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STABILITY IN PRO-HOMOTOPY THEORY

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If $\mathscr{C}$ is a category, an object of pro-$\mathscr{C}$ is *stable* if it is isomorphic in pro-$\mathscr{C}$ to an object of $\mathscr{C}$. A local condition on such a pro-object, called *strong-movability*, is defined, and it is shown in various contexts that this condition is equivalent to stability. Also considered, in the case $\mathscr{C}$ is a suitable model category, is the stability problem in the homotopy category Ho(pro-$\mathscr{C}$), where pro-$\mathscr{C}$ has the induced closed model category structure defined by Edwards and Hastings [**6**].

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1. Introduction

Let $\mathscr{C}$ be any category and pro-$\mathscr{C}$ the category of pro-objects in $\mathscr{C}$ ([**1**, appendix]). There is a full embedding of $\mathscr{C}$ in pro-$\mathscr{C}$ obtained by regarding an object of $\mathscr{C}$ as a pro-object indexed by the trivial, one point, category. The *stability problem* in pro-$\mathscr{C}$ is then the following question. *When is an object of pro-$\mathscr{C}$ isomorphic in pro-$\mathscr{C}$ to an object of $\mathscr{C}$?*

Let $\mathbf{S}$ be the category of simplicial sets, and $\mathbf{S}_0$ the associated category of pointed, connected objects in $\mathbf{S}$. There is a homotopy category $Ho(\mathbf{S}_0)$ obtained from the standard closed model category structure on $\mathbf{S}$ (see, for example, [**3**, §2]). Largely because of its significance for shape theory, the stability problem in pro-$Ho(\mathbf{S}_0)$ (or equivalently, the pro-category of spaces having the pointed homotopy type of pointed CW-complexes) has been extensively studied (e.g. [**4, 5, 6, 9**]). Typical results have given conditions on algebraic invariants associated to pro-homotopy types which imply stability. For example, Edwards and Geoghegan prove:

Theorem 0. ([**4**], see also [**6**, Theorem 5.5.5.]). *Let* $\mathbf{X}$ *be an object of pro-*$\mathbf{S}_0$, *and suppose that* $\pi_k(\mathbf{X})$ *is stable in the category of pro-groups for each* $k \geqq 1$. *Then* $\mathbf{X}$ *is stable in pro-*$Ho(\mathbf{S}_0)$ *if* $\sup_i\{\dim X_i\} < \infty$.

We note two extensions that can be made to Theorem 0. Firstly, if $\mathbf{S}p$ is the category of simplicial spectra, and $\mathbf{S}p_0$ the associated category of pointed, connected objects in $\mathbf{S}p$, then there is a canonical closed model category structure on $\mathbf{S}p$, yielding a homotopy category $Ho(\mathbf{S}p_0)$. Then it is noted in [**6**, Theorem 5.5.5], that Theorem 0 also holds with $\mathbf{S}_0$ replaced by $\mathbf{S}p_0$.

Secondly, for any suitably nice closed model category $\mathscr{C}$ ($\mathbf{S}$ and $\mathbf{S}p$ are "nice"), Edwards and Hastings define a natural closed model category structure on pro-$\mathscr{C}$, [**6**]. Thus, we obtain a *strong pro-homotopy category*, Ho(pro-$\mathscr{C}$). The canonical functor

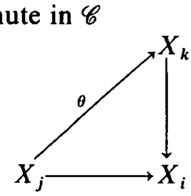
$\mathscr{C} \to Ho(\mathscr{C})$, extends to a functor pro-$\mathscr{C} \to$pro-$Ho(\mathscr{C})$ which factors through Ho(pro-$\mathscr{C}$). We therefore refer to pro-$Ho(\mathscr{C})$ as the *weak pro-homotopy category*. The categories Ho(pro-$\mathscr{C}$) are fruitfully exploited by the above authors in various geometrical contexts ([**6**, §§6, 7 and 8]). For present purposes, we note that the stability problem may also be studied in Ho(pro-$\mathscr{C}_0$); that is, we can ask when an object of Ho(pro-$\mathscr{C}_0$) is isomorphic in Ho(pro-$\mathscr{C}_0$) to an object of $Ho(\mathscr{C}_0)$, (again, $Ho(\mathscr{C}_0)$ is embedded as a full subcategory of Ho(pro-$\mathscr{C}_0$) in an obvious way). In [**4**] (see also [**6**, 5.5.7]), it is shown that the conclusion of Theorem 0 holds in Ho(pro-$\mathbf{S}_0$) if $\mathbf{X}$ is a *tower* (i.e., a pro-object indexed by the natural numbers).

Theorem 0 is proved by showing that the homotopy limit, holim $\mathbf{X}$ ([**6**, §4]), is finite under the given hypothesis, and then applying a Whitehead theorem in pro-$Ho(\mathbf{S}_0)$ ([**6**, 5.5.3]), to show that the natural map $\mathbf{h}$:holim $\mathbf{X} \to \mathbf{X}$ is an isomorphism in pro-$Ho(\mathbf{S}_0)$. Since a strong Whitehead theorem is now available ([**11**, Theorem A]), the same proof yields immediately (see also Theorem 2.6 below):

Theorem 1. *Let $\mathscr{C} = \mathbf{S}$ or $\mathbf{Sp}$. Then the conclusion of Theorem 0 holds in Ho(pro-$\mathscr{C}_0$).*

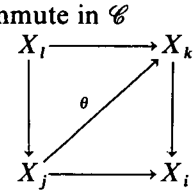
Theorems 0 and 1 study the stability problem in terms of algebraic invariants of pro-homotopy types. In this paper, we shall approach the question from a different point of view. Thus, we shall consider *local* conditions on a pro-homotopy type which guarantee stability.

One such local condition which has been widely studied and accepted as a desirable property of pro-homotopy types is the notion of *movability* (e.g. [**6**, §2]). If $\mathscr{C}$ is any category, an object $\mathbf{X}$ in pro-$\mathscr{C}$ is *movable*, if for each object $i \in I$ (a small, cofiltering index category for $\mathbf{X}$), there is a $j \to i$ such that, for each $k \to j$ there is a morphism θ in $\mathscr{C}$ making the following diagram commute in $\mathscr{C}$



(the unlabelled arrows are bonding maps). However, it is known that a movable object of pro-$Ho(\mathbf{S}_0)$ need *not* be stable (e.g., [**6**, Example 5.5.16]). We must therefore look for something stronger than movability. The basic local condition to be considered in this paper is:

Definition 1.1. An object $\mathbf{X}$ (indexed by I) in pro-$\mathscr{C}$ is *strongly-movable* if, for each object $i \in I$, there is a $j \to i$ such that, for each $k \to j$, there is an $l \to k$ and a morphism θ in $\mathscr{C}$ making the following diagram commute in $\mathscr{C}$



We shall show in various contexts that strong-movability and stability are equivalent.

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2. Statement of results

We shall first study the stability problem in pro-$\mathscr{C}$ when $\mathscr{C}$ is sufficiently well behaved. Thus,

Definition 2.1. $\mathscr{C}$ is an *EM-category* if each morphism $f:A\to B$ can be factorized in the form $A\xrightarrow{\bar f}C\xrightarrow{u}B$, with $\bar f$ an epimorphism and u a monomorphism. Further, this factorization is unique in the sense that, if $A\xrightarrow{\bar f'}C'\xrightarrow{u'}B$ is another such, then there is a unique isomorphism $\gamma:C\to C'$ with $\bar f'=\gamma\circ\bar f$ and $u=u'\circ\gamma$.

Remark 2.2. It is routine to deduce from the uniqueness property that, given a solid arrow diagram

$$\begin{array}{ccccc} A & \xrightarrow{\bar f} & C & \xrightarrow{u} & B \\ \downarrow & & \downarrow & & \downarrow \\ A' & \xrightarrow{\bar f'} & C' & \xrightarrow{u'} & B' \end{array}$$

in which the bottom horizontal line is a factorization of $f':A'\to B'$, then the dotted arrow exists and is unique.

We can now state the first of our main theorems.

Theorem 2.3. *Let $\mathscr{C}$ be an EM-category. Then* **X** *in pro-$\mathscr{C}$ is stable in pro-$\mathscr{C}$ if and only if* **X** *is strongly-movable.*

For non EM-categories, more is required. We extend Theorem 2.3 to the general case as follows.

Theorem 2.4. *Let $\mathscr{C}$ be any category, and* **X** *an object of pro-$\mathscr{C}$ indexed by a small, cofiltering category J. Then* **X** *is stable in pro-$\mathscr{C}$ if and only if* **X** *is strongly-movable, and the functor from $\mathscr{C}$ to* **Sets** *given by $K\mapsto\lim_{j\in J}\mathscr{C}(K:X_j)$, is representable (i.e., if $\lim_{j\in J}X_j$ exists in $\mathscr{C}$).*

We apply Theorem 2.4 to homotopy categories (which, of course, are not EM).

Theorem 2.5. *Let $\mathscr{C}$ be a closed model category, and suppose Brown's representability theorem holds in $Ho(\mathscr{C}_0)$ (i.e., pointed-set-valued homotopy functors on $Ho(\mathscr{C}_0)$ are representable). Then an object* **X** *in pro-$Ho(\mathscr{C}_0)$ is stable in pro-$Ho(\mathscr{C}_0)$ if and only if* **X** *is strongly-movable.*

Finally, we extend Theorem 2.5 to obtain a result holding in the strong pro-homotopy category.

Theorem 2.6. *Let $\mathscr{C} = \mathbf{S}$ or $\mathbf{Sp}$. Then an object $\mathbf{X}$ in pro-$\mathscr{C}_0$ is stable in Ho(pro-$\mathscr{C}_0$) if and only if it is stable in pro-Ho($\mathscr{C}_0$).*

We mention three simple consequences of our theorems. The first is immediate from Theorems 2.4, 2.6 and [**8,** Corollary 1, p. 227], and the second is immediate from Theorem 2.3 and [**6,** Theorem C, p. 144].

Proposition 2.7. *Under the hypotheses of Theorem 2.4, if $\mathbf{X}$ is dominated by an object $P \in \mathscr{C}$, then $\mathbf{X}$ is stable in pro-$\mathscr{C}$. Under the hypoptheses of Theorem 2.6, if $\mathbf{X} \in$ pro-$\mathscr{C}_0$ is dominated in pro-Ho($\mathscr{C}_0$) by an object $P \in \mathscr{C}_0$, then $\mathbf{X}$ is stable in Ho(pro-$\mathscr{C}_0$).*

Proposition 2.8. *Let $\mathbf{G}$ be a strongly-movable pro-group indexed by J. Then.*

(i) $\lim_J \mathbf{G} \to \mathbf{G}$ *is an isomorphism of pro-groups;*

(ii) $\lim_J^1 \mathbf{G} = 0$;

(iii) $\lim_J^s \mathbf{G} = 0$ *for $s > 0$ if each G_j is abelian.*

In [**6,** 9.3], Edwards and Hastings ask the question: if $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ is a morphism in Ho(pro-$\mathbf{S}_0$) which is invertible in pro-$Ho(\mathbf{S}_0)$, is $\mathbf{f}$ invertible in Ho(pro-$\mathbf{S}_0$)? The answer to this question is unknown even on the full subcategory tow-$\mathbf{S}_0$ generated by the tower objects. However, for towers, Edwards and Hastings prove that, if $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ in Ho(tow-$\mathbf{S}_0$) is an isomorphism in tow-$Ho(\mathbf{S}_0)$, then there is *some* isomorphism $\mathbf{g}: \mathbf{X} \to \mathbf{Y}$ in Ho(tow-$\mathbf{S}_0$) whose image in tow-$Ho(\mathbf{S}_0)$ coincides with that of $\mathbf{f}$ [**6,** Theorem 5.2.9]. But it is not known whether we can take $\mathbf{g} = \mathbf{f}$ in general. The above result suffices to obtain the important corollary that the isomorphism classification of objects is the same in tow-$Ho(\mathbf{S}_0)$ and Ho(tow-$\mathbf{S}_0$) [**6,** Corollary 5.2.17].

Unfortunately, because it is not possible to give a sufficiently rigid construction of $\mathbf{g}$ above, we cannot, with the techniques presently at our disposal, generalize the above result from towers to arbitrary pro-objects. However, using Theorem 2.6 above, we can generalise Corollary 5.2.13 of [**6**], as follows.

Proposition 2.9. *Let $\mathscr{C} = \mathbf{S}$ or $\mathbf{Sp}$ and $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ be a morphism in Ho(pro-$\mathscr{C}_0$) which is an isomorphism in pro-Ho($\mathscr{C}_0$). If either $\mathbf{X}$ or $\mathbf{Y}$ is stable in pro-Ho($\mathscr{C}_0$), then $\mathbf{f}$ is an isomorphism in Ho(pro-$\mathscr{C}_0$).*

We remark that a notion of strong-movability was first introduced by Borsuk [**2**] in the context of the shape theory of compacta. This was subsequently developed by Mardes[**7**] and Dydak [**3**]. For applications to shape theory, see [**8,** Theorem 8, p. 230, and Remark 2, p. 235]. The abstract definition (1.1) of strong-movability may also be found in [**8,** p. 226]. Theorem 2.5, with C the category of polyhedra, has been proved by Dydak [**3**]—see also [**8,** Theorem 7, p. 228]. Of course, Brown's representability theorem holds for polyhedra, so that Dydak's result follows immediately from ours. However, Theorems 2.4 and 2.5, taken together, bring out what is essential to theorems of this type (unlike the more specific and direct argument given by Dydak), and have much more general applicability. For example, we can take $\mathscr{C} = \mathbf{Sp}$, or, more generally, a category of diagrams (say, in **S**) in Theorem 2.5 (these being categories for which a Brown representability theorem is available).

Theorem 2.6 is also proved in a more general form than stated above. In fact, what is required is that $\mathscr{C}$ be "nice" in the sense of Edwards and Hastings [**6**]; see also [**11**], so that $Ho(\text{pro-}\mathscr{C}_0)$ exists, and that a Whitehead theorem (see 7.7 below) and a Brown representability theorem hold in $Ho(\mathscr{C}_0)$. Note also that Theorem 2.6 implies that the shape theory stability results derived from Theorem 2.5 and mentioned earlier, hold in the sense of "strong shape theory", where the strong shape of a space X is defined as the isomorphism class in $Ho(\text{pro-}\mathbf{S}_0)$ of any object of pro-$\mathbf{S}_0$ associated to X (e.g., the Vietoris nerve of X).

The remainder of the paper is arranged as follows. In Section 3, we prove the "only if" assertion of our theorems, and also a special case of Theorem 2.4 (Proposition 3.6). In Section 4, we prove Theorem 2.3 by reducing it to this special case. Theorem 2.4 is proved in Section 5. The main step here is a "change of limits" theorem (Proposition 5.1) and its Corollaries (5.2 and 5.4), which effectively allow us to reduce Theorem 2.4 to Theorem 2.3. However, owing to its length, the proof of the technical Proposition 5.1 is deferred to Section 8. In Section 6, we apply Brown's representability theorem for homotopy functors to deduce Theorem 2.5. In Section 7, we apply the "induction up from towers' technique" pioneered in [**11**] to obtain Theorem 2.6; we also prove Proposition 2.9. Finally, in an appendix, we collect together the technical results on pro-objects and morphisms which we use.

3. Strong movability

We first introduce some convenient notation.

Notation 3.1. Let $\mathbf{X}$ be a strongly-movable object of pro-$\mathscr{C}$ indexed by $I \in Cofilt$ (see Appendix). Given $i \in I$, a morphism $j \to i$ in I will be called a *morphism of strong movability for i* (with respect to $\mathbf{X}$), if $j \to i$ satisfies the condition of Definition 1.1. We abbreviate the above terminology to *MOSM for i.*

Clearly, if $\mathbf{X}$ is strongly-movable, then given $i \in I$, there is a MOSM for i. The following is obvious from the definitions.

Lemma 3.2. *If $i \to i'$ is any morphism in I, and $j \to i$ is a MOSM for i with respect to* $\mathbf{X}$, *then the composite $j \to i'$ is a MOSM for i'.*

The following proposition is immediate from [**8,** Theorem 5, p. 227].

Proposition 3.3. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be objects of pro-$\mathscr{C}$ with* $\mathbf{X} \cong \mathbf{Y}$. *Then if* $\mathbf{Y}$ *is strongly-movable, so is* $\mathbf{X}$.

We can now easily deduce the "only if" part of our main theorems, namely

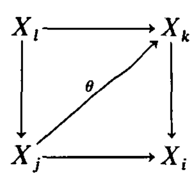
Corollary 3.4. *Let* $\mathbf{X}$ *be an object of pro-*$\mathscr{C}$. *Then if* $\mathbf{X}$ *is stable in pro-*$\mathscr{C}$, *it is strongly-movable.*

Proof. Let $\mathbf{X} \cong X$ in pro-$\mathscr{C}$, with X in $\mathscr{C}$. Regarded as a pro-object indexed by the trivial, one point category, X is clearly strongly-movable, and so the corollary follows from Proposition 3.3.

Finally, in this section, we shall prove a special case of Theorem 2.3 (though without the EM assumption).

Lemma 3.5. *Let* $\mathbf{X}$ *be a strongly-movable object of pro-*$\mathscr{C}$, *and suppose that each bonding morphism* $X_j \to X_i$ *is an epimorphism. Then, given* i, *there is a* $j \to i$ *such that, for each* $k \to j$, *the induced bonding morphism* $X_k \to X_j$ *is an isomorphism.*

Proof. Choose $j \to i$ to be a MOSM for i. Then, given $k \to j$, we may construct a diagram



Consider the composite $\theta \circ (X_k \to X_j): X_k \to X_k$. We have:

$$\theta \circ (X_k \to X_j) \circ (X_l \to X_k) = \theta \circ (X_l \to X_j) = (X_l \to X_k).$$

Since $X_l \to X_k$ is an epimorphism, we deduce that $\theta \circ (X_k \to X_j) = \text{identity on } X_k$. Conversely, consider $(X_k \to X_j) \circ \theta: X_j \to X_j$. We have:

$$(X_k \to X_j) \circ \theta \circ (X_l \to X_j) = (X_k \to X_j) \circ (X_l \to X_k) = (X_l \to X_j).$$

Again, $X_l \to X_j$ an epimorphism implies that $(X_k \to X_j) \circ \theta = \text{identity on } X_j$. Thus, $X_k \to X_j$ is an isomorphism with inverse θ.

Proposition 3.6. *Let* $\mathbf{X}$ *be a strongly-movable object of pro-*$\mathscr{C}$ *all of whose bonding morphisms* $X_j \to X_i$ *are epimorphisms. Then* $\mathbf{X}$ *is stable in pro-*$\mathscr{C}$.

Proof. Let $\mathbf{X}$ be indexed by $I \in \mathit{Cofilt}$. Choose any $i_0 \in I$. By Lemma 3.5, we may choose $j_0 \to i_0$ such that, for each $j \to j_0$, the induced map $X_j \to X_{j_0}$ is an isomorphism.

Let $S(j_0) \subset I$ be the full subcategory of I generated by the successors of j_0; that is, by those $j \in I$ for which there exists a $j \to j_0$. Then, given $i \in I$, we may construct a diagram, $j_0 \leftarrow j \to i$. Thus, $j \in S(j_0)$ and there is a $j \to i$. It follows that $S(j_0)$ is cofiltering and the inclusion $S(j_0) \subset I$ is cofinal (cf. [**11**, Proposition 1.1(ii)]). Hence, by A.1, $\mathbf{X}$ is isomorphic

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to the restricted pro-object indexed by $S(j_0)$. But this latter object has all its bonding maps isomorphisms, and so is obviously stable.

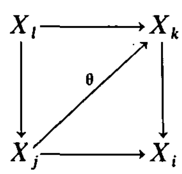
4. Proof of Theorem 2.3

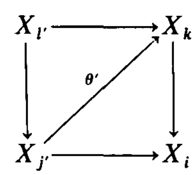
Throughout this section, $\mathscr{C}$ will be an EM-category (Definition 2.1).

Let **X** be a strongly-movable object of pro-$\mathscr{C}$ indexed by $I \in Cofilt$. Given $i \in I$, let $j \to i$ be a MOSM for i (Section 3). Decompose the bonding morphism $X_j \to X_i$ into a composite $X_j \xrightarrow{\beta_i} C_i \xrightarrow{\alpha_i} X_i$ with β_i an epimorphism and α_i a monomorphism.

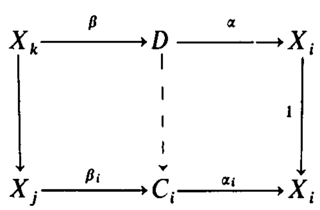
Lemma 4.1. *C_i is independent (up to unique isomorphism preserving α_i) of the choice of MOSM for i.*

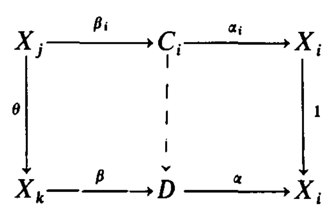
Proof. Let $j' \to i$ be another MOSM for i, and $X_{j'} \xrightarrow{\beta'_i} C'_j \xrightarrow{\alpha'_i} X_i$ the associated decomposition of $X_{j'} \to X_i$. Choose a diagram in I, $j' \leftarrow k \to j$. Since $j \to i$ and $j' \to i$ are MOSMs for i, we may construct diagrams





Decompose $X_k \to X_i$ as $X_k \xrightarrow{\beta} D \xrightarrow{\alpha} X_i$ with β an epimorphism and α a monomorphism. By uniqueness of such decompositions, it follows that the dotted arrows in the diagrams below exist and are unique (see Remark 2.2)





Let $\gamma: D \to D$ be the composite of the two dotted arrows. Then $\alpha \circ \gamma = 1_{X_i} \circ \alpha = \alpha \circ 1_D$, whence $\gamma = 1_D$ since α is a monomorphism.

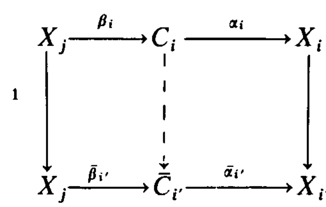
Similarly, the composite $C_i \to D \to C_i$ is the identity on C_i, and so the unique map $C_i \to D$ is an isomorphism. Again, by the same argument, we have a unique isomorphism $D \to C'_i$ satisfying $\alpha_i \circ (D \to C'_i) = \alpha$. Thus, the composite $C_i \to D \to C'_i$ is a unique isomorphism satisfying $\alpha_i = \alpha'_i \circ (C_i \to C'_i)$, which proves the lemma.

For each $i \in I$, choose, once and for all, a MOSM for i. This choice then gives a definite assignment $i \mapsto C_i$, from the objects of I to the objects of $\mathscr{C}$, and a definite assignment $i \mapsto \alpha_i$, from the objects of I to the morphisms of $\mathscr{C}$.

Lemma 4.2. *The assignments* $i \mapsto C_i$ *and* $i \mapsto \alpha_i$ *extend uniquely to a functor* $\mathbf{C}: I \to \mathscr{C}$, *and a natural transformation* $\boldsymbol{\alpha}: \mathbf{C}, \underset{\sim}{\mathbf{X}}$. *Further, all the bonding morphisms of* $\mathbf{C}$ *are epimorphisms.*

Proof. Given $i \to i'$ in I, let $j \to i$ and $j' \to i'$ be the chosen MOSMs for i and i' which define C_i, $C_{i'}$, α_i and $\alpha_{i'}$. By Lemma 3.2, the composite $j \to i \to i'$ is also a MOSM for i'. If the bonding map $X_j \to X_{i'}$ decomposes as $X_j \xrightarrow{\bar{\beta}_{i'}} \bar{C}_{i'} \xrightarrow{\bar{\alpha}_{i'}} X_{i'}$, then it follows from Lemma 4.1 that there is a unique isomorphism $\bar{C}_{i'} \xrightarrow{\cong} C_{i'}$ satisfying $\alpha_{i'} \circ (\bar{C}_{i'} \to C_{i'}) = \bar{\alpha}_{i'}$.

Now, by Remark 2.2, the dotted arrow in the following diagram exists and is unique.

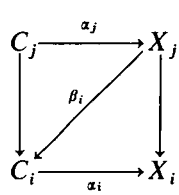


Hence, there is a unique morphism $C_i \to \bar{C}_{i'} \xrightarrow{\cong} C_{i'}$ satisfying $\alpha_{i'} \circ (C_i \to C_{i'}) = (X_i \to X_{i'}) \circ \alpha_i$. Functoriality now follows by uniqueness.

Finally, since $(C_i \to \bar{C}_{i'}) \circ \beta_i = \bar{\beta}_{i'}$, and $\bar{\beta}_{i'}$ is an epimorphism, it follows that $C_i \to \bar{C}_{i'}$ is an epimorphism. Hence, so is $C_i \to C_{i'}$.

Lemma 4.3. $\boldsymbol{\alpha}: \mathbf{C} \to \mathbf{X}$ *is an isomorphism in pro-*$\mathscr{C}$.

Proof. Given $i \in I$, let $j \to i$ be the MOSM for i which defines C_i and α_i. Consider the diagram



The lower right triangle commutes by definition of C_i, α_i and β_i. Now, the bonding map $C_j \to C_i$ is uniquely characterized by the property, $\alpha_i \circ (C_j \to C_i) = (X_j \to X_i) \circ \alpha_j$ (because α_i is a monomorphism). We therefore have,

$$\alpha_i \circ (C_j \to C_i) = (X_j \to X_i) \circ \alpha_j = \alpha_i \circ \beta_i \circ \alpha_j,$$

and so by uniqueness we must have $(C_j \to C_i) = \beta_i \circ \alpha_j$, showing that the upper left triangle also commutes. That $\boldsymbol{\alpha}: \mathbf{C} \to \mathbf{X}$ is an isomorphism in pro-$\mathscr{C}$ now follows from Proposition A.3.

4.4. Proof of Theorem 2.3. If $\mathbf{X}$ in pro-$\mathscr{C}$ is stable, then $\mathbf{X}$ is strongly-movable by Corollary 3.4. Conversely, if $\mathscr{C}$ is EM and $\mathbf{X}$ is strongly-movable, then $\mathbf{X} \cong \mathbf{C}$ by Lemma 4.3. Hence, $\mathbf{C}$ is strongly-movable by Proposition 3.3. But $\mathbf{C}$ has all its bonding maps epimorphisms (Lemma 4.2), and so is stable by Proposition 3.6. It now follows that $\mathbf{X}$ is stable, as required.

5. Interchange of limits and Theorem 2.4

Most of this section is taken up with change of limit theorems. At the end of the section we shall use these theorems to deduce Theorem 2.4.

Our central proposition is the following: the proof will be given in Section 8.

Proposition 5.1. *Let I be any small category and $J \in Cofilt$. Let $\mathbf{F}: I \times J \to \mathscr{C}$ be a bifunctor, and $\hat{\mathbf{F}}: J \to \mathscr{C}^I$ the "adjoint" of $\mathbf{F}$. Regard $\hat{\mathbf{F}}$ as an object of pro-$\mathscr{C}^I$. Then, if $\hat{\mathbf{F}}$ is stable in pro-$\mathscr{C}^I$, the canonical change of limits map*

$$\kappa: \operatorname*{colim}_{i \in I} \lim_{j \in J} F_{ij} \to \lim_{j \in J} \operatorname*{colim}_{i \in I} F_{ij}$$

is an isomorphism in $\mathscr{C}$ (provided the limits and colimits exist).

Now let $\mathscr{C}$ be an EM-category. If $\gamma: F \to G$ is a morphism in $\mathscr{C}^I$, then we may decompose each $\gamma_i: F_i \to G_i$ uniquely in the form $F_i \xrightarrow{\tilde{\gamma}_i} C_i \xrightarrow{u_i} G_i$, with $\tilde{\gamma}_i$ an epimorphism and u_i a monomorphism. By Remark 1.2, if $i \to i'$ is a morphism in I, then there is a unique morphism $C_i \to C_{i'}$ making the following diagram commute

$$\begin{array}{ccccc} F_i & \xrightarrow{\tilde{\gamma}_i} & C_i & \xrightarrow{u_{i'}} & G_i \\ \downarrow & & \downarrow & & \downarrow \\ F_{i'} & \xrightarrow{\tilde{\gamma}_{i'}} & C_{i'} & \xrightarrow{u_{i'}} & G_{i'} \end{array}$$

It follows that $i \mapsto C_i$ defines an object C in $\mathscr{C}^I$, and that the u_i and $\tilde{\gamma}_i$ define morphisms in $\mathscr{C}^I$, $F \xrightarrow{\tilde{\gamma}} C \xrightarrow{u} G$, with $u \circ \tilde{\gamma} = \gamma$. Further, each $\tilde{\gamma}_i$ an epimorphism and each u_i a monomorphism implies that $\tilde{\gamma}$ is an epimorphism and u is a monomorphism in $\mathscr{C}^I$. The uniqueness of the above decomposition of γ in $\mathscr{C}^I$, follows from that in $\mathscr{C}$. Thus, if $\mathscr{C}$ is EM, so is $\mathscr{C}^I$.

Corollary 5.2. *Let $\mathscr{C}$ be EM. Let I be a small category and $J \in Cofilt$, and $\mathbf{F}: I \times J \to \mathscr{C}$ a functor. Then, if $\hat{\mathbf{F}}$ is strongly-movable in $\mathscr{C}^I$, the canonical map*

$$\kappa: \operatorname*{colim}_{i \in I} \lim_{j \in J} F_{ij} \to \lim_{j \in J} \operatorname*{colim}_{i \in I} F_{ij}$$

is an isomorphism in $\mathscr{C}$.

Proof. By the above remarks, $\mathscr{C}^I$ is EM. Hence, by Theorem 2.3, $\hat{\mathbf{F}}$ is stable in $\mathscr{C}^I$, and so the corollary follows from Proposition 5.1.

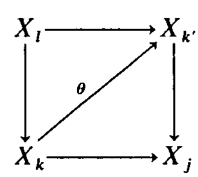
Now let $\mathscr{C}$ be any category, not necessarily EM. Let $\mathbf{X}$ and $\mathbf{K}$ be objects of pro-$\mathscr{C}$ indexed by J and I, respectively. Define a functor

$$\mathbf{F}: I^{op} \times J \to Sets$$

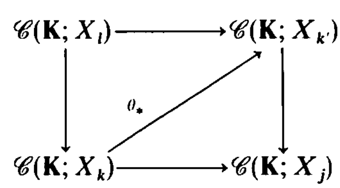
by $F_{ij} = \mathscr{C}(K_i; X_j)$.

Lemma 5.3. *If* $\mathbf{X}$ *is strongly-movable in pro-*$\mathscr{C}$*, then* $\hat{\mathbf{F}}$ *is strongly-movable in pro-*$(Sets)^{I^{op}}$.

Proof. We have $\hat{F}_j = \mathscr{C}(K; X_j)$ for $j \in J$. Given $j \in J$, *let* $k \to j$ be a MOSM for j with respect to $\mathbf{X}$. Then, given $k' \to k$, we may find $l \to k'$ and a θ such that the following diagram commutes



This yields a diagram in $(Sets)^{I^{op}}$



Thus, $\theta_*: \hat{F}_k \to \hat{F}_{k'}$ exhibits $k \to j$ as a MOSM for j with respect to the pro-object $\hat{\mathbf{F}}$.

Corollary 5.4. *Let* $\mathscr{C}$ *be any category and* $\mathbf{X}$ *and* $\mathbf{K}$ *in pro-*$\mathscr{C}$ *be indexed by* J *and* I, *respectively. Then, if* $\mathbf{X}$ *is strongly-movable in pro-*$\mathscr{C}$, *the canonical map*

$$\kappa: \operatorname*{colim}_{i \in I} \lim_{j \in J} \mathscr{C}(K_i;, X_j) \to \lim_{j \in J} \operatorname*{colim}_{i \in I} \mathscr{C}(K_i; X_j) = pro\text{-}\mathscr{C}(\mathbf{K}; \mathbf{X})$$

is an isomorphism of sets.

Proof. *Sets* is EM and admits all small limits and colimits. The corollary now follows from Lemma 5.3 and Corollary 5.2.

5.5. Proof of Theorem 2.4 First suppose that $\mathbf{X}$ is stable in pro-$\mathscr{C}$. Then $\mathbf{X}$ is strongly-movable by Corollary 3.4. Let $\mathbf{h}: X \to \mathbf{X}$ be an isomorphism in pro-$\mathscr{C}$ with X in $\mathscr{C}$. Then, for any $\mathbf{K}$ in pro-$\mathscr{C}$, $\mathbf{h}_*$:pro-$\mathscr{C}(\mathbf{K}; X) \to$ pro-$\mathscr{C}(\mathbf{K}; \mathbf{X})$ is an isomorphism. Taking $\mathbf{K} = K$ in $\mathscr{C}$, this reduces to

$$\mathbf{h}_*; \mathscr{C}(K; X) \xrightarrow{\cong} \text{pro-}\mathscr{C}(K; \mathbf{X}) = \lim_J \mathscr{C}(K; \mathbf{X})$$

which shows that the functor $K \mapsto \lim_J \mathscr{C}(K; \mathbf{X})$ is representable on $\mathscr{C}$.

Conversely, suppose that $\mathbf{X}$ is strongly-movable and that X in $\mathscr{C}$ represents the functor $K \mapsto \lim_J \mathscr{C}(K; \mathbf{X})$. Thus, there is an isomorphism, $\mathscr{C}(K; X) \cong \lim_{j \in J} \mathscr{C}(K; X_j) =$ *pro*-$\mathscr{C}(K; \mathbf{X})$. Taking $K = X$ gives a canonical morphism in pro-$\mathscr{C}$, $\mathbf{h}: X \to \mathbf{X}$. We shall show that $\mathbf{h}$ is an isomorphism, and hence that $\mathbf{X}$ is stable.

Let $\mathbf{K}$ be any object of pro-$\mathscr{C}$ indexed by I. Then we have natural isomorphisms of sets induced by $\mathbf{h}$

$$\phi_i: \mathscr{C}(K_i; X) \xrightarrow{\cong} \lim_{j \in J} \mathscr{C}(K_i; X_j).$$

These yield an isomorphism

$$\phi = \operatorname*{colim}_{i \in I} \phi_i : \operatorname*{colim}_{i \in I} \mathscr{C}(K_i; X) \xrightarrow{\cong} \operatorname*{colim}_{i \in I} \lim_{j \in J} \mathscr{C}(K_i; X_j).$$

But, $\operatorname{colim}_{i \in I} \mathscr{C}(K_i; X) = \text{pro-}\mathscr{C}(\mathbf{K}; X)$, and by Corollary 5.4,

$$\kappa: \operatorname*{colim}_{i \in I} \lim_{j \in J} \mathscr{C}(K_i; X_j) \to \lim_{j \in J} \operatorname*{colim}_{i \in I} \mathscr{C}(K_i; X_j) = \textit{pro-}\mathscr{C}(\mathbf{K}; \mathbf{X})$$

is an isomorphism. Hence, we have a natural isomorphism

$$\kappa \circ \phi : \text{pro-}\mathscr{C}(\mathbf{K}; X) \xrightarrow{\cong} \text{pro-}\mathscr{C}(\mathbf{K}; \mathbf{X})$$

which, by construction and naturality is just $\mathbf{h}_*$. It now follows formally that $\mathbf{h}$ is an isomorphism in pro-$\mathscr{C}$.

Remark 5.6. If $\mathscr{C}$ is a pointed category (i.e., it has an initial-terminal object *), then the functor $K \mapsto \lim_{j \in J} \mathscr{C}(K; X_j)$ takes values in $Sets_0$, the category of pointed sets. By the same proof, we than obtain a pointed version of Theorem 2.4.

6. Homotopy categories and Theorem 2.5

Let $\mathscr{C}$ be a closed model category, and $Ho(\mathscr{C})$ its associated homotopy category, [10]. Let $\mathscr{C}_0$ be the associated category of pointed, connected objects in $\mathscr{C}$.

A homotopy functor on $\mathscr{C}_0$ is a contravariant functor $H:Ho(\mathscr{C}_0)\to Sets_0$, satisfying the following axioms:

6.1 (i) If $\vee$ denotes the coproduct in $\mathscr{C}_0$, then for any small indexing set A, the canonical map

$$u:H\left(\bigvee_{\alpha\in A} K_\alpha\right)\to\prod_{\alpha\in A} H(K_\alpha)$$

is an isomorphism of based sets.

(ii) Given a pushout diagram in $Ho(\mathscr{C}_0)$

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \tilde{g}} \\ Y & \xrightarrow{\tilde{f}} & P \end{array}$$

the induced diagram of based sets

$$\begin{array}{ccc} H(P) & \xrightarrow{\tilde{f}^*} & H(Y) \\ {\scriptstyle \tilde{g}^*}\downarrow & & \downarrow{\scriptstyle g^*} \\ H(X) & \xrightarrow{f^*} & H(A) \end{array}$$

has the Mayer–Vieteris property; i.e., given $x\in H(X)$, $y\in H(Y)$ such that $f^*(x)=g^*(y)$, then there exists $z\in H(P)$ such that $\tilde{g}^*(z)=x$ and $\tilde{f}^*(z)=y$.

We shall assume that Brown's Representability Theorem holds for homotopy functors on $\mathscr{C}_0$; i.e.

6.2. Brown's Representability Theorem *If H is a homotopy functor on $\mathscr{C}_0$, then H is representable; i.e., there is an object X in $\mathscr{C}_0$ and a natural isomorphism of based-set-valued functors on $Ho(\mathscr{C}_0)$*

$$T:Ho(\mathscr{C}_0)(\_\_;X)\xrightarrow{\cong} H(\_\_).$$

The main step in our proof of Theorem 2.5 is the following.

Lemma 6.3. *Let $\mathscr{C}_0$ be as above, and suppose that $\mathbf{X}$ is a strongly-movable object of pro-$Ho(\mathscr{C}_0)$ indexed by J. Then the functor $K\mapsto\lim_J Ho(\mathscr{C}_0)(K;\mathbf{X}):Ho(\mathscr{C}_0)\to Sets_0$, is representable.*

Proof. By Brown's Representability Theorem, it suffices to show that $K \mapsto \lim_J Ho(\mathscr{C}_0)(K;\mathbf{X})$ is a homotopy functor.

(i) We have

$$\lim_J Ho(\mathscr{C}_0)\left(\bigvee_\alpha K_\alpha;\mathbf{X}\right) \to \lim_J \prod_\alpha Ho(\mathscr{C}_0)(K_\alpha;\mathbf{X})$$

is an isomorphism since $K \mapsto Ho(\mathscr{C}_0)(K;X_j)$ is a homotopy functor for each $j \in J$. Axiom 6.1(i) now follows from the fact that limits commute.

(ii) Let Δ be a finite category and $D:\Delta \to Ho(\mathscr{C}_0)$ a Δ-diagram in $Ho(\mathscr{C}_0)$. For Y in $Ho(\mathscr{C}_0)$, the composite of D with $Ho(\mathscr{C}_0)(\_\_;Y):Ho(\mathscr{C}_0) \to Sets_0$, defines a Δ^{op}-diagram in $Sets_0$, which we denote by $Ho(\mathscr{C}_0)(D;Y)$. Thus, $j \mapsto Ho(\mathscr{C}_0)(D;X_j)$ defines a pro-object in the category of Δ^{op}-diagrams in $Sets_0$; i.e., an object of pro-$(Sets_0)^{\Delta^{op}}$. Further, by Lemma 5.3, $\mathbf{X}$ strongly-movable in pro-$Ho(\mathscr{C}_0)$ implies that $Ho(\mathscr{C}_0)(D;\mathbf{X})$ is strongly-movable in pro-$(Sets_0)^{\Delta^{op}}$. Thus, since $Sets_0$, and hence $(Sets_0)^{\Delta^{op}}$ (see the remarks preceding Corollary 5.2), is EM, it follows from Theorem 2.3 that $Ho(\mathscr{C}_0)(D;\mathbf{X})$ is stable in pro-$(Sets_0)^{\Delta^{op}}$. Hence, the natural map

$$h:\lim_{j\in J} Ho(\mathscr{C}_0)(D;X_j) \to Ho(\mathscr{C}_0)(D;\mathbf{X})$$

is an isomorphism in pro-$(Sets_0)^{\Delta^{op}}$. It follows that, given $j \in J$, there is a $k \to j$ and a morphism θ in $(Sets_0)^{\Delta^{op}}$ such that the following diagram commutes.

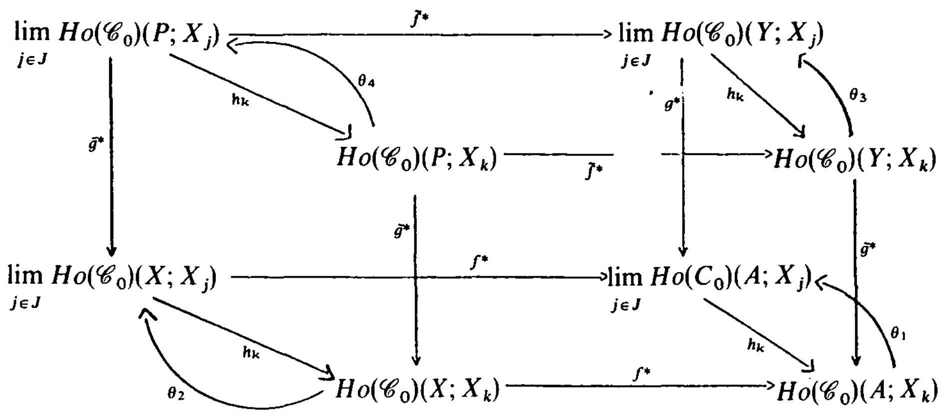
$$\begin{array}{ccc}
\lim_{j\in J} Ho(\mathscr{C}_0)(D;X_j) & \xrightarrow{h_k} & Ho(\mathscr{C}_0)(D;X_k) \\
\downarrow 1 & \swarrow \theta & \downarrow \\
\lim_{j\in J} Ho(\mathscr{C}_0)(D;X_j) & \xrightarrow[h_j]{} & Ho(\mathscr{C}_0)(D;X_j)
\end{array} \qquad (6.4)$$

Now take Δ to be the diagram type

$$\Delta = \left\{\begin{array}{ccc} 1\bullet & \longrightarrow & \bullet 2 \\ \downarrow & & \downarrow \\ 3\bullet & \longrightarrow & \bullet 4 \end{array}\right\}$$

and D to be the Δ-diagram of Axiom 6.1(ii). Then we obtain a commutative diagram in $Ho(\mathscr{C}_0)$ from (6.4.)—see overleaf. Here, the θ's split the projections h_k; i.e., $\theta_i \circ h_k = \text{identity}$, $1 \leqq i \leqq 4$.

Now, suppose given $x \in \lim_{j\in J} Ho(\mathscr{C}_0)(X;X_j)$ and $y \in \lim_{j\in J} Ho(\mathscr{C}_0)(Y;X_j)$ which have the same image in $\lim_{j\in J} Ho(\mathscr{C}_0)(A;X_j)$. Then $h_k(x) \in Ho(\mathscr{C}_0)(X;X_k)$ and $h_k(y) \in Ho(\mathscr{C}_0)(Y;X_k)$ have the same image in $Ho(\mathscr{C}_0)(A;X_k)$. Thus, since $Ho(\mathscr{C}_0)(\_\_;X_k)$ is a homotopy functor, and hence satisfies Axiom 6.1(ii), there is a $\tilde{z} \in Ho(\mathscr{C}_0)(P;X_k)$



(6.5)

satisfying $\tilde{g}^*(\tilde{z})=h_k(x)$ and $\tilde{f}^*(\tilde{z})=h_k(y)$. Take $z=\theta_4(\tilde{z})\in\lim_{j\in J} Ho(\mathscr{C}_0)(P;X_j)$. Then the commutativity of the diagram 6.5 together with the fact that $\theta_i\circ h_k=\text{identity}$, implies that $\tilde{g}^*(z)=x$ and $\tilde{f}^*(z)=y$. This shows that $\lim_{j\in J} Ho(\mathscr{C}_0)(\_\_;X_j)$ satisfies Axiom 6.1(ii), as required.

6.6. Proof of Theorem 2.5. *If* **X** *is stable, then* **X** *is strongly-movable by Corollary 3.4. Conversely, suppose that* **X** *is strongly-movable. Then* **X** *is stable by Lemma 6.3, Theorem 2.4 and Remark 5.6.*

7. Strong homotopy categories and Theorem 2.6

We shall deduce Theorem 2.6 from Theorem 2.5 using the induction technique developed in [**11**]. For the reader's convenience we begin by reviewing this technique.

Let $\mathscr{C}$ be a nice closed model category ([**6**, Section 2]). $\mathscr{C}=\mathbf{S}$ or $\mathscr{C}=\mathbf{S}p$ will do. Let $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ be a morphism in $\mathscr{C}^I$ for some $I\in CSDS$ (see appendix). We wish to give conditions under which **f** is an isomorphism in $Ho(\text{pro-}\mathscr{C}_0)$.

Let tow-$\mathscr{C}_0$ be the full subcategory of pro-$\mathscr{C}_0$ generated by the tower objects (i.e., the pro-objects indexed by the natural numbers $\mathbb{N}$). The induction technique is based on the following idea. Suppose that P is some property of morphisms in pro-$\mathscr{C}_0$ for which we are able to show that any morphism in tow-$\mathscr{C}_0$ which satisfies P is an isomorphism in $Ho(\text{pro-}\mathscr{C}_0)$. Then, if **f** is a morphism in pro-$\mathscr{C}_0$ which satisfies P, under what circumstances can we conclude that **f** is an isomorphism in $Ho(\text{pro-}\mathscr{C}_0)$? Thus, we are asking when we can "induct up" from a known result for towers to the same result in the general case.

To state what is required, we introduce the category $\tilde{I}$ of *towers in I* ([**11**, 1.3]). The objects of $\tilde{I}$ are towers in I, $\mathbf{i}=(i_0\leftarrow i_1\leftarrow\cdots\leftarrow i_n\leftarrow\cdots)$, and morphisms in $\tilde{I}$ are level preserving ladder diagrams. There is an evaluation functor $e:\mathbb{N}\times\tilde{I}\to I$, given on objects by $e(n,\mathbf{i})=i_n$. Then $\tilde{I}\in Cofilt$ and e is cofinal ([**11**, Proposition 1.4]).

For each $\mathbf{i} \in \tilde{I}$, define a morphism $\mathbf{f}_\mathbf{i}: \mathbf{X}_\mathbf{i} \to \mathbf{Y}_\mathbf{i}$ in $\mathscr{C}_0^{\mathbb{N}}$, by $(\mathbf{f}_\mathbf{i})_n = f_{i_n}: X_{i_n} \to Y_{i_n}$, $n \geqq 0$. Let $\tilde{I}(\mathbf{f}) \subseteq \tilde{I}$ be the full subcategory of $\tilde{I}$ generated by those $\mathbf{i} \in \tilde{I}$ for which $\mathbf{f}_\mathbf{i}$ is an isomorphism in $Ho(\text{pro-}\mathscr{C}_0)$. Then our induction technique is based on the following *Induction Theorem.*

Theorem 7.1. ([**11**, Theorem 3.1]). If $\tilde{I}(\mathbf{f}) \subseteq \tilde{I}$ is cofinal, then $\mathbf{f}$ is an isomorphism in $Ho(\text{pro-}\mathscr{C}_0)$.

Remarks 7.2. (i) For $\mathbf{i} \in \tilde{I}$, define $e_\mathbf{i}: \mathbb{N} \to I$ by $e_\mathbf{i}(n) = i_n$. Then observe that $\mathbf{f}_\mathbf{i} = e_\mathbf{i}^*(\mathbf{f}): e_\mathbf{i}^*(\mathbf{X}) \to e_\mathbf{i}^*(\mathbf{Y})$.

(ii) Suppose $\mathbf{X} = X$ is an object of $\mathscr{C}$. We may regard X as an object of $\mathscr{C}^I$ having all bonding maps the identity on X. Then $\mathbf{f}: X \to \mathbf{Y}$ consists of a collection of morphisms $f_i: X \to Y_i$ satisfying $(Y_i \to Y_j) \circ f_i = f_j$, for each $i \to j$ in I. In this case, for $\mathbf{i} \in \tilde{I}$, $\mathbf{f}_\mathbf{i}: X \to \mathbf{Y}_\mathbf{i} = e_\mathbf{i}^*(\mathbf{Y})$ is just the composite $X \xrightarrow{f} \mathbf{Y} \xrightarrow{\mu_\mathbf{i}} \mathbf{Y}_\mathbf{i}$, where $\mu_\mathbf{i}$ is the canonical morphism in pro-$\mathscr{C}_0$ given by A.1.

With these preliminaries at hand, we can now take the main step in the proof of Theorem 2.6.

Lemma 7.3. *Let* $\mathbf{X}$ *in pro-*$\mathscr{C}_0$ *be strongly-movable as an object of pro-*$Ho(\mathscr{C}_0)$*. Suppose* $\mathbf{X}$ *is indexed by* $I \in CSDS$*, and let* $\tilde{I}(\mathbf{X})$ *be the full subcategory of* $\tilde{I}$ *generated by those* $\mathbf{i} \in \tilde{I}$ *for which the canonical morphism* $\mu_\mathbf{i}; \mathbf{X} \to e_\mathbf{i}^*(\mathbf{X}) = \mathbf{X}_\mathbf{i}$*, is an isomorphism in pro-*$Ho(\mathscr{C}_0)$*. Then the inclusion* $\tilde{I}(\mathbf{X}) \subseteq \tilde{I}$ *is cofinal.*

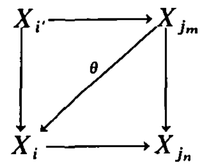
Proof. Given $\mathbf{i} \in \tilde{I}$, it suffices to construct a morphism $\mathbf{j} \to \mathbf{i}$ in $\tilde{I}$ with $\mathbf{j} \in \tilde{I}(\mathbf{X})$ (cf. [**11**, Lemma 1.1(ii)]).

We first show that $\mathbf{j} \to \mathbf{i}$ can be chosen so that each $j_n \to j_{n-1}$ is a MOSM for j_{n-1} (with respect to $\mathbf{X}$). Take $j_0 = i_0$. Suppose that $j_0 \leftarrow j_1 \leftarrow \cdots \leftarrow j_{n-1}$, and compatible morphisms $j_r \to i_r$, $0 \leqq r \leqq n-1$, have been constructed. Choose $j_{n-1} \leftarrow j'_n \to i_n$ so that the following square commutes

$$\begin{array}{ccc} j'_n & \longrightarrow & i_n \\ \downarrow & & \downarrow \\ j_{n-1} & \longrightarrow & i_{n-1} \end{array}$$

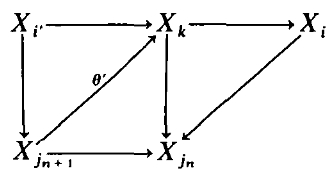
Now take $j_n \to j'_n$ to be a MOSM for j'_n (such exists since $\mathbf{X}$ is strongly-movable). Then, by Lemma 3.2, $j_n \to j_{n-1}$ is a MOSM for j_{n-1}. Thus, $\mathbf{j} \to \mathbf{i}$ can be constructed inductively with the required property.

We next show that $\mathbf{j} \in \tilde{I}(\mathbf{X})$; i.e., that $\mu_\mathbf{j}: \mathbf{X} \to e_\mathbf{j}^*(\mathbf{X}) = \mathbf{X}_\mathbf{j}$ is an isomorphism in pro-$Ho(\mathscr{C}_0)$. By Proposition A.5 and Remarks 7.2, it suffices to show that, for each morphism $i \to j_n$ in I, there is an $m \geqq n$, morphisms $i' \to i$ and $i' \to j_m$ in I, and θ in $Ho(\mathscr{C}_0)$ such that the following diagram commutes



(7.4)

Take $m=n+1$. Choose morphisms in I, $j_{n+1}\leftarrow k\rightarrow i$. Then, since $j_{n+1}\rightarrow j_n$ is a MOSM for j_n (by construction of **j**), there is an $i'\rightarrow k$ and a morphism θ' in $Ho(\mathscr{C}_0)$ such that the following diagram commutes



(note: the right-hand triangle commutes since $I\in CSDS$). Take $\theta=(X_k\rightarrow X_i)\circ\theta'$, then we obtain the required diagram 7.4.

Recall that, in [**6**, §4], there is defined a homotopy limit functor.

$$\text{holim}: Ho(\text{pro-}\mathscr{C}_0)\rightarrow Ho(\mathscr{C}_0) \tag{7.5}$$

which is right adjoint to the inclusion $Ho(\mathscr{C}_0)\rightarrow Ho(\text{pro-}\mathscr{C}_0)$. This adjointness implies that, for each **X** in $Ho(\text{pro-}\mathscr{C}_0)$, we have a canonical natural morphism in $Ho(\text{pro-}\mathscr{C}_0)$

$$\mathbf{h}: \text{holim}\,\mathbf{X}\rightarrow\mathbf{X}. \tag{7.6}$$

We shall need to assume that a Whitehead Theorem holds in $Ho(\mathscr{C}_0)$.

Whitehead Theorem 7.7. *A map $f: X\rightarrow Y$ in $Ho(\mathscr{C}_0)$ which induces isomorphisms, $f_{\#}:\pi_k(X)\rightarrow\pi_k(Y)$ for each $k\geqq 1$, is an isomorphism in $Ho(\mathscr{C}_0)$.*

Such a theorem holds for $\mathscr{C}_0=\mathbf{S}_0$ or $\mathbf{S}p_0$.

The following properties of the map **h** (7.6) are well known, and easy to deduce from the Bousfield–Kan spectral sequence ([**6**, §4.9]), together with 7.7. We omit the proof.

Lemma 7.8. (i) *If* **X** *in $Ho(\text{pro-}\mathscr{C}_0)$ is stable in pro-$Ho(\mathscr{C}_0)$, then* **h** *is an isomorphism in pro-$Ho(\mathscr{C}_0)$.*

(ii) *If* **X** *in $Ho(\text{pro-}\mathscr{C}_0)$ is stable in $Ho(\text{pro-}\mathscr{C}_0)$, then* **h** *is an isomorphism in $Ho(\text{pro-}\mathscr{C}_0)$.*

7.9. Proof of Theorem 2.6. To prove Theorem 2.6, we need only assume that $Ho(\mathscr{C}_0)$ satisfies 6.2 and 7.7. This is the case for $\mathscr{C}_0=\mathbf{S}_0$ or $\mathbf{S}p_0$.

If $\mathbf{X}$ is stable in $Ho(\text{pro-}\mathscr{C}_0)$, then trivially it is stable in pro-$Ho(\mathscr{C}_0)$. Suppose, then that $\mathbf{X}$ in $Ho(\text{pro-}\mathscr{C}_0)$ is stable in pro-$Ho(\mathscr{C}_0)$. Then $\mathbf{X}$ is strongly-movable in pro-$Ho(\mathscr{C}_0)$ by Theorem 2.5. Also, by Lemma 7.8(i), $\mathbf{h}:\text{holim}\,\mathbf{X}\to\mathbf{X}$ is an isomorphism in pro-$Ho(\mathscr{C}_0)$.

By the appendix, $\mathbf{X}$ is canonically isomorphic in pro-$\mathscr{C}_0$ to an object indexed by $I\in CSDS$. Clearly, if $\mathbf{X}$ is stable in pro-$Ho(\mathscr{C}_0)$, so is any object isomorphic to it. Hence, we may assume that $\mathbf{X}$ is indexed by $I\in CSDS$.

Let $\tilde{I}(\mathbf{X})$ be as in Lemma 7.3. Then, for each $\mathbf{j}\in\tilde{I}(\mathbf{X})$, the composite $\text{holim}\,\mathbf{X}\xrightarrow{\mathbf{h}}\mathbf{X}\xrightarrow{\mu_j}\mathbf{X}_\mathbf{j}$ is an isomorphism in pro-$Ho(\mathscr{C}_0)$. Regard $\mu_\mathbf{j}\circ\mathbf{h}$ as a morphism in $Ho(\text{tow-}\mathscr{C}_0)$. Then, since $\text{holim}\,\mathbf{X}$ is stable, it follows from [**6**, Corollary 5.2.13, page 178], that $\mu_\mathbf{j}\circ\mathbf{h}$ is an isomorphism in $Ho(\text{tow-}\mathscr{C}_0)$, and hence in $Ho(\text{pro-}\mathscr{C}_0)$.

Now consider the full subcategory $\tilde{I}(\mathbf{h})\subseteq\tilde{I}$, defined in the remarks preceding Theorem 7.1. By Remarks 7.2, $\tilde{I}(\mathbf{h})$ is the full subcategory generated by those $\mathbf{i}\in\tilde{I}$ for which $\mu_\mathbf{i}\circ\mathbf{h}:\text{holim}\,\mathbf{X}\to\mathbf{X}_\mathbf{i}$ is an isomorphism in $Ho(\text{pro-}\mathscr{C}_0)$. We have therefore shown that $\tilde{I}(\mathbf{X})\subseteq\tilde{I}(\mathbf{h})\subseteq\tilde{I}$. But, by Lemma 7.3, $\tilde{I}(\mathbf{X})\subseteq\tilde{I}$ is cofinal, and hence, so it $\tilde{I}(\mathbf{h})\subseteq\tilde{I}$. That $\mathbf{h}$ is an isomorphism in $Ho(\text{pro-}\mathscr{C}_0)$ now follows from Theorem 7.1. Thus, $\mathbf{X}$ is stable in $Ho(\text{pro-}\mathscr{C}_0)$, and the proof of Theorem 2.6 is complete.

7.10. Proof of Proposition 2.9. By hypothesis, both $\mathbf{X}$ and $\mathbf{Y}$ are stable in pro-$Ho(\mathscr{C}_0)$. Hence, by Theorem 2.6, they are stable in $Ho(\text{pro-}\mathscr{C}_0)$. We then have, by the naturality of $\mathbf{h}$ (7.6), a commutative diagram in $Ho(\text{pro-}\mathscr{C}_0)$

$$\begin{array}{ccc} \text{holim}\,\mathbf{X} & \xrightarrow{\text{holim}\,\mathbf{f}} & \text{holim}\,\mathbf{Y} \\ {\scriptstyle \mathbf{h}}\big\downarrow\cong & & \cong\big\downarrow{\scriptstyle \mathbf{h}} \\ \mathbf{X} & \xrightarrow{\mathbf{f}} & \mathbf{Y} \end{array} \tag{7.11}$$

Since $\mathbf{f}$ is an isomorphism in pro-$Ho(\mathscr{C}_0)$, it is a weak pro-homotopy equivalence (i.e., it induces isomorphisms of pro-groups on pro-homotopy groups). Hence, by [**11**, Theorem B], $\text{holim}\,\mathbf{f}$ is an isomorphism in $Ho(\mathscr{C}_0)$, and therefore in $Ho(\text{pro-}\mathscr{C}_0)$. That $\mathbf{f}$ is an isomorphism in $Ho(\text{pro-}\mathscr{C}_0)$ now follows from diagram (7.11).

8. Proof of Proposition 5.1

Recall that, if $\mathbf{X}$ is an object $\mathscr{D}^J$ for some category $\mathscr{D}$, and $\mathbf{X}$ is stable in pro-$\mathscr{D}$, then $\lim_J\mathbf{X}$ exists in $\mathscr{D}$, and the canonical morphism $h:\lim_J\mathbf{X}\to\mathbf{X}$ is an isomorphism in pro-$\mathscr{D}$. Thus, if $\hat{\mathbf{F}}$ is stable in pro-$\mathscr{C}^I$, the canonical morphism $\mathbf{h}:\lim_J\hat{\mathbf{F}}\to\hat{\mathbf{F}}$ is an isomorphism in pro-$\mathscr{C}^I$.

Now $\lim_J\hat{\mathbf{F}}$ is the object of $\mathscr{C}^I$ given by $i\to\lim_{j\in J}F_{ij}$. Hence, by Proposition A.3,

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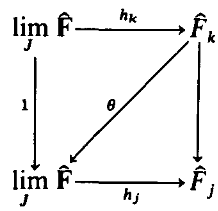
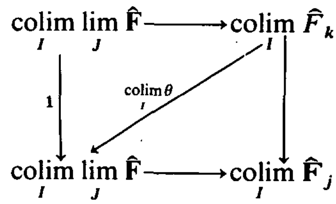
given $j \in J$, there is a $k \to j$ and a morphism θ in $\mathscr{C}^I$ such that the following diagram commutes in $\mathscr{C}^I$



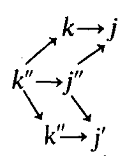
Take the colimit over I of this diagram to obtain a diagram in $\mathscr{C}$



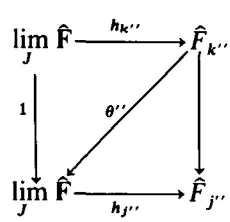
Now let $p_j\colon \lim_J \operatorname{colim}_I \hat{\mathbf{F}} \to \operatorname{colim}_I \hat{F}_j$ be the projection and define $\phi\colon \lim_J \operatorname{colim}_I \hat{\mathbf{F}} \to \operatorname{colim}_I \lim_J \hat{\mathbf{F}}$ by: $\phi = (\operatorname{colim}_I \theta) \circ p_k$.

We claim that ϕ is independent of the choice of $k \to j$ and θ. Suppose that $k' \to j'$ and θ' are another such choice. We must show that $(\operatorname{colim}_I \theta') \circ p_{k'} = (\operatorname{colim}_I \theta) \circ p_k$.

Embed $k \to j$ and $k' \to j'$ in a diagram



We may assume that k'' is chosen so that there is a θ'' making the following diagram commute in $\mathscr{C}^I$



However, we do *not* know that $\theta \circ (\hat{F}_{k''} \to \hat{F}_k) = \theta'' = \theta' \circ (\hat{F}_{k''} \to \hat{F}_{k'})$. We must show that θ'' can be chosen so that this is the case.

By construction, both θ and θ'' represent $\mathbf{h}^{-1}\colon \hat{\mathbf{F}} \to \lim_J \hat{\mathbf{F}}$ in pro-$\mathscr{C}^I$. Hence, the maps

$$\mathscr{C}^I(\hat{F}_k; \lim_J \hat{\mathbf{F}}) \to \operatorname{colim}_{j \in J} \mathscr{C}^I(\hat{F}_j; \lim_J \hat{\mathbf{F}}) = \text{pro-}\mathscr{C}^I(\hat{\mathbf{F}}; \lim_J \hat{\mathbf{F}})$$

and

$$\mathscr{C}^I(\hat{F}_{k''};\lim_J \hat{\mathbf{F}}) \to \operatorname{colim}_{j\in J} \mathscr{C}^I(\hat{F}_j;\lim_J \hat{\mathbf{F}}) = \text{pro-}\mathscr{C}^I(\hat{\mathbf{F}};\lim_J \hat{\mathbf{F}})$$

send θ and θ'' to $\mathbf{h}^{-1}$, respectively. It follows that there is a diagram in J, $k \leftarrow l \rightarrow k''$, such that the two composites $\theta \circ (\hat{F}_l \to \hat{F}_k)$ and $\theta'' \circ (\hat{F}_l \to \hat{F}_{k''})$, are equal.

Similarly, we may find a diagram, $k' \leftarrow l' \rightarrow k''$, such that $\theta' \circ (\hat{F}_{l'} \to \hat{F}_{k'}) = \theta'' \circ (\hat{F}_{l'} \to \hat{F}_{k''})$. Now choose a diagram, $l \leftarrow l'' \rightarrow l'$. Then we have

$$\theta \circ (\hat{F}_{l''} \to \hat{F}_k) = \theta'' \circ (\hat{F}_{l''} \to \hat{F}_{k''}) = \theta' \circ (\hat{F}_{l''} \to \hat{F}_{k'})$$

in $\mathscr{C}^I$. Taking colimits over I then gives

$$(\operatorname{colim}_I \theta) \circ (\operatorname{colim}_I \hat{F}_{l''} \to \operatorname{colim}_I \hat{F}_k) = (\operatorname{colim}_I \theta') \circ (\operatorname{colim}_I \hat{F}_{l''} \to \operatorname{colim}_I \hat{F}_{k''})$$

$$= (\operatorname{colim}_I \theta') \circ (\operatorname{colim}_I \hat{F}_{l''} \to \operatorname{colim}_I \hat{F}_{k'}). \tag{8.1}$$

We also have, by definition of the maps p_j, that

$$(\operatorname{colim}_I \hat{F}_{l''} \to \operatorname{colim}_I \hat{F}_k) \circ p_{l''} = p_k$$

and

$$(\operatorname{colim}_I \hat{F}_{l''} \to \operatorname{colim}_I \hat{F}_{k'}) \circ p_{l''} = p_{k'}.$$

Combining these with (8.1) gives the result $(\operatorname{colim}_I \theta) \circ p_k = (\operatorname{colim}_I \theta'') \circ p_{l''} = (\operatorname{colim}_I \theta') \circ p_{k'}$, as required.

To prove the proposition, we shall show that $\phi = (\operatorname{colim}_I \theta) \circ p_k$ is the inverse of κ. Now κ is uniquely characterized by the property that, for each $j \in J$, $p_j \circ \kappa : \operatorname{colim}_I \lim_J \hat{\mathbf{F}} \to \operatorname{colim}_I \hat{F}_j$ is the colimit over I of the canonical projection $h_j : \lim_J \hat{\mathbf{F}} \to \hat{F}_j$, in $\mathscr{C}^I$. We therefore have

$$\phi \circ \kappa = (\operatorname{colim}_I \theta) \circ p_k \circ \kappa = (\operatorname{colim}_I \theta) \circ (\operatorname{colim}_I h_k)$$

$$= \operatorname{colim}_I (\theta \circ h_k) = \operatorname{colim}_I (1_{\lim_J \hat{\mathbf{F}}}) = 1_{\operatorname{colim}_I \lim_J \hat{\mathbf{F}}}.$$

Conversely, given $j \in J$, we may represent ϕ in the form $\phi = (\operatorname{colim}_I \theta) \circ p_k$ for some $k \to j$ and θ (since ϕ is independent of which representation we choose). Hence,

$$p_j \circ \kappa \circ \phi = p_j \circ \kappa \circ (\mathrm{colim}_I\, \theta) \circ p_k$$

$$= (\mathrm{colim}_I\, h_j) \circ (\mathrm{colim}_I\, \theta) \circ p_k$$

$$= (\mathrm{colim}_I\, (h_j \circ \theta)) \circ p_k$$

$$= (\mathrm{colim}_I\, \hat{F}_k \to \mathrm{colim}_I\, \hat{F}_j) \circ p_k = p_j.$$

By the universal property of limits, it now follows that $\kappa \circ \phi =$ identity on $\lim_J \mathrm{colim}_I\, \hat{\mathbf{F}}$. This proves the proposition.

Appendix

Let *Cofilt* be the category whose objects are small cofiltering (or left filtering) categories, with cofinal functors as morphisms. $I \in$ *Cofilt* is a *strongly-directed-set* if there is at most one morphism between any two objects of I, and if there exist morphisms $i \to j$ and $j \to i$, then $i = j$. A strongly-directed-set is *cofinite* if there are only finitely many morphisms out of each object. Let *CSDS* denote the full subcategory of *Cofilt* generated by the cofinite strongly-directed-sets.

Let $\mathscr{C}$ be any category. Recall [**1**, appendix] that a pro-object in $\mathscr{C}$ is a functor $\mathbf{X}: I \to \mathscr{C}$ for some $I \in$ *Cofilt*, with morphism set given by:

$$\text{pro-}\mathscr{C}(\mathbf{X}; \mathbf{Y}) = \lim_{j \in J} \operatorname*{colim}_{i \in I} \mathscr{C}(X_i; Y_j)$$

where $\mathbf{Y}: J \to \mathscr{C}$. With these definitions we obtain a category pro-$\mathscr{C}$, of pro-objects in $\mathscr{C}$.

Next recall ([**6**, §2], [**11**, 1.2 and 1.5]), that there is a functor M:pro-$\mathscr{C} \to$ pro-$\mathscr{C}$ (the "Mardešić construction") and a natural isomorphism $\mathbf{X} \overset{\equiv}{\to} M\mathbf{X}$, with $M\mathbf{X}$ indexed by a cofinite strongly-directed set.

If $\phi: I \to J$ is a morphism in *Cofilt*, then there is a natural isomorphism in pro-$\mathscr{C}$.

$$\mu_\phi: \mathbf{X} \to \phi^* \mathbf{X}, \tag{A.1}$$

for any pro-object $\mathbf{X}$ indexed by J, where $\phi^*\mathbf{X} = \mathbf{X} \circ \phi$ is the induced pro-object indexed by I ([**1**, appendix]).

Recall that any morphism $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ may be "uniformly reindexed" ([**1**, appendix]). We shall need the details of this construction. If $\mathbf{X}$ is indexed by I and $\mathbf{Y}$ by J, we say that a morphism in $\mathscr{C}$, $g: X_i \to Y_j$, *represents* $\mathbf{f}$ if the image of g under $\mathscr{C}(X_i; Y_j) \to$

$\operatorname{colim}_{i\in I}\mathscr{C}(X_i; Y_j)$ coincides with the image of **f** under $\text{pro-}\mathscr{C}(\mathbf{X};\mathbf{Y}) = \lim_{j\in J}\operatorname{colim}_{i\in I}\mathscr{C}(X_i; Y_j) \to \operatorname{colim}_{i\in I}\mathscr{C}(X_i; Y_j)$. If $h: X_{i'} \to Y_{j'}$ also represent **f**, then a morphism: $g \to h$ is a pair of morphisms $i \to i'$ and $j \to j'$, such that the following square commutes

$$\begin{array}{ccc} X_i & \xrightarrow{g} & Y_j \\ \downarrow & & \downarrow \\ X_{i'} & \xrightarrow{h} & Y_{j'} \end{array}$$

We obtain, therefore, a category $\Delta(\mathbf{f})$ whose objects are the morphisms representing **f**. Define $\phi: \Delta(\mathbf{f}) \to I$ and $\psi: \Delta(\mathbf{f}) \to J$ by $\phi(g) = i$ and $\psi(g) = j$. Then it can be shown that $\Delta(\mathbf{f})$ is small and cofiltering, and ϕ and ψ are cofinal. Now define a "uniformly indexed" morphism $\tilde{\mathbf{f}}: \phi^*\mathbf{X} \to \psi^*\mathbf{Y}$, by $\tilde{f}_g = g: X_{\phi(g)} \to Y_{\psi(g)}$, $g \in \Delta(\mathbf{f})$. Then, by the naturality of (A.1), we have a commutative diagram in pro-$\mathscr{C}$

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{\mathbf{f}} & \mathbf{Y} \\ \mu_* \downarrow \cong & & \cong \downarrow \mu_* \\ \phi^*\mathbf{X} & \xrightarrow{\tilde{\mathbf{f}}} & \psi^*\mathbf{Y} \end{array} \tag{A.2}$$

With the above machinery, we can now classify isomorphisms in pro-$\mathscr{C}$. The proposition we shall give is slightly more general than is usual (cf. [**11**, Proposition 1.8]; [**6**, Lemma 5.5.4]).

Proposition A.3. *Let* $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ *be a morphism in pro-*$\mathscr{C}$*, with* **X** *indexed by* I *and* **Y** *by* J. *Then* **f** *is an isomorphism in pro-*$\mathscr{C}$ *if and only if, for each morphism* $g: X_i \to Y_j$ *representing* **f**, *there are morphisms* $k \to i$ *in* I *and* $l \to j$ *in* J, *a morphism* $h: X_k \to Y_l$ *representing* **f**, *and a morphism* $\theta: Y_l \to X_i$, *such that the following diagram commutes.*

$$\begin{array}{ccc} X_k & \xrightarrow{h} & Y_l \\ \downarrow & \overset{\theta}{\swarrow} & \downarrow \\ X_i & \xrightarrow{g} & Y_j \end{array}$$

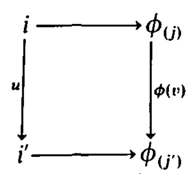
Proof. Let $\Delta(\mathbf{f})$, ϕ and ψ be as described above. By (A.2), **f** is an isomorphism if and only if $\hat{f}$ is. But $\tilde{f}$ is a uniformly indexed morphism, and the proposition for such morphisms is well known (e.g. [**11**, Proposition 1.8]; [**6**, Lemma 5.5.4]). This latter proposition applied to $\tilde{f}$ translates into the given statement.

Now let $\phi: J \to I$ be any functor (not necessarily cofinal) with $I, J \in \mathit{Cofilt}$, and let $\mathbf{X}$ be a pro-object indexed by I. We may still construct a canonical morphism $\mu_\phi: \mathbf{X} \to \phi^*\mathbf{X}$, to be the image of $1_{\mathbf{X}}$ under the canonical map

$$\text{pro-}\mathscr{C}(\mathbf{X};\mathbf{X}) = \lim_{i' \in I} \operatorname{colim}_{i \in I} \mathscr{C}(X_i; X_{i'}) \to \lim_{j \in J} \operatorname{colim}_{i \in I} \mathscr{C}(X_i; X_{\phi(j)}) = \text{pro-}\mathscr{C}(\mathbf{X}; \phi^*\mathbf{X}),$$

(though if ϕ is not cofinal, this map need not be an isomorphism). We wish to find conditions under which μ_ϕ is an isomorphism.

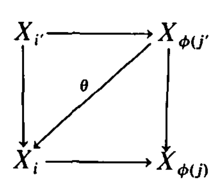
First observe that, if $i \to \phi(j)$ is a morphism in I, then the induced map $X_i \to X_{\phi(j)}$ represents μ_ϕ. Now define a category Δ_ϕ to have as objects the morphisms $i \to \phi_{(j)}$ in I, and as morphisms the commutative squares



Define a functor $\rho: \Delta_\phi \to \Delta(\mu_\phi)$ by $\rho(i \to \phi(j)) = (X_i \to X_{\phi(j)})$ on objects, and similarly on morphisms. We then have the following lemma, the proof of which is routine and left to the reader.

Lemma A.4. *Δ_ϕ is cofiltering and ρ is cofinal.*

Proposition A.5. *Let $\phi: J \to I$ be any functor with I and $J \in \mathit{Cofilt}$, and let $\mathbf{X}$ be a pro-object indexed by I. Then the canonical morphism $\mu_\phi: \mathbf{X} \to \phi^*\mathbf{X}$, is an isomorphism in pro-$\mathscr{C}$ if and only if, for each $i \to \phi(j)$ in I, there are morphisms $i' \to i$ in I, $j' \to j$ in J, $i' \to \phi(j')$ in I, and $\theta: X_{\phi(j')} \to X_i$ in $\mathscr{C}$, such that the following diagram commutes*



Proof. This follows immediately from Proposition A.3 and Lemma A.4.

Remark. If ϕ is cofinal, then the morphism θ in the above diagram can be chosen to be induced from a morphism $\phi(j') \to i$ in I.

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