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ON COMPLETE INTERSECTIONS AND THEIR HILBERT FUNCTIONS

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ABSTRACT. We deal here with the existence of *half-way nonzero divisors* for complete intersections and with related properties of their Hilbert functions.

In this note we discuss the Hilbert function $H = H(d_1, \ldots, d_n)$ $(1 \leq d_1 \leq d_2 \leq \cdots \leq d_n)$ of a complete intersection $k[X_1, \ldots, X_n]/(F_1, \ldots, F_n)$, where k is a field, X_i are indeterminates of degree 1, $\deg(F_i) = d_i$, and $(F_1, \ldots, F_n)$ is a regular sequence. As indicated by the notation, H depends only on $(d_1, d_2, \ldots, d_n)$. We first study H by considering the particular example $A = k[X_1, \ldots, X_n]/(X_1^{d_1}, \ldots, X_n^{d_n})$. Our main conclusions are that H is strictly increasing, indeed is a differentiable O-sequence, up to a certain integer u (which we determine explicitly), then is constant for awhile, and by symmetry decreases to 0. These results are trivial for $n = 1$, well known and easy for $n = 2$, but appear to be much more difficult if $n > 2$.

Richard Stanley has indicated the following argument: suppose that $k = \mathbb{C}$. Then $A \cong H^*(\mathbb{P}_{\mathbb{C}}^{d_1} \times \cdots \times \mathbb{P}_{\mathbb{C}}^{d_n}, \mathbb{C})$ (after dividing all degrees on the right by 2). The class of the hyperplane section (for some embedding of $\mathbb{P}_{\mathbb{C}}^{d_1} \times \cdots \times \mathbb{P}_{\mathbb{C}}^{d_n}$ in $\mathbb{P}_{\mathbb{C}}^{N}$) can be taken to be (the canonical image in A of) $S = X_1 + \cdots + X_n$. By the hard Lefschetz Theorem (see [S1]), multiplication by S^{t-2i} gives an isomorphism from A_i to A_{t-i}. (t is the largest integer such that $H(t) \neq 0$). Our results on H described above (except for the explicit determination of u) follow, in characteristic 0, from this property of S. However the hard Lefschetz Theorem seems to be rather heavy machinery for such seemingly elementary results. We call elements with the same property as S *strongly faithful*. In Theorem 5 we give an elementary ring theoretic proof that S is strongly faithful in A if the characteristic of k is 0. (Richard Stanley informs us that he has obtained an elementary combinatorial proof, based on [PSS]. See, for example, the discussion at the end of [S1, Section 2]). We then discuss the question of whether every complete intersection in characteristic 0 has a linear strongly faithful element, present some partial results and conclude by stating some conjectures.

Throughout this paper k is a field, $1 \leq d_1 \leq d_2 \leq \cdots \leq d_n$ are integers and $(F_1, \ldots, F_n)$ is a sequence of homogeneous polynomials in $k[X_1, \ldots, X_n]$ of degrees $d_1, \ldots, d_n$ respectively. By an algebra we mean a standard graded k-algebra as defined

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in [S] (that is, a k-algebra of the form $k[X_1, \dots, X_n]/I$ where the X_i have degree one and I is a homogeneous ideal). We set $t = \sum_{i=1}^{n}(d_i - 1)$. We denote elements in $k[X] = k[X_1, \dots, X_n]$ by capital letters and their canonical images in A by the corresponding lower case letters.

We now give an elementary proof of the above description of H, working in the algebra A. Adopting the convention that $H(i) = 0$ for $i < 0$, we have $H(i) = H(t-i)$ for all integers i. In particular, $H(t) = 1$ and $H(i) = 0$ for $i > t$. (For a given r, if $M = X_1^{a_1} \cdots X_n^{a_n}$ with $0 \le a_i \le d_i - 1$ and $\sum_{i=1}^{n} a_i = r$, then $M \leftrightarrow (X_1^{d_1-1} \cdots X_n^{d_n-1})/M$ gives a bijection between bases of A_r and A_{t-r}). Thus H is symmetric about $t/2$.

Let B any graded k-algebra and consider $R = B[X]/(X^m)$, where X has degree 1. Let $(b_i)_{i \ge 0}$ be the Hilbert function of B ($b_i = 0$ for $i < 0$, by the usual convention). Then it is clear that $H_R(i) = b_i + \cdots + b_{i-m+1}$ (m terms). We can visualize H_R in the following manner: graph H_B and imagine a caterpillar with m segments climbing this *hill* from left to right. For given i, the j^{th} segment of the caterpillar is represented by a dot at the point $\big(i-j+1, H_A(i-j+1)\big)$, the segment $j = 1$ being the head, which is at x coordinate i. Then $H_R(i)$ is the sum of the heights of the segments. For example let $B = k[X_1, X_2]/(X_1^3, X_2^3)$, and $A = k[X_1, X_2, X_3]/(X_1^3, X_2^3, X_3^4)$, thus $A = B[X]/(X^4)$, where $X = X_3$. Then H_B is the sequence 1 2 3 2 1 0 0 . . . and the caterpillar has 4 segments. We read in the graph the corresponding heights and sum them up: $H_A(3) = 2 + 3 + 2 + 1 = 8$.

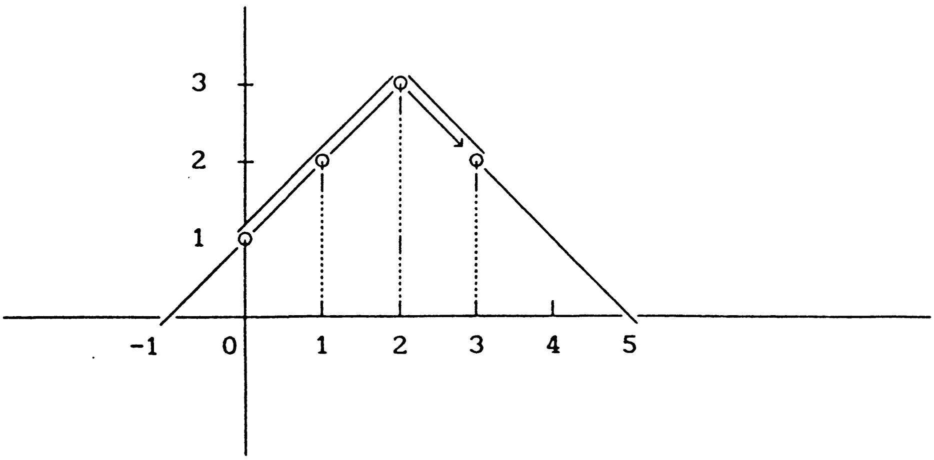

Figure 1

We use the caterpillar image in our proofs, their formalization being obvious.

For a, b in $\mathbb{R}$, let $[a, b] = \{i \in \mathbb{Z} \mid a \le i \le b\}$. We say that a sequence H is *increasing* (respectively *strictly increasing*) in the interval $[a, b]$ if $i < j$ implies $H(i) \le H(j)$ (respectively $H(i) < H(j)$) for all i, j in $[a, b]$. The words *decreasing* and *strictly decreasing* will be used in a similar manner. Now we prove the following.

THEOREM 1. *Let $A = k[X_1, \ldots, X_n]/(X_1^{d_1}, \ldots, X_n^{d_n})$, where k is a field and $1 \le d_1 \le d_2 \cdots \le d_n$. Let $t := \sum_{i=1}^{n}(d_i - 1)$.*

Suppose $d_n \le (t+1)/2$. If t is even, then H_A is strictly increasing in the interval $[0, t/2]$ and strictly decreasing in the interval $[t/2, t+1]$. If t is odd, then H_A is strictly increasing in the interval $[0, (t-1)/2]$, constant in the interval $[(t-1)/2), (t+1)/2]$ and strictly decreasing in the interval $[(t+1)/2, t+1]$.

Suppose $d_n > (t+1)/2$. Then H_A is strictly increasing up to a flat top which begins at $t' := \sum_{i=1}^{n-1}(d_i - 1)$ and ends at $d_n - 1$; afterwards it is strictly decreasing to 0.

PROOF. The result is clear if $n = 1$, since $H_A(i) = 1$ for $0 \le i \le d_1 - 1 = t$ and 0 otherwise. We proceed by induction on n, applying the caterpillar argument with $B = k[X_1, \ldots, X_{n-1}]/(X_1^{d_1}, \ldots, X_{n-1}^{d_{n-1}})$, $R = A$ and $m = d_n$. By induction, H_B strictly increases to a flat top of length at most d_{n-1} and so at most d_n (by the length of the flat top, we mean the number of integers where H_B attains its maximum). Assume first that $d_n \le t' = \sum_{i=1}^{n-1}(d_i - 1)$ (equivalently $d_n \le (t+1)/2$). The caterpillar cannot fit inside the flat top of H_B, so H_A increases strictly until the caterpillar is symmetrically situated about the peak of H_B, and the flat top is of length 1 or 2, according to whether t is even or odd respectively. If $d_n > t'$ (that is $d_n > (t+1)/2$), the flat top of H_A occurs when the caterpillar completely covers the graph of H_B between 0 and t' and so the flat top begins at t' and ends at $d_n - 1$. This completes the proof of the theorem. ∎

If a is a real number, then $[a]$ denotes the largest integer $\le a$.

By taking into account the symmetry of H_A and by direct computation, we obtain the following reformulation of Theorem 1.

COROLLARY 2. *Let A, t, t' be as in Theorem 1. Let $u := \min(t', [t/2])$, $v := \max\big(d_n - 1, [(t+1)/2]\big)$. Then H_A is strictly increasing in the interval $[0, u]$, constant in the interval $[u, v]$ and strictly decreasing in $[v, t+1]$. We have: $v = t - u$. The flat top of H_A is of length $\max(d_n - t', 1)$ for t even and $\max(d_n - t', 2)$ for t odd.* ∎

We say that a O-sequence H is differentiable on $[0, a]$ if the sequence ∇H defined by $\nabla H(i) = H(i) - H(i-1)$ is an O-sequence on $[0, a]$ (that is, the defining condition 2.2 (iii) of [S, page 61] is satisfied for $n \in [0, a-1]$). (Recall that $H(i) = 0$ for $i < 0$).

LEMMA 3. *Let H be a differentiable O-sequence on $[0, a]$. Let $d > 0$ be a positive integer. Let G be defined by $G(i) = H(i) - H(i-d)$. Then G is an O-sequence on $[0, a]$.*

PROOF. We can assume that a is an integer. Let H' be defined by $H'(i) = H(i)$ for $i \le a$ and $H'(i) = H(a)$ for $i \ge a$. Then H' is a differentiable O-sequence, so by [GMR] there is a reduced graded k-algebra D with Hilbert function H, where k is an infinite field. Thus, D has a non-zero-divisor f of degree 1. Then on $[0, a]$, G is the Hilbert function of D/f^d, completing the proof of the lemma. ∎

THEOREM 4. *Let the notation be as in Theorem 1 and Corollary 2. Then H_A is differentiable in the interval $[0, v]$.*

PROOF. By [G, (141.(9e))], we have for all i, $\nabla H_A(i) = H_B(i) - H_B(i - d_n)$ (this can be seen easily in the caterpillar image. As the argument increases from $i - 1$ to i, H_B gains $H_B(i)$ at the head and loses $H_B(i - d_n)$ at the tail).

We assume by induction on n that H_B is differentiable on $[0, t'/2]$. H_A is then differentiable on $[0, t'/2]$ by Lemma 3. ∇H_A is decreasing in $[t'/2, t-u]$. Indeed, in this interval $H_B(i)$ is decreasing and $H_B(i - d_n)$ is increasing so $\nabla H_A(i) = H_B(i) - H_B(i - d_n)$ is decreasing. (In the caterpillar image, the head is descending and the tail is still climbing). Thus H_A is differentiable in $[0, t - u] = [0, v]$. ■

For $i \leq t/2$, we have $\nabla H_A(i) = 0 \iff H_B(i) = H_B(i - d_n) = 0$ (indeed, if $\nabla H_A(i) = 0$ and $i \leq t/2$, then $u + 1 \leq i \leq t/2$, so $t' < i$ and $H_B(i) = 0$).

We have proved that $H = H_A$ is differentiable in its interval of increase. It is not clear if a similar result holds for differentiability of second order of H, i.e. is H twice differentiable in the interval of increase of ∇H?

It follows from the formula $\nabla H_A(i) = H_B(i) - H_B(i - d_n)$ that $\nabla H_A(i) = H_B(i)$ for $i < d_n$. Hence, H_A is twice differentiable in $[0, \min(d_n, v')]$, where $[0, v']$ is the interval of differentiability of H_B.

We now turn to the proof of the Stanley observation.

THEOREM 5. *Let k be a field of zero characteristic, $A = k[X_1, \ldots, X_n]/(X_1^{d_1}, \ldots, X_n^{d_n})$ $= k[x_1, \ldots, x_n]$ for $1 \leq d_1 \leq \cdots \leq d_n$. Let $m \geq 1$ and let f be a nonzero homogeneous element in A such that $(x_1 + \cdots + x_n)^m f = 0$. Then,* $\deg f \geq (t - m + 1)/2$, *where* $t = \sum_{i=1}^{n}(d_i - 1)$.

PROOF. By induction on $\deg f$. First let $\deg f = 0$, so $(x_1 + \cdots + x_n)^m = 0$. Assume that $\deg f < (t - m + 1)/2$, that is $t \geq m$. As $t = \sum_{i=1}^{n}(d_i - 1)$ and $\operatorname{char} k = 0$, in the expansion of $(X_1 + \cdots + X_n)^m$ there occurs a monomial $X_1^{i_1} \cdots X_n^{i_n}$ with $i_j \leq d_j - 1$ for all j, thus $(x_1 + \cdots + x_n)^m \neq 0$, a contradiction. It follows that $\deg f \geq (t - m + 1)/2$.

Let $\deg f > 0$. Set $B = k[X_1, \ldots, X_{n-1}]/(X_1^{d_1}, \ldots, X_{n-1}^{d_{n-1}})$, $X = X_n$, $R = B[X]$ and $d = d_n$, so $A = R/RX^d$. Let $S = x_1 + \cdots + x_{n-1} + X$ and let F be a homogeneous element of R whose canonical image in A is f. (Notice that $R = B[X]$ has a natural grading which is determined by the grading of B, X being homogeneous of degree 1). We have $S^m F = GX^d$, where $G \in R$. Differentiate with respect to X to obtain: $(S^m F)' = mS^{m-1}F + S^m F' = G'X^d + dGX^{d-1}$. Hence, $S^{m-1}(mF + SF') \equiv 0 \bmod (X^{d-1})$. Multiply by S to obtain that $S^{m+1}F' \equiv 0 \bmod (X^{d-1})$. As $\operatorname{char} k = 0$, by the inductive assumption on the degree, we obtain: $\deg f = \deg F = 1 + \deg F' \geq 1 + \big((t-1) - (m+1) + 1\big)/2 = (t - m + 1)/2$. ■

COROLLARY 6. *Let k be a field of zero characteristic, $A = k[X_1, \ldots, X_n]/(X_1^{d_1}, \ldots, X_n^{d_n}) = k[x_1, \ldots, x_n]$ for $1 \leq d_1 \leq \cdots \leq d_n$. Let f be a nonzero homogeneous element in A such that $(x_1 + \cdots + x_n)f = 0$. Then,* $\deg f \geq \max(t/2, d_n - 1)$, *where* $t = \sum_{i=1}^{n}(d_i - 1)$.

PROOF. By Theorem 5 (for $m = 1$), we have: $\deg f \geq t/2$. We have as in the proof of Theorem 5: $A = B[X_n]/(X_n^{d_n})$, where $B = k[X_1, \ldots, X_{n-1}]/(X_1^{d_1}, \ldots, X_{n-1}^{d_{n-1}})$. Let F be a homogeneous element in $B[X_n]$ which has f as its canonical image. Then,

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$(X_1 + \cdots + X_n)F = GX_n^{d_n}$ for some $G \in B[X_n]$, so $\deg F + 1 \geq d_n$. It follows that $\deg f \geq d_n - 1$. ∎

The previous corollary implies part of the results presented above, namely that H_A is differentiable in the interval $[0, u]$ (see Theorem 4).

Obviously, we may replace the element $(x_1 + \cdots + x_n)$ in Corollary 6 by any element of the form $c_1x_1 + \cdots + c_nx_n$, where $c_1, \ldots, c_n$ are nonzero scalars.

As remarked by Ed Davis (SUNY at Albany), Corollary 6 is false in the case k is of finite characteristic. Indeed, let k be of finite characteristic p, let d_n be a power of p, so $f^{d_n} = 0$ for any homogeneous element f of positive degree in A. We can certainly have $d_n \leq t/2$, so the annihilator of f contains the element $f^{d_n - 1}$, which is of degree $< t/2$ (if $\deg f = 1$).

Motivated by the previous presentation, especially by Theorem 5, we introduce the following

DEFINITION. Let A be a graded k-algebra. Let s be a nonzero homogeneous element of degree d. The element s is called *faithful* if for all $i \geq 0$, the map $A_i \longrightarrow A_{i+d}$ induced by multiplication by s is an injection or a surjection. The element s is *strongly faithful* if s^i is faithful for all $i \geq 0$.

By definition a strongly faithful element is faithful.

In case $(F_1, \ldots, F_n)$ is a complete intersection in $k[X_1, \ldots, X_n]$ and F is a nonzero homogeneous polynomial, we say that F is faithful (strongly faithful) with respect to the given complete intersection if its canonical image in $k[X_1, \ldots, X_n]/(F_1, \ldots, F_n)$ is faithful (strongly faithful).

Let A be a 0-dimensional (graded standard) Gorenstein k-algebra. Let t be the maximal integer for which $A_t \neq 0$, so that $H_A(i) = H_A(t-i)$ for all i. Assume that H_A is increasing on $[0, t/2]$ (as is the case if A is a complete intersection). Let s be a nonzero homogeneous element of degree d in A. Using the perfect pairing $A_i \times A_{t-i} \longrightarrow A_t \cong k$ induced by multiplication, we see that multiplication by s induces an injection $A_i \longrightarrow A_{i+d}$ if and only if it induces a surjection $A_{t-i-d} \longrightarrow A_{t-i}$. Since $H_A(i) \leq H_A(i+d)$ if and only if $i \leq (t-d)/2$, we have that s is faithful if and only if it induces injections $A_i \longrightarrow A_{i+d}$ for $i \leq (t-d)/2$, equivalently it induces surjections $A_i \longrightarrow A_{i+d}$ for $i \geq (t-d)/2$.

For x in A, multiplication by x will be denoted by T_x.

For a given positive integer m, we denote by $\alpha = \alpha(m)$ the greatest integer which is strictly less that $m/2$ and by $\beta = \beta(m)$ the least integer which is strictly greater than $m/2$. Thus $\alpha + \beta = m$. Moreover $\beta - \alpha$ equals 1 if m is odd and equals 2 if m is even. Using the previous remarks, we obtain

PROPOSITION 7. *Let A be a 0-dimensional Gorenstein algebra such that H_A is increasing on $[0, t/2]$. Let s be a nonzero homogeneous element of degree d in A. Set $\alpha = \alpha(t + d - 1)$ and $\beta = \beta(t + d - 1)$. Let B be the graded algebra A/As. The following conditions are equivalent:*

(0) The element s is faithful in A.

(1) $B_\beta = 0$.

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(2) $B_i = 0$ *for all* $i \geq (t+d)/2$.
(3) *The map* $T_s|A_{\beta-d}$ *is surjective.*
(4) *The map* $T_s|A_i$ *is surjective for all* $i \geq (t-d)/2$.
(5) *The map* $T_s|A_{\alpha-d+1}$ *is injective.*
(6) *The map* $T_s|A_i$ *is injective for all* $i \leq (t-d)/2$.
(7) *If* a *is a nonzero homogeneous element in* A *such that* $sa = 0$, *then* $\deg a > (t-d)/2$ *(equivalently* $\deg a \geq (t-d+1)/2$*).*

PROOF. The equivalences (0) $\iff$ (4) $\iff$ (6) and (3) $\iff$ (5) were explained above. We have (1) $\iff$ (2) because B is a standard k-algebra. Clearly (2) $\iff$ (4), (1) $\iff$ (3) and (6) $\iff$ (7). ■

In particular, we conclude from the previous proposition that if $(F_1, \ldots, F_n)$ is a complete intersection in $k[X_1, \ldots, X_n]$ and F_{n+1} is a nonzero homogeneous polynomial, then F_{n+1} is faithful with respect to this complete intersection if and only if the algebra $k[X_1, \ldots, X_n]/(F_1, \ldots, F_{n+1})$ is zero in degree $\beta = \beta\big(t(d_1, \ldots, d_{n+1})\big)$. Note that this condition is symmetric with respect to the polynomials $F_1, \ldots, F_{n+1}$.

From Proposition 7 we obtain

PROPOSITION 8. *Let* A *be a 0-dimension Gorenstein algebra such that* H_A *is increasing on* $[0, t/2]$. *Let* s *be a nonzero element in* A_1. *Set* $\alpha = \alpha(t)$ *and* $\beta = \beta(t)$. *Let* B *be the graded algebra* A/As. *The following conditions are equivalent:*

(0) *The element* s *is faithful in* A.
(1) $B_\beta = 0$.
(2) $B_i = 0$ *for all* $i > t/2$.
(3) *The map* $T_s|A_{\beta-1}$ *is surjective.*
(4) *The map* $T_s|A_{i-1}$ *is surjective for all* $i > t/2$.
(5) *The map* $T_s|A_\alpha$ *is injective.*
(6) *The map* $T_s|A_i$ *is injective for all* $0 \leq i < t/2$.
(7) *If* a *is a nonzero homogeneous element in* A *such that for some* $m \geq 1$ *we have* $s^m a = 0$, *then* $\deg a > \big((t+1)/2\big) - m$. ■

Similarly we have

PROPOSITION 9. *Let* A *be a 0-dimensional Gorenstein algebra such that* H_A *is increasing on* $[0, t/2]$. *Let* s *be a nonzero element in* A_1. *The following conditions are equivalent:*

(0) *The element* s *is strongly faithful in* A.
(1) *The map* $T_{s^{t-2i}} : A_i \longrightarrow A_{t-i}$ *is bijective for all* $0 \leq i < t/2$.
(2) *The map* $T_{s^m} : A_i \longrightarrow A_{m+i}$ *is surjective for all* $m + 2i \geq t$.
(3) *The map* $T_{s^m} : A_i \longrightarrow A_{m+i}$ *is injective for all* $m + 2i \leq t$.
(4) *For* $1 \leq m \leq t$, *if* $C^{(m)}$ *is the graded algebra* A/As^m, *then* $C_i^{(m)} = 0$ *for* $i \geq (t+m)/2$.

(5) *If a is a nonzero homogeneous element in A such that for some $m \geq 1$ we have $s^m a = 0$, then* $\deg a > (t-m)/2$ *(equivalently* $\deg a \geq (t-m+1)/2$*).* ∎

For general 0-dimensional Gorenstein algebras linear faithful elements do not necessarily exist even in zero characteristic [S, Example 4.3]: indeed, using the notations of this example, for a given $s \geq 4$, if there exists in R a linear faithful element $v = (w, \phi)$ then multiplication by v induces a surjection $R_1 \longrightarrow R_2$, so it follows from the definitions that multiplication by w in S induces a surjection $S_1 \longrightarrow S_2$, which is impossible since $\dim_k S_1 < \dim_k S_2$. In case of finite characteristic, we have already seen that linear faithful elements do not necessarily exist even for complete intersections of the form $(X_1^{d_1}, \ldots, X_n^{d_n})$. On the other hand, if the characteristic is 0, Theorem 5 states that for complete intersections of the type $(X_1^{d_1}, \ldots, X_n^{d_n})$, any linear element $c_1x_1 + \ldots c_nx_n$ with all the coefficients c_i nonzero, is strongly faithful.

Obviously, if s is a strongly faithful element of degree 1 in a 0-dimensional Gorenstein algebra A, then $s^t \neq 0$. Nevertheless, for a faithful element of degree 1, this extra condition does not imply that the element is strongly faithful. Indeed, for any field k and $3 \leq m \leq n$ consider the complete intersection $k[x,y] = k[X,Y]/\left(X^m, (X+Y)^{n-m+2}Y^{m-2}\right)$. Let $s = x+y$. We have $k[x,y]/(s) \cong k[X]/(X^m)$, thus this algebra is zero in degree m. But $t = m+n-2$ and $m \leq n$, so $m \leq \beta(t)$ and s is faithful by Proposition 8, (1) $\Longleftrightarrow$ (0). If $s^t = 0$, then for $S = X + Y$, there are homogeneous polynomials f and g such that $S^{n+m-2} = fX^m + gS^{n-m+2}Y^{m-2}$. It follows that S^{n-m+2} divides f, so S^{2m-4} belongs to the ideal generated by X^m and Y^{m-2}. Now $t(m, m-2) = 2m-4$, so this contradicts the fact that S is strongly faithful with respect to the complete intersection (X^m, Y^{m-2}). We conclude that $s^t \neq 0$. The element s is not strongly faithful because $s^{n-m+2}y^{m-2} = 0$ and $(n-m+2) + 2(m-2) = t$ (Proposition 9, (0) $\Longleftrightarrow$ (5)).

We now express the property of $(F_1, \ldots, F_n)$ being a complete intersection in terms of the coefficients of the polynomials $F_1, \ldots, F_n$. Let $1 \leq d_1 \leq \cdots \leq d_n$ be given integers. Let $(F_1, \ldots, F_n)$ be homogeneous polynomials in $k[X_1, \ldots, X_n]$ of degrees $d_1, \ldots, d_n$ respectively. For a given j, let $\mathcal{M}_j$ be the set of all monomials of degree j. For any i, let $F_i = \sum_{M \in \mathcal{M}_{d_i}} c_{i,M}M$ with $c_{i,M}$ in k. Let $(M_1, \ldots, M_r)$ be all monomials of degree $t+1$ with $r = \binom{n+t}{t+1}$. For any sequence of r polynomials of the form MF_i with M a monomial of degree $t+1-d_i$ consider the matrix of the coefficients of these polynomials with respect to $(M_1, \ldots, M_r)$. Let $\mathcal{A}$ be the set of all such matrices, so $\mathcal{A}$ is finite. Clearly the ideal $I = (F_1, \ldots, F_n)$ is a complete intersection if and only if $\dim I_{t+1} = r$, equivalently $\det C \neq 0$ for some matrix C in $\mathcal{A}$. Thus the set of complete intersections is open with respect to the Zariski topology in the affine space defined by the coefficients of the polynomials F_i.

Generally, we say that a property of a sequence of polynomials $(F_1, \ldots, F_n)$ of degrees $d_1, \ldots, d_n$ respectively, is *generic* over k if it holds for an open subset of the affine space defined by the coefficients which contains a rational point over k. Thus generically, a sequence of polynomials $(F_1, \ldots, F_n)$ of given degrees is a complete intersection (for arbitrary characteristic).

Let $s = e_1x_1 + \cdots + e_nx_n$ be a linear element in $A = k[x_1, \ldots, x_n] = k[X_1, \ldots, X_n]/(F_1, \ldots, F_n)$ with e_i in k. We express the property of s being faithful in A in terms of the coefficients e_j and $c_{i,M}$. Assume that $(F_1, \ldots, F_n)$ is a complete intersection. By Proposition 8(1), the element s is faithful if and only if the algebra $k[X_1, \ldots, X_n]/(F_1, \ldots, F_n, s)$ is zero in degree $\beta = \beta(t)$. Equivalently, this means that for the ideal $J = (F_1, \ldots, F_n, s)$ we have $J_\beta = k[X_1, \ldots, X_n]_\beta$. Similarly to the argument proving the genericity of complete intersection, we obtain that the set of complete intersections with linear faithful elements is an open subset in the Zariski topology (of the affine space defined by the coefficients e_j and $c_{i,M}$).

We see that for any field k and given integers $(d_1, \ldots, d_n)$, a complete intersection of type $(d_1, \ldots, d_n)$ has generically a linear faithful element if and only if there exists a complete intersection over k with a linear faithful element.

For a given complete intersection $(F_1, \ldots, F_n)$ there is a finite set of polynomials $\mathcal{P}$ in $k[X_1, \ldots, X_n]$ such that an element $e_1x_1 + \cdots + e_nx_n$is faithful if and only if $P(e_1, \ldots, e_n) \neq 0$ for some polynomial P in $\mathcal{P}$. Hence if k is infinite this property holds if and only if for algebraically independent elements $u_1, \ldots, u_n$ over k, $P(u_1, \ldots, u_n) \neq 0$ for some polynomial P in $\mathcal{P}$. This means that the existence of linear faithful elements is equivalent to the element $u_1x_1 + \cdots + u_nx_n$ being a faithful element in the algebra $A \otimes_k k(u_1, \ldots, u_n)$.

For a given algebraically closed field k and integers $1 \leq d_1 \leq \cdots \leq d_n$, let $k[\boldsymbol{C}, \boldsymbol{E}] = k[C_{i,M}, E_1, \ldots, E_n]$ be a polynomial ring over k, the indeterminates corresponding to the coefficients $c_{i,M}$ and $e_1, \ldots, e_n$ in the previous notation. We have seen that there is a determinantal ideal J_1 in $k[\boldsymbol{C}]$ which defines the complement of the set of complete intersections of type $(d_1, \ldots, d_n)$. Also there is a determinantal ideal J_2 in $k[\boldsymbol{C}, \boldsymbol{E}]$ which determines among the set of complete intersection of type $(d_1, \ldots, d_n)$ those with no linear faithful elements. Thus any complete intersection of type $(d_1, \ldots, d_n)$ over k has a linear faithful element if and only if $J_1 \subseteq \sqrt{J_2}$. Moreover this criterion is algorithmic. (The ideals J_1 and J_2 are in fact defined over the prime field).

Similar results hold for linear strongly faithful elements.

In case $\operatorname{char} k = 0$, by Theorem 5, the set of complete intersections which have linear strongly faithful elements contains a rational point over k. Hence we sum up part of the previous discussion as follows:

THEOREM 10. *If* $\operatorname{char} k = 0$, *then a sequence of n polynomials in $k[X_1, \ldots, X_n]$ of given degrees is generically a complete intersection with linear strongly faithful elements.*

If k is infinite the existence of a linear (strongly) faithful element for a given complete intersection is equivalent to the element $u_1x_1 + \ldots u_nx_n$ being (strongly) faithful with respect to the algebra $A \otimes_k k(u_1, \ldots, u_n)$, where $u_1, \ldots, u_n$ are algebraically independent elements over k. ∎

In case $n = 1$, i.e. $A = k[X]/X^{t+1}$, every nonzero linear element is strongly faithful. For $n = 2$ we have

PROPOSITION 11. *Let $A = k[X_1, X_2]/(F_1, F_2)$ be a complete intersection. Then, if $d_1 = d_2$, every nonzero linear element of A is faithful. If $d_1 < d_2$, then a linear element s is faithful $\iff S \not| F_1$, in $k[X_1, X_2]$.*

PROOF. First assume that $d_1 = d_2$, so $t/2 = d_1 - 1$. We obtain from degree considerations that every nonzero linear element of A is faithful.

Now assume that $d_1 < d_2$. Clearly s is not faithful if S divides F_1. Assume that S does not divide F_1 and let $SG \in (F_1, F_2)$ for some homogeneous polynomial G of degree $< t/2$, thus $\deg G < d_2 - 1$. It follows from degree considerations that $SG \in (F_1)$, so F_1 divides G and s is faithful. ∎

We can partly generalize Proposition 11 as follows: for any complete intersection in $k[X_1, \ldots, X_n]$ such that $(t/2) + 1 < d_n$, a nonzero linear element is faithful if and only if it is not a zero-divisor $\operatorname{mod}(F_1, \ldots, F_{n-1})$. As a result of Proposition 11 we obtain

COROLLARY 12. *Let $A = k[X_1, X_2]/(F_1, F_2)$ be a complete intersection. There are no linear faithful elements in A if and only if $d_1 < d_2$, the field k is finite and $X_1^q X_2 - X_2^q X_1$ divides F_1, where q is the number of elements of k.* ∎

For the next theorem we need

LEMMA 13. *Let k be a field of zero characteristic and set $R = k[X, Y]$. For given m and n, let W be an m-dimensional subspace of R_{m+n}. Let u be a transcendental element over k. For any k-vector space V, denote $V \otimes_k k(u)$ by $V(u)$. We have*

$$R_{m+n}(u) = W(u) \oplus (X + uY)^m R_n(u).$$

PROOF. We consider the ring $T = k[u, X, Y]$ with the grading inherited from $k[X, Y]$, so $T_0 = k[u]$. Let $S = X + uY$. From dimension considerations it is enough to show that $S^m T_n \cap k[u]W = (0)$. Let G be a nonzero element in T_n such that $S^m G$ belongs to this intersection. Write $G = S^r G_0$ with G_0 not divisible by S in T. Consider $k[u, X, Y]$ as a polynomial ring in u over $k[X, Y]$. Differentiate $S^m G = S^{m+r} G_0$ with respect to u : $(S^m G)' = S^{m+r-1} G_1$, where $G_1 = (m + r)YG_0 + SG_0'$ is not divisible by S in T. Thus we obtain inductively polynomials G_i not divisible by S such that $(S^m G)^{(i)} = S^{m+r-i} G_i$ for $0 \le i \le m$. Now, the $m + 1$ forms $(S^m G)^{(i)}$ belong to $W(u)$ and they are linearly independent over $k(u)$: otherwise for some $0 \le j \le m$ and for $0 \le i \le j$ there are polynomials α_i in $k[u]$ such that $\alpha_j \neq 0$ and, $\sum_{i=0}^{j} \alpha_i (S^{m+r-i} G_i) = 0$. It follows that $S^{m+r-j+1}$ divides $\alpha_j S^{m+r-j} G_j$, so S divides G_j, a contradiction. The linear independence of these $m + 1$ forms contradicts the fact that $\dim_k W = m$. ∎

We obtain from the previous lemma by specializing the element u over k that there is an element c in k such that $R_{m+n} = W \oplus (X + cY)^m R_n$.

THEOREM 14. *If k is a field of zero characteristic, then any complete intersection in $k[X_1, X_2]$ has a linear strongly faithful element.*

PROOF. Denote X_1 by X and X_2 by Y. Let $I = (F, G)$ be a complete intersection in $R = k[X, Y]$ and set $t = \deg F + \deg G - 2$ as usual. For a given $i < (t/2)$ set $W = I_{t-i}$,

$m = \dim_k W$ and $n = t - i - m$. By the previous lemma and in the same notation, we have $R(u)_{t-i} = W(u) \oplus S^m R(u)_n$. Therefore $s^m: A_n \longrightarrow A_{t-i}$ is a surjection. (As usual $A = R/(F, G)$). Now, $m = \dim_k I_{t-i} \geq t - 2i$ because if H is the Hilbert function of the given complete intersection, we have $1 + t - i - \dim_k I_{t-i} = H(t-i) = H(i) \leq i + 1$. Thus $n = t - i - m \leq i$ so $s^{t-2i}: A_i \longrightarrow A_{t-i}$ is also a surjection, hence an isomorphism because $H_A(i) = H_A(t-i)$. Therefore condition (1) of Proposition 9 is satisfied and so S is a strongly faithful element over $k(u)$. By Theorem 10, the complete intersection (F, G) has a linear strongly faithful element over k. ∎

PROPOSITION 15. *Let k be an infinite field and $n \geq 1$. The existence of linear faithful elements for any complete intersection in $k[X_1, \ldots, X_{n+1}]$ implies the existence of linear strongly faithful elements for any complete intersection in $k[X_1, \ldots, X_n]$.*

PROOF. Let $(F_1, \ldots, F_n)$ be a complete intersection in $k[X_1, \ldots, X_n]$, and let $u_1, \ldots, u_{n+1}$ be algebraically independent elements over k. Set $K = k(u_1, \ldots, u_{n+1})$ and $S = u_1X_1 + \cdots + u_{n+1}X_{n+1}$. Let $m \geq 1$. By Theorem 10, S is faithful with respect to the algebra $K[X_1, \ldots, X_{n+1}]/(F_1, \ldots, F_n, X_{n+1}^m)$. Since the algebras

$$K[X_1, \ldots, X_{n+1}]/(F_1, \ldots, F_n, X_{n+1}^m, S) \text{ and}$$
$$K[X_1, \ldots, X_n]/\left(F_1, \ldots, F_n, (u_1X_1 + \cdots + u_nX_n)^m\right)$$

are isomorphic it follows from Propositions 8(2) and 9(4) that $u_1X_1 + \ldots u_nX_n$ is a strongly faithful element for $K[X_1, \ldots, X_n]/(F_1, \ldots, F_n)$. By Theorem 10, the k-algebra $k[X_1, \ldots, X_n]/(F_n, \ldots, F_n)$ has linear strongly faithful elements. ∎

PROPOSITION 16. *The existence of linear strongly faithful elements for generic complete intersections of any type in $k[X_1, \ldots, X_n]$ implies the existence linear faithful elements for generic complete intersections of any type in $k[X_1, \ldots, X_{n+1}]$.*

PROOF. If S is a linear strongly faithful element for a complete intersection $(F_1, \ldots, F_n)$ of type $(d_1, \ldots, d_n)$ in $k[X_1, \ldots, X_n]$, then for any integer d_{n+1}, $X_{n+1} + S$ is a faithful element for the complete intersection $(F_1, \ldots, F_n, X_{n+1}^{d_{n+1}})$ in $k[X_1, \ldots, X_{n+1}]$. Since this is a complete intersection of type $(d_1, \ldots, d_{n+1})$ with a faithful element, this implies that generic complete intersections of type $(d_1, \ldots, d_{n+1})$ have faithful linear elements. ∎

Proposition 16 is a new result just for the case of finite characteristic; for the case of zero characteristic we know already that generic complete intersections have linear strongly faithful elements. We conjecture that the same holds for the case of finite characteristic; to show this it is enough to produce an example of a complete intersection with a linear strongly faithful element for any given type over a field of order p for any prime p.

We conjecture that any complete intersection over a field of zero characteristic has linear faithful elements. By Proposition 15 it is equivalent to conjecture the existence of linear strongly faithful elements.

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