalgebra of A.

Corollary 2.5. *Every idempotent of A_1 is central in A.*

Proof. Let $e \in A_1$ be an idempotent and let x be an arbitrary element of A. The elements $\alpha = ex(1-e)$ and $\beta = (1-e)xe$ are such that $\alpha^2 = \beta^2 = 0$.

Write e as a linear combination $e = \sum_i l_i u_i$ of elements $u_i \in \Delta(\mathcal{U}A)$ with coefficients $l_i \in K$. By the previous lemma, both α and β commute with every u_i and thus with e. Hence

$$\begin{aligned} e\alpha &= \alpha e = 0, \\ e\beta &= \beta e = 0. \end{aligned}$$

Now, $e\alpha = ex(1-e)$ and thus $ex = exe$. In a similar way we obtain that $xe = exe$ and thus $xe = ex$, as claimed. □

Theorem 2.6. *Let A be an algebra generated by its units, as a linear space over an infinite field K such that $\mathcal{U}A$ is FC. Then every idempotent and every nilpotent element are central in A.*

Moreover, if A is generated by its torsion units, as a linear space over K, then $\mathcal{U}A$ is FC if and only if A is commutative.

Proof. The first part of the statement follows immediately from Corollary 2.5 and item (iii) of Theorem 2.2 while the second is a consequence of item (ii) of the same theorem. □

3. Orders

Let D be a domain, K its field of fractions and let A be a K-algebra. By a D-order Λ in A we mean a D-subalgebra of A such that $A = K\Lambda$. Notice that this implies that Λ contains a K-basis of A. Of course, $\Delta(\mathcal{U}\Lambda) \subset \Delta(\mathcal{U}A) \cap \Lambda$, but, in general, equality does not hold. To see this take, for example, $K_8 = \langle a, b \mid a^4 = 1,\ a^2 = b^2,\ bab^{-1} = a^{-1} \rangle$ and set $\Lambda = \mathbb{Z}K_8$ and $A = \boldsymbol{Q}K_8$. Then the element $x = 1 + a + a^3$ lives in Λ, is central and invertible in A, with inverse $x^{-1} = \frac{1}{3}(1 + a - 2a^2 + a^3)$, but $x \notin \mathcal{U}\Lambda$.

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Proposition 3.1. *Let D be an infinite domain and let K be its field of fractions. Let A be a K-algebra and Λ a D-order in A. If $x \in \Delta(\mathcal{U}\Lambda)$ and $y \in A$ is nilpotent then $xy = yx$.*

Proof. Let $y \in A$ be a nilpotent element. Since Λ is a D-order in A, there exists an element $d \in D$ such that $y_1 = dy \in \Lambda$.

For each $\alpha \in D$ set $z_\alpha = 1 + \alpha y_1$. Then z_α is a unit in Λ whose inverse is

$$z_\alpha^{-1} = \sum_{i=0}^{n} (-1)^i (\alpha y_1)^i.$$

As in item (iii) of Theorem 2.2 we can conclude that there exists an infinite set S of D such that the set $\{z_\alpha^{-1} x z_\alpha \mid \alpha \in S\}$ consists of a single element. If $xy_1 \neq y_1 x$ taking a K-basis $\mathcal{B}$ of A contained in Λ and a fixed scalar $\beta \in S$ we can obtain, as before, a non-zero polynomial $-w(b_0) + \alpha v_1(b_0) + \alpha^2 v_2(b_0) + \cdots$ that has infinitely many roots in S, a contradiction.

Hence, $xy_1 = y_1 x$ and thus also $xy = yx$. □

As a consequence of Proposition 3.1 we obtain the following theorem.

Theorem 3.2. *Let D be an infinite domain, K its field of fractions, A a finite-dimensional K-algebra, Λ a D-order in A, $\mathcal{J} = \mathcal{J}(A)$ the 'Jacobson radical' of A, and $\bar{A} = A/\mathcal{J}$. Assume that $\mathrm{Hom}_A(P_i, P_j) = 0$ for every pair of non-isomorphic principal modules P_i, P_j of multiplicity 1 in A. If every minimal ideal of $\bar{A}$ which is a division ring is isomorphic to K, then*

$$\Delta(\mathcal{U}\Lambda) \subset \mathcal{Z}(A),$$

Proof. Let $\bar{A} = M_{n_1}(\mathcal{D}_1) \times \cdots \times M_{n_s}(\mathcal{D}_s)$ be the Wedderburn decomposition of $\bar{A}$, let $\mathcal{V}_i$ be the ith irreducible $\bar{A}$-module and $\mathcal{P}_i$ the principal A-module corresponding to $\mathcal{V}_i$. Then we have an A-module isomorphism

$$A \cong n_1\mathcal{P}_1 \oplus \cdots \oplus n_s\mathcal{P}_s, \tag{3.1}$$

and, by the Peirce decomposition (see [**4**, p. 26]), we obtain that A is isomorphic to the algebra of matrices of the form

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1s} \\ a_{21} & a_{22} & \dots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s1} & a_{s2} & \dots & a_{ss} \end{bmatrix},$$

where $a_{ij} \in \mathrm{Hom}_A(n_j\mathcal{P}_j, n_i\mathcal{P}_i)$. Notice that $\mathrm{End}_A(n_i\mathcal{P}_i) \cong M_{n_i}(\mathrm{End}_A(\mathcal{P}_i))$.

We shall denote by $e_{ij}(a_{ij})$ a matrix whose entry in position (i,j) is equal to a_{ij} and all other entries are equal to zero.

Let $x \in \Delta(\mathcal{U}\Lambda)$. By Proposition 3.1 we have that x commutes with all nilpotent elements of A. In particular, if $i \neq j$, it commutes with every matrix $e_{ij}(a_{ij})$. Thus it remains to show that x centralizes the diagonal subalgebra $\mathrm{End}_A(n_1\mathcal{P}_1) \times \cdots \times \mathrm{End}_A(n_s\mathcal{P}_s)$.

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Let $x_{ij} \in A$ be the entry of x belonging to $\mathrm{Hom}_A(n_j\mathcal{P}_j, n_i\mathcal{P}_i)$. We wish to show that x is a diagonal matrix. Assume that, in the decomposition of A given in (3.1) above, we have that $n_i > 1$ if $1 \leqslant i \leqslant t$ and $n_i = 1$ if $t+1 \leqslant i \leqslant s$. It follows directly from our assumption on the principal modules of multiplicity 1 in A that x is of the form

$$x = \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,t} & x_{1,t+1} & x_{1,t+2} & \cdots & x_{1,s} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,t} & x_{2,t+1} & x_{2,t+2} & \cdots & x_{2,s} \\ & & \cdots & & & & \cdots & \\ x_{t,1} & x_{t,2} & \cdots & x_{t,t} & x_{t,t+1} & x_{t,t+2} & \cdots & x_{t,s} \\ x_{t+1,1} & x_{t+1,2} & \cdots & x_{t+1,t} & x_{t+1,t+1} & 0 & \cdots & 0 \\ x_{t+2,1} & x_{t+2,2} & \cdots & x_{t+2,t} & 0 & x_{t+2,t+2} & \cdots & 0 \\ & & \cdots & & & & \cdots & \\ x_{s,1} & x_{s,2} & \cdots & x_{s,t} & 0 & 0 & \cdots & x_{s,s} \end{bmatrix}.$$

For an index $i \leqslant t$ and every nilpotent element $a \in \mathrm{End}_A(n_iP_i)$, by Proposition 3.1 we have that $e_{ii}(a)x = xe_{ii}(a)$. A straightforward computation shows that

$$ax_{ij} = 0 = x_{ij}a \quad \text{for all } j \neq i. \tag{3.2}$$

We claim that this implies $x_{ij} = x_{ji} = 0$, for all $j \neq i$, $1 \leqslant i \leqslant s$. In fact, recall that $\mathrm{End}_A(n_iP_i) = M_{n_i}(\mathrm{End}_A(P_i))$ and set $a = e_{kl}(1) \in M_{n_i}(\mathrm{End}_A(P_i))$ with $k \neq l$. For an arbitrary element $y \in n_jP_j$ we compute $x_{ij}(y) \in n_iP_i$, so, if we consider n_iP_i as column matrices with entries in P_i, we can write $x_{ij}(y)$ in the form

$$x_{ij}(y) = \begin{pmatrix} x^1_{ij}(y) \\ x^2_{ij}(y) \\ \cdots \\ x^{n_i}_{ij}(y) \end{pmatrix}.$$

Then

$$0 = ax_{ij}(y) = e_{kl}(1) \begin{pmatrix} x^1_{ij}(y) \\ x^2_{ij}(y) \\ \cdots \\ x^{n_i}_{ij}(y) \end{pmatrix} = \begin{pmatrix} 0 \\ \cdots \\ x^l_{ij}(y) \\ \cdots \\ 0 \end{pmatrix},$$

where $x^l_{ij}(y)$ is the kth entry of the column.

This implies that $x^l_{ij}(y) = 0$ for $l \neq k$. Since k and l are arbitrary distinct, we have that $x_{ij}(y) = 0$, for all $y \in n_jP_j$ and thus $x_{ij} = 0$. A similar argument shows that also $x_{ji} = 0$. Consequently, x is diagonal.

Fix an index i with $1 \leqslant i \leqslant t$ (and, thus, $n_i > 1$). Consider the element $y = e_{ii}(e_{km}(1))$ of the diagonal subalgebra where $k \neq m$ and the elementary matrix $e_{km}(1)$ belongs to $M_{n_i}(\mathrm{End}_A(\mathcal{P}_i))$. Then $y^2 = 0$ and the equality $xy = yx$ implies that x_{ii} commutes with $e_{km}(1)$. Since k and m are arbitrary, it follows that x_{ii} must be scalar, $x_{ii} = aI$, $a \in \mathrm{End}_A(\mathcal{P}_i)$. Moreover, for all $b \in \mathrm{End}_A(\mathcal{P}_i)$ and $e_{km}(b) \in \mathrm{End}_A(\mathcal{P}_i)$, we have that

$$aIe_{km}(b) = abe_{km}(1) = e_{ij}(b)aI = bae_{km}(1).$$

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Consequently, $a \in \mathcal{Z}(\mathrm{End}_A(\mathcal{P}_i))$. Thus x_{ii} centralizes $E_A(n_i\mathcal{P}_i)$ with $n_i > 1$.

Now assume $i > t$ and thus $n_i = 1$.

In this case, $M_{n_i}(D_i) = D_i$ is a division ring, so, by our hypothesis, $D_i \cong \mathrm{End}_{\bar{A}}(\mathcal{V}_i) \cong K$. Therefore, $\mathrm{End}_A(\mathcal{P}_i)/\mathcal{J}(\mathrm{End}_A(\mathcal{P}_i))$ is also isomorphic to K. Hence

$$\mathrm{End}_A(\mathcal{P}_i) = Ke_i \oplus \mathcal{J}(\mathrm{End}_A(\mathcal{P}_i)),$$

a direct sum of K-vector spaces. Since we have shown that x is diagonal, it follows immediately that it commutes with the elements of Ke_i and, as x centralizes $\mathcal{J}(\mathrm{End}_A(\mathcal{P}_i))$, we conclude that $x \in \mathcal{C}_A(\mathrm{End}_A(\mathcal{P}_i))$, which completes the proof. □

Remark 3.3. Notice that the restriction that $\mathrm{Hom}_A(P_i, P_j) = 0$ for every pair of non-isomorphic principal modules P_i, P_j of multiplicity 1 in A is always verified in the case of semisimple algebras, by Schur's Lemma. On the other hand, we observe that it is essential in the non-semisimple case, as shown by the following example.

Take

$$A = \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{Q} \\ 0 & \boldsymbol{Q} \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix},$$

and set

$$x = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

It is easy to see that the conjugacy class of x in $\mathcal{U}\Lambda$ is of order 2, so x is non-central but $x \in \Delta(\mathcal{U}\Lambda)$.

Corollary 3.4. *Let D and K be as above, A be a finite-dimensional K-algebra and Λ an order in A. Assume that $\mathrm{Hom}_A(P_i, P_j) = 0$ for every pair of non-isomorphic principal modules P_i, P_j of multiplicity 1 in A. If K is a splitting field for A, then*

$$\Delta(\mathcal{U}\Lambda) \subset \mathcal{Z}(A).$$

Corollary 3.5. *Let D and K be as above, A be a semisimple finite-dimensional K-algebra and Λ a D-order in A. If A has no minimal ideal that is a non-commutative division ring, then*

$$\Delta(\mathcal{U}\Lambda) \subset \mathcal{Z}(A).$$

Proof. The proof of the theorem shows that $x \in \Delta(\mathcal{U}\Lambda)$ centralizes each Wedderburn component $M_{n_i}(\mathcal{D}_i)$ of A with $n_i > 1$. On the other hand, by our assumption, $n_i = 1$ implies that $\mathcal{D}_i$ is a field. Hence $x \in \mathcal{Z}(A)$. □

Theorem 3.6. *Let D be an infinite domain and R a D-algebra.*

(i) *If R is torsion free as a D-module, then*

$$\Delta(GL_n(R)) = \Delta(\mathcal{U}R)I,$$

where I is the identity matrix of $M_n(R)$.

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(ii) *If* $\mathrm{char}(D) = 0$ *and* $n > 1$, *then*

$$\Delta(GL_n(R)) = \mathcal{Z}(GL_n(R)).$$

Proof. Let $\{e_{ij} : 1 \leqslant i,\ j \leqslant n\}$ be the basis of elementary matrices of $M_n(R)$ and let

$$a = \sum_{i,j} a_{ij} e_{ij} \in \Delta(GL_n(R)).$$

Fix $1 \leqslant i_0,\ j_0 \leqslant n$ with $i_0 \neq j_0$ and for each $r \in D$ set

$$a_r = (I - re_{i_0 j_0}) a (I + re_{i_0 j_0}).$$

Since D is infinite and the conjugacy class of a is finite, there exist $r, s \in D$, $r \neq s$, such that $a_r = a_s$. This implies that a commutes with $I + (r - s)e_{i_0 j_0}$ and hence with $e_{i_0 j_0}$, as R is torsion free over D. It follows that $a_{j_0 j} = a_{i i_0} = 0$ for all $j \neq j_0$ and $i \neq i_0$ and that $a_{i_0 i_0} = a_{j_0 j_0}$. Since this holds for all $1 \leqslant i_0,\ j_0 \leqslant n$, $i_0 \neq j_0$, we conclude that $a = a_{11} I$ and $a_{11} \in \Delta(R)$ and (i) follows.

Now suppose that $\mathrm{char}(D) = 0$. For $i \neq j$ set $u_{ij} = I + e_{ij}$. Then u_{ij} is a unit. If $a \in \Delta(GL_n(R))$ then there exists a positive integer k such that u_{ij}^k centralizes a. Note that $u_{ij}^k = I + ke_{ij}$. Since $\mathrm{char}(D) = 0$, it follows easily that $ae_{ij} = e_{ij}a$ if $i \neq j$ and, as $e_{ii} = e_{ij} \cdot e_{ji}$, we conclude that a commutes with all the matrices of the basis of $M_n(R)$ and thus a is a scalar matrix, i.e. of the form $a = \lambda_0 I$, where I is the identity matrix and $\lambda_o \in R$. Finally, set $u = I + \lambda e_{12}$ with $\lambda \in R$. Since $u^k = I + k\lambda e_{12}$, an argument similar to the one above shows that $\lambda_0 \in \mathcal{Z}(R)$. □

Notice that the arguments in the proof above do not depend on the fact that the given matrix a is invertible. Hence, if for a given ring R we denote by $\Delta(R)$ the set of elements in R who have finitely many conjugates under the action of $\mathcal{U}R$, we actually have the following.

Corollary 3.7. *Let D be an infinite domain and R a D-algebra.*

(i) *If R is torsion free as a D-module then*

$$\Delta(M_n(R)) = \Delta(R) I,$$

where I is the identity matrix of $M_n(R)$.

(ii) *If* $\mathrm{char}(D) = 0$ *and* $n > 1$ *then*

$$\Delta(M_n(R)) = \mathcal{Z}(M_n(R)).$$

4. Group rings

In this section we shall apply our previous results to the case of group rings. First, we notice that if G is a finite group such that the group algebra $\boldsymbol{Q}G$ has no minimal ideal which is a non-commutative division ring, then Corollary 3.5 shows that

$\Delta(\mathcal{U}(\boldsymbol{Z}G)) \subset \mathcal{Z}(\boldsymbol{Q}G)$. We remark that there are many important classes of groups which satisfy this condition, as all finite simple groups, nilpotent groups of odd order [**13**, Corollary 20.7], and groups which have no non-abelian homomorphic image that is fixed-point free, as considered in [**10**].

We begin with some technical lemmas.

Lemma 4.1. *Let K be a field and let G be a subgroup of $GL(2, K)$. Then*

(i) *if $a \in GL(2, K)$ is non-central, its centralizer in $GL(2, K)$ is abelian, and*

(ii) *either $\Delta(G) = \mathcal{Z}(G)$ or G is abelian-by-finite.*

Proof. To prove (i) we may assume, without loss of generality, that K is algebraically closed. Then the statement follows directly, considering the Jordan normal form of a.

To prove (ii), notice that if $\Delta(G) \neq \mathcal{Z}(G)$, taking $a \in \Delta(G)$ non-central, we have that $[G : \mathcal{C}_G(a)]$ is finite and the argument above showed that $\mathcal{C}_G(a)$ is abelian so G is abelian-by-finite, as desired. □

Notice that, if K_8 is the quaternion group of order 8, it is well-known that $\mathcal{U}(\mathbb{Z}K_8) = \pm K_8$ and thus an analogue of Herstein's result does not hold for the order $\mathbb{Z}K_8$ in $\boldsymbol{Q}K_8$. However, we have the following.

Proposition 4.2. *Let $G = K_8 \times \langle c \rangle$, where c is an element of order p, an odd prime, and $K_8 = \langle a, b \rangle$ is the quaternion group of order 8. Then*

$$\Delta(\mathcal{U}(\mathbb{Z}G)) = \mathcal{Z}(\mathcal{U}(\mathbb{Z}G)).$$

Proof. Let $K_8 = \langle a, b : a^4 = 1,\ a^2 = b^2,\ b^{-1}ab = a^{-1} \rangle$. Consider the ring representation $\psi : \mathbb{Z}G \to \boldsymbol{M}_2(\boldsymbol{C})$ given by

$$a \to \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \qquad b \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad c \to \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix},$$

where ξ is a primitive pth root of unity.

If we write $u = x_0 + x_1 a + x_2 b + x_3 ab \in \mathbb{Z}G$ with

$$x_t = \alpha_{0_t} + \alpha_{1_t} c + \cdots + \alpha_{p-1_t} c^{p-1} \in \mathbb{Z}(\langle c \rangle),$$

then

$$\psi(u) = \begin{pmatrix} y_0 + y_1 \mathrm{i} & -(y_2 + y_3 \mathrm{i}) \\ y_2 - y_3 \mathrm{i} & y_0 - y_1 \mathrm{i} \end{pmatrix},$$

with

$$y_t = \alpha_{0_t} + \alpha_{1_t} \xi + \cdots + \alpha_{p-1_t} \xi^{p-1} \in \mathbb{Z}[\xi]$$

and $\operatorname{Ker} \psi = \hat{c}\mathbb{Z}G$.

If we restrict ψ to $\mathcal{U}(\mathbb{Z}G)$ we have a group homomorphism

$$\psi| : \mathcal{U}(\mathbb{Z}G) \to GL(2, \boldsymbol{C}),$$

whose kernel is $\operatorname{Ker}\psi| = (1+\hat{c}\mathbb{Z}G)\cap\mathcal{U}(\mathbb{Z}G)$, where $\hat{c} = 1+c+\cdots+c^{p-1}$. We will show that there are no units of $\mathbb{Z}G$ of this form, other than 1, that is, the restriction of ψ is an injection of $\mathcal{U}(\mathbb{Z}G)$ into $GL(2,\boldsymbol{C})$. For this let $\varphi : G \to K_8$ be a group homomorphism defined by $a \to a$, $b \to b$ and $c \to 1$. Extend it linearly to $\varphi : \mathbb{Z}G \to \mathbb{Z}K_8$. Now let $x = 1+\hat{c}y \in \mathcal{U}(\mathbb{Z}G)$ and observe that we may assume that $y \in \mathbb{Z}K_8$. So $\varphi(x) = 1+py$ is a unit in $\mathbb{Z}K_8$ and then it must be trivial. Hence $y = 0$.

Now suppose that $\Delta(\mathcal{U}(\mathbb{Z}G))$ is not contained in the centre of $\mathcal{U}(\mathbb{Z}G)$. Let $u \in \Delta(\mathcal{U}(\mathbb{Z}G))$, $u \notin \mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$. Then $\psi(u) \notin \mathcal{Z}(GL(2,\boldsymbol{C}))$ and, by Lemma 4.1, $\psi(\mathcal{U}(\mathbb{Z}G))$ is abelian-by-finite.

On the other hand, by a Theorem of Hartley and Pickel (see [**13**, Theorem 5.1]), we have that $\mathcal{U}(\mathbb{Z}G)$ contains a non-cyclic free group, therefore $\psi(\mathcal{U}(\mathbb{Z}G)) \cong \mathcal{U}(\mathbb{Z}G)$ cannot be abelian-by-finite, a contradiction. Hence $\Delta(\mathcal{U}(\mathbb{Z}G)) = \mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$. □

Lemma 4.3. *Let G be a group and let $g_0 \in T\Delta(\mathcal{U}(\mathbb{Z}G))$ and $g \in G$. Then the commutator $[g_0, g]$ is an element of $\langle g_0\rangle \bigcap \langle g\rangle$.*

Proof. Since $T\Delta(\mathcal{U}(\mathbb{Z}G))$ is a periodic normal subgroup of $\mathcal{U}(\mathbb{Z}G)$, it follows from [**12**, Theorem II.5.1] that $T\Delta(\mathcal{U}(\mathbb{Z}G)) \subset G$ and that all its subgroups are also normal. In particular, g normalizes the group $\langle g_0\rangle$, so we only need to prove that g_0 also normalizes the group $\langle g\rangle$.

Let $\alpha = (1-g)g_0\hat{g}$. If α is zero then the claim follows easily. If not, since α is nilpotent, Proposition 3.1 tells us that $\alpha g_0 = g_0\alpha$ so it also follows that g_0 normalizes $\langle g\rangle$, as desired. □

Proposition 4.4. *Let G be a finite group and suppose that $T\Delta(\mathcal{U}(\mathbb{Z}G))$ is non-abelian. Then G is a 2-group.*

Proof. Since $T\Delta(\mathcal{U}(\mathbb{Z}G))$ is a torsion normal subgroup, it follows again from [**12**, Theorem II.5.1] that it is contained in G and that every subgroup of $T\Delta(\mathcal{U}(\mathbb{Z}G))$ is normal. As we are assuming that it is not abelian, it must be a Hamiltonian group and thus contains a subgroup H isomorphic to K_8. Suppose that there exists a prime $p \geqslant 3$ dividing the order of G and let $x \in G$ be an element of order p. As $H \cap \langle x\rangle = 1$, it follows from Lemma 4.3 that x centralizes H. Hence $G_1 = H \times \langle x\rangle$ is a subgroup of G. Thus $H \subset T\Delta(\mathcal{U}(\boldsymbol{Z}G_1))$, which, according to Proposition 4.2, should be central, a contradiction. □

Our next result first appeared in [**16**, Theorem 1] and alternative proofs were given in [**15**]. The proof we offer is shorter than the previous ones.

Theorem 4.5. *Let G be a periodic group. If $T\Delta(\mathcal{U}(\mathbb{Z}G))$ has a non-central element g_0 then there exist an element $x \in G$ of order 4 and an abelian subgroup A such that $G = \langle A, x \mid x^2 = g_0^2$ and $x^{-1}ax = a^{-1}\forall a \in A\rangle$.*

Proof. Assume that there exist elements $g_0 \in T\Delta(\mathcal{U}(\mathbb{Z}G))$, $g \in G$, such that $g_0 g \neq g g_0$. Then Lemma 4.3 shows that $\langle g_0\rangle$ and $\langle g\rangle$ are both normal in $\langle g, g_0\rangle$, so every cyclic subgroup of this group is normal and, thus, the group is Hamiltonian. As it has only two generators, it must be isomorphic to K_8. Since the only element of order 2 in K_8 is central, it follows that $o(g) = o(g_0) = 4$ and also that $g^2 = g_0^2$ and $g^{-1}g_0g = g_0^{-1}$.

Let A denote the centralizer of g_0 in G. If $g \in G$ is not in A, then, for each element $a \in A$, we have that $ag \notin A$, so $(ag)g_0 \neq g_0(ag)$ and, by the argument in the above paragraph, we have that $(ag)^{g_0} = g^{-1}a^{-1}$. On the other hand, $(ag)^{g_0} = ag^{g_0} = ag^{-1}$, so we also conclude that $gag^{-1} = a^{-1}$ for all $a \in A$. This implies that A is abelian.

Finally, let us observe that, if x and y are two elements that do not commute with g_0, we have that $x^{-1}y^{-1}g_0xy = x^{-1}g_0^{-1}x = g_0$ so $y \in xA$ and thus $G = \langle A, x\rangle$, as claimed. □

We can also use Proposition 4.1 to give an example in the case of infinite groups.

Example 4.6. Consider the infinite dihedral group $\boldsymbol{D} = \langle a, b \mid b^2 = 1,\ bab = a^{-1}\rangle$ and let R be an integral domain of characteristic 0. We claim that $\varDelta(\mathcal{U}(R\boldsymbol{D})) = \mathcal{Z}(\mathcal{U}(R\boldsymbol{D}))$.

In fact, it is well known that, if N is a subgroup of a group G with $[G : N] = n$, then RG can be imbedded in the full matrix ring $M_n(RN)$. So, as $\langle a\rangle$ is torsion free, abelian, we have that $R\langle a\rangle$ is an integral domain and it follows that $R\boldsymbol{D}$ can be imbedded in $M_2(K)$, where K denotes the field of fractions of $R\langle a\rangle$.

Once again, a result of Hartley and Pickel [**13**, Theorem 5.1] shows that $\mathcal{U}(R\boldsymbol{D})$ contains a free group on two generators. Hence Proposition 4.1 implies that $\varDelta(\mathcal{U}(R\boldsymbol{D})) = \mathcal{Z}(\mathcal{U}(R\boldsymbol{D}))$, as claimed.

Proposition 4.7. *Let G be a finite group such that $\boldsymbol{Q}G$ has no Wedderburn component which is a non-commutative division ring and let H be a free abelian group. Then*

$$\varDelta(\mathcal{U}(\mathbb{Z}[G \times H])) = \mathcal{Z}(\mathcal{U}(\mathbb{Z}[G \times H])).$$

Proof. Set $G_1 = G \times H$. Then $\boldsymbol{Q}G_1 = \boldsymbol{Q}G \otimes \boldsymbol{Q}H$ and, if $\boldsymbol{Q}G \cong \oplus_i M_{n_i}(D_i)$ is the Wedderburn decomposition of $\boldsymbol{Q}G$, we have that

$$\boldsymbol{Q}G_1 \cong \boldsymbol{Q}H \otimes (\oplus_i M_{n_i}(D_i)) \cong \oplus_i M_{n_i}(\boldsymbol{Q}H \otimes D_i).$$

Notice that Proposition 3.1 implies that an element $x \in \varDelta(\mathcal{U}(\mathbb{Z}[G \times H]))$ commutes with every nilpotent element in each component. This shows that the components of x are scalar matrices. Also, since they commute with every matrix of the form $e_{ij}(d)$, for all $d \in D$, we have that each component of x is central, so the result follows. □

We conclude this section by showing that our main theorem also allows us to obtain some results in the case when the characteristic is positive.

Corollary 4.8. *Let R be an infinite domain of positive characteristic and let G be a finite group. If $\operatorname{char}(R)$ does not divide $|G|$, then $\varDelta(RG) = \mathcal{Z}(RG)$.*

Proof. Let K be the field of fractions of R and suppose that p does not divide $|G|$. Then KG is semisimple.

Assume that a Wedderburn component of KG is a division ring D. Then D is generated, as a K-vector space, by the image G_1 of G under the projection to D. But G_1 is a finite subgroup of D and a theorem of Herstein [**5**] shows that G_1 is cyclic so D is commutative. The result now follows from Corollary 3.5. □

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