ON THE GENERALISED TODD GENUS OF FLAG BUNDLES

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(Received 14th February 1973)

1. Introduction

Let V be a complex algebraic variety. Given integers $a_1, \ldots, a_m$ such that

$$0 \leqq a_1 < a_2 < \ldots < a_m = n,$$

one defines a $(a_1, \ldots, a_m)$-flag as a nested system

$$S: \; S_{a_1} \subset S_{a_2} \subset \ldots \subset S_{a_m}, \; \dim_C S_{a_i} = a_i,$$

of subspaces of S_n, the n-dimensional complex projective space. The set of all such flags is called an incomplete flag-manifold in S_n, and is denoted by $W(a_1, \ldots, a_m)$. Also let E be a complex n-dimensional vector bundle over V. Then we denote by $E(a_1, \ldots, a_{m-1}, n; \; V)$ an associated fibre bundle of E with fibre $W(a_1-1, \ldots, a_{m-1}-1, n-1)$. $E(a_1, \ldots, a_{m-1}-1, n; V)$ is called an incomplete flag bundle of E over V (cf. (**2**), (**3**)). In Section 10.3 and Section 14.4 of (**1**), the generalised Todd genus $T_y(W(0, n))$ and $T_y(W(0, 1, \ldots, n))$ of the n-dimensional projective space $W(0, n)$ and the flag manifold $W(0, 1, \ldots, n)$ (or $F(n+1)$) were calculated. Here we compute $T_y(W(a_1, \ldots, a_m))$ and also $T_y(E(a_1, \ldots, a_{m-1}, \mathrm{n}; \; V))$.

Notation. We shall interpret the expression

$$\sum_{i_1=0}^{n} t^{i_1} \sum_{i_0=0}^{i_1} t^{i_0}$$

to mean

$$1 + t \, . \sum_{i_0=0}^{1} t^{i_0} + t^2 \, . \sum_{i_0=0}^{2} t^{i_0} + \ldots + t^n \, . \sum_{i_0=0}^{n} t^{i_0}.$$

By iteration, the expression

$$\sum_{i_m=0}^{n} t^{i_m} . \sum_{i_{m-1}=0}^{i_m} t^{i_{m-1}} \ldots \sum_{i_0=0}^{i_1} t^{i_0}$$

will be interpreted similarly. We shall denote this last expression by

$$\prod_{j=0}^{m} \sum_{i_j=0}^{i_{j+1}} t^{i_j}, \text{ where } i_{m+1} = n.$$

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2. Generalised Todd Genus

Lemma 2.1.

$$\frac{(1-t^{2(m+2)})(1-t^{2(m+3)})\dots(1-t^{2(n+1)})}{(1-t^2)(1-t^4)\dots(1-t^{2(n-m)})} \equiv \prod_{j=0}^{m} \sum_{i_j=0}^{i_{j+1}} t^{2i_j},$$

where $i_{m+1} = n-m$.

Proof. A simple inductive argument on m shows that the right-hand side enumerates all partitions $(i_0, \dots, i_m)$ such that

$$0 \leqq i_j \leqq n-m \quad (j=0, \dots, m).$$

But this is equal to the left-hand side from (7) and Section 26 of (6).

Corollary 2.2. *The Poincaré polynomial of a Grassmannian is given by*

$$P(W(m, n)\colon t) = \prod_{j=0}^{m} \sum_{i_j=0}^{i_{j+1}} t^{2i_j},$$

where $i_{m+1} = n-m$.

Proof. The proof follows immediately from the Lemma and from the formula of Hirsch (cf. (6)).

Theorem 2.3. *The generalised Todd genus of* $W(a_1, \dots, a_{m-1}, n)$ *is given by*

$$T_y(W(a_1, \dots, a_{m-1}, n)) = \prod_{k=1}^{m-1} \left[\prod_{j=0}^{a_k-a_{k-1}-1} \sum_{i_j=0}^{i_{j+1}} (-1)^{i_j} y^{i_j} \right],$$

where $a_0 = -1$, $i_{a_k - a_{k-1}} = n - a_k$.

Proof. We first prove the theorem in the case of a Grassmannian, $W(m, n)$. Consider the following flag bundles:

$$F(n+1) \xrightarrow[F(n-m)]{} W(0, 1, \dots, m, n) \xrightarrow[F(m+1)]{} W(m, n).$$

From Section 14 of (1), the T_y-genus behaves multiplicatively with respect to flag bundles and so

$$T_y(W(m, n)) = \frac{T_y(F(n+1))}{T_y(F(m+1))T_y(F(n-m))}$$
$$= \frac{(1-y+y^2-\dots+(-1)^{m+1}y^{m+1})\dots(1-y+y^2-\dots+(-1)^n y^n)}{(1-y)(1-y+y^2)\dots(1-y+y^2-\dots+(-1)^{n-m-1}y^{n-m-1})}.$$

Poincaré polynomials also behave multiplicatively with respect to flag bundles (cf. (4)). Thus

$$P(W(m, n)) = \frac{P(F(n+1))}{P(F(m+1)).P(F(n-m))}$$
$$= \frac{(1+y^2+\dots+y^{2(m+1)}).\dots.(1+y^2+\dots+y^{2n})}{(1+y^2).\dots.(1+y^2+\dots+y^{2(n-m)})}.$$

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But from Lemma 2.1,

$$P(W(m, n)) = \prod_{j=0}^{m} \sum_{i_j=0}^{i_{j+1}} t^{2i_j}, \quad \text{where } i_{m+1} = n-m.$$

Thus by comparing the various polynomials, we find that the generalised Todd genus of a Grassmannian is given by

$$T_y(W(m, n)) = \prod_{j=0}^{m} \sum_{i_j=0}^{i_{j+1}} (-1)^{i_j} y^{i_j}, \quad \text{where } i_{m+1} = n-m. \tag{2.4}$$

Now to find the generalised Todd genus of $W(a_1, \ldots, a_{m-1}, n)$, we consider the following sequence of flag bundles:

$$W(a_1, \ldots, a_{m-1}, n) \xleftarrow{F(a_1+1)} W(0, \ldots, a_1, a_2, \ldots, a_{m-1}, n) \xleftarrow{F(a_2-a_1)} \ldots \xleftarrow{F(n-a_{m-1})} F(n+1).$$

Hence

$$T_y(W(a_1, \ldots, a_{m-1}, n)) = \frac{T_y(F(n+1))}{T_y(F(a_1+1))T_y(F(a_2-a_1))\ldots T_y(F(n-a_{m-1}))}$$

$$= \frac{T_y(F(n+1))}{T_y(F(a_1+1)).T_y(F(n-a_1))} \cdots \frac{T_y(F(n-a_{n-2}))}{T_y(F(a_{m-1}-a_{m-2})).T_y(F(n-a_{m-1}))}$$

$$= T_y(W(a_1, n)).T_y(W(a_2-a_1-1, n-a_1-1)).$$

$$\ldots T_y(W(a_{m-1}-a_{m-2}-1, n-a_{m-2}-1)).$$

The theorem now follows from (2.4).

Corollary 2.5. *The Todd genus of all flag manifolds is equal to* 1, *i.e.*

$$T(W(a_1, \ldots, a_{m-1}, n)) = 1.$$

Proof. Put $y = 0$ in the theorem since for a variety V,

$$T(V) = T_0(V) \text{ (cf. Section 10 of (1))}.$$

Corollary 2.6. $T_y(E(a_1, \ldots, a_{m-1}, n;\ V)) = T_y(V).T_y(W(a_1-1, \ldots, n-1))$.

Proof. $E(a_1, \ldots, a_{m-1}, n;\ V) \longrightarrow V$ is a fibre bundle, fibre

$$W(a_1-1, \ldots, a_{m-1}-1, n-1).$$

From Section 14 of (**1**) and the theorem it follows that the T_y-genus behaves multiplicatively with respect to incomplete flag bundles and so the corollary follows.

Corollary 2.7. $T(E(a_1, \ldots, a_{m-1}, n;\ V)) = T(V)$.

Proof. Put $y = 0$ in Corollary 2.6 and use Corollary 2.5.

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