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Metric Compactifications and Coarse Structures

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Abstract. Let **TB** be the category of totally bounded, locally compact metric spaces with the C_0 coarse structures. We show that if X and Y are in **TB**, then X and Y are coarsely equivalent if and only if their Higson coronas are homeomorphic. In fact, the Higson corona functor gives an equivalence of categories $\mathbf{TB} \to \mathbf{K}$, where **K** is the category of compact metrizable spaces. We use this fact to show that the continuously controlled coarse structure on a locally compact space X induced by some metrizable compactification $\widetilde{X}$ is determined only by the topology of the remainder $\widetilde{X} \setminus X$.

1 Introduction

When studying "large-scale" or "asymptotic" structures of metric spaces, one is often led to consider a kind of "boundary at infinity" of them, for example, the boundary sphere $\partial_\infty \mathbb{H}^n = S^{n-1}$ of the Poincaré ball $\mathbb{H}^n$. This boundary sphere reflects the geometry of $\mathbb{H}^n$ in the sense that the isometries of $\mathbb{H}^n$ are in one-to-one correspondence with the Möbius transformations of S^{n-1}. In many situations we can associate a boundary at infinity with a metric space, and in the optimal case, the large-scale structure in question is recovered from the boundary. Results in this direction are pursued by several authors, including Paulin [9], Bonk–Schramm [2], Buyalo–Schroeder [4], and Jordi [8]. As an example, let X and Y be Gromov hyperbolic geodesic spaces and let $\partial_\infty X$ and $\partial_\infty Y$ be their boundaries at infinity. We can define a visual metric on each of these boundaries, which is an analogue of the angle metric on $\partial_\infty \mathbb{H}^n = S^{n-1}$ (see [3, Chapter III.H]). Then under some niceness condition (for example, it is satisfied by Cayley graphs of Gromov hyperbolic groups and their boundaries), the metric spaces X and Y are quasi-isometric if and only if $\partial_\infty X$ and $\partial_\infty Y$ are quasi-Möbius equivalent [4, 8].

In this paper, we prove another such correspondence in more topological settings. Let X be a locally compact, totally bounded metric space. Then our main result states that a large-scale structure called the C_0 *coarse structure* on X introduced by Wright [14] (see §2) is completely recovered from the topology of the boundary $\widetilde{X} \setminus X$, where $\widetilde{X}$ stands for the completion of X (Theorem 4.2). Before introducing our results in more details, we informally review the notion of coarse structure (see §2 for formal definitions). "Large-scale" properties of spaces, such as quasi-isometry invariant properties of finitely generated groups, can be described by coarse structures. A

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coarse structure on a set X is given by a collection of *controlled* subsets of $X \times X$ satisfying several axioms. When $E \subset X \times X$ is a fixed controlled subset, one thinks of x and y as "close uniformly" for all $(x, y) \in E$. Thus a typical coarse structure on a metric space X is the *bounded coarse structure*, where $E \subset X \times X$ is controlled if and only if there exists $C > 0$ such that $d(x, y) \leq C$ for all $(x, y) \in E$. In this structure, the phrase "close uniformly" above has its usual meaning. The C_0 coarse structure on a locally compact metric space, mentioned above, is another kind of coarse structure. Roughly, the phrase "close uniformly" in the C_0 structure actually means "becoming closer and closer as points approach infinity".

Given a suitable coarse structure on a locally compact Hausdorff space, we can define the *Higson compactification* hX of X (see §2), a compactification of X defined in terms of the ring of "slowly oscillating" functions called the Higson functions. The remainder $\nu X = hX \smallsetminus X$ is called the *Higson corona*, and νX can be regarded as a boundary of X. The corona νX is a coarse invariant, in the sense that "coarsely equivalent" coarse spaces have homeomorphic Higson coronas [10, Corollary 2.42]. Then it is natural to ask whether the converse holds: if νX and νY are homeomorphic, then are X and Y coarsely equivalent? As we mentioned earlier, an analogous statement is true for Gromov hyperbolic groups.

The paper of Cuchillo-Ibáñez, Dydak, Koyama, and Morón [5] gives an affirmative answer to this question about Higson coronas in some special case. They considered Z-sets (which are "thin" closed subsets in some sense) in the Hilbert cube and their complements, where each Z-set can be regarded as the Higson corona of the complement equipped with the C_0 structure. Their result then states that the category of Z-sets in the Hilbert cube (and the continuous maps between them) is isomorphic to the category of the C_0 coarse spaces formed by their complements.

In this paper, we extend the argument in [5] to general locally compact metric spaces equipped with the C_0 structure. Formally stated, our main result claims an equivalence of categories $\mathbf{TB} \to \mathbf{K}$, where $\mathbf{TB}$ is the category of totally bounded locally compact metric spaces and C_0 coarse maps modulo closeness, and $\mathbf{K}$ is the category of compact metrizable spaces and continuous maps (Theorem 4.2). This equivalence is realized by the Higson corona functor, which in this case reduces to the operation of taking the complement in the completion. As a consequence of the equivalence $\mathbf{TB} \simeq \mathbf{K}$, it follows that the C_0 coarse structure on $M \smallsetminus Z$, where Z is a nowhere dense closed set in a compact metric space M, is determined (up to coarse equivalence) only from the topological type of Z, regardless of the space M or how Z is embedded in M (Corollary 4.3).

A compactification $\widetilde{X}$ of a (locally compact Hausdorff) space X in general induces a natural coarse structure on X, called the *continuously controlled coarse structure* (see §2). Since this structure can be regarded as a C_0 coarse structure with the Higson compactification $\widetilde{X}$ (see Corollary 3.10 and Remark 4.1), we have that the continuously controlled structure on X is determined, up to coarse equivalence, by the topological type of the remainder $\widetilde{X} \smallsetminus X$ (Corollary 4.4).

2 Preliminaries on Coarse Structures and Higson Coronas

We refer the reader to Roe's monograph [10] as a basic reference for this section.

A *coarse structure* on a set X is defined as a collection $\mathcal{E}$ of subsets of $X \times X$, called *controlled sets,* satisfying the following five conditions:

(i) the diagonal $\Delta_X = \{(x,x) \mid x \in X\}$ belongs to $\mathcal{E}$,
(ii) if $E \in \mathcal{E}$ and $E' \subset E$ then $E' \in \mathcal{E}$,
(iii) if $E \in \mathcal{E}$ then its inverse $E^{-1} = \{(x,y) \in X \times X \mid (y,x) \in E\}$ belongs to $\mathcal{E}$,
(iv) if $E, F \in \mathcal{E}$ then the composition

$$E \circ F = \{(x,z) \in X \times X \mid \text{there exists } y \in X \text{ such that } (x,y) \in E \text{ and } (y,z) \in F\}$$

belongs to $\mathcal{E}$, and
(v) if $E, F \in \mathcal{E}$ then the union $E \cup F$ belongs to $\mathcal{E}$.

The pair $(X, \mathcal{E})$ (or briefly X) is then called a *coarse space*. A subset $B \subset X$ is called *bounded* in the coarse space X if $B \times B$ is controlled.

Let X and Y be coarse spaces. We can define a class of maps from X to Y that respect coarse structures, namely the coarse maps, as follows. A map $f: X \to Y$ is called *proper* if the inverse image $f^{-1}(B)$ is bounded for every bounded set B of Y. The map f is called *bornologous* if $(f \times f)(E) \subset Y \times Y$ is controlled for every controlled set $E \subset X \times X$. Then we say that $f: X \to Y$ is a *coarse map* if it is both proper and bornologous. A coarse map $f: X \to Y$ is called a *coarse equivalence* if there exists a coarse map $g: Y \to X$ such that both $g \circ f$ and $f \circ g$ are close to their respective identities. Here maps $h, k: S \to Z$ from a set S to a coarse space Z are called *close* if the set $\{(h(s), k(s)) \mid s \in S\}$ is controlled. Coarse spaces X and Y are then called *coarsely equivalent*.

A coarse structure on a paracompact Hausdorff space X is called *proper* (in which case we say that X is a *proper coarse space*) if (1) there is a controlled neighborhood of the diagonal Δ_X and (2) every bounded subset has compact closure. For a proper coarse space X, the converse statement of (2) is also true if X is *coarsely connected*, that is, each singleton $\{(x,y)\}$ is controlled (see [10, Proposition 2.23]). Notice also that a proper coarse space is necessarily locally compact.

As mentioned in the introduction, a standard example of a coarse structure is the *bounded coarse structure* on a metric space (X,d), where $E \subset X \times X$ is defined to be controlled if there exists $C > 0$ such that $d(x,y) \le C$ for every $(x,y) \in E$. In this structure, the bounded sets are exactly the bounded sets in the metric sense. The bounded coarse structure on X is proper if and only if X is *proper* as a metric space, that is, every closed bounded subset of X is compact. It is not difficult to show that two geodesic metric spaces with the bounded coarse structures are coarsely equivalent if and only if they are quasi-isometric.

For a locally compact metric space (X,d), we can define a coarse structure other than the bounded structure, called the C_0 *coarse structure*, which was introduced by Wright [14]. In the C_0 coarse structure, a subset E of $X \times X$ is defined to be controlled if for every $\varepsilon > 0$ we can find a compact set $K \subset X$ such that $d(x,y) < \varepsilon$ for every $(x,y) \in E \setminus K \times K$. The following proposition is proved for completeness.

Proposition 2.1 *Let (X,d) be a locally compact metric space. Then the above definition of the C_0 coarse structure indeed gives a coarse structure on X, where a subset is bounded if and only if it has compact closure. In case X is separable, this structure is proper.*

Proof Let (X,d) be a locally compact metric space. It is easy to verify conditions (i), (ii), (iii), and (v). To see (iv), take any controlled sets E, F and $\varepsilon > 0$. We prove that $E \circ F$ is also controlled. Since $E \cup F$ is controlled, we can choose a compact set K_0 of X such that $d(x,y) < \varepsilon/2$ whenever $(x,y) \in (E \cup F) \setminus K_0 \times K_0$. Since X is locally compact, there is an $\varepsilon' > 0$ with $\varepsilon' \le \varepsilon/2$ such that the closed ε'-neighborhood $\overline{N}(K_0, \varepsilon')$ of K_0 is compact. Then we can choose a compact set K of X containing $\overline{N}(K_0, \varepsilon')$ such that $d(x,y) < \varepsilon'$ whenever $(x,y) \in (E \cup F) \setminus K \times K$.

We claim that $d(x,y) < \varepsilon$ holds for every $(x,y) \in (E \circ F) \setminus K \times K$. Given $(x,y) \in (E \circ F) \setminus K \times K$, we can find a $z \in X$ such that $(x,z) \in E$ and $(z,y) \in F$. Since $(x,y) \notin K \times K$, either $x \notin K$ or $y \notin K$ holds. We first consider the case when $x \notin K$. Then we see from $(x,z) \in E \setminus K \times K$ that $d(x,z) < \varepsilon'$. Since $\overline{N}(K_0, \varepsilon') \subset K$, we have $z \notin K_0$, and in particular, $(z,y) \in F \setminus K_0 \times K_0$. This in turn implies that $d(z,y) < \varepsilon/2$, and hence $d(x,y) \le d(x,z) + d(z,y) < \varepsilon' + \varepsilon/2 \le \varepsilon$. Since the case when $y \notin K$ can be treated in a similar way, condition (iv) is verified.

It is clear from the definition of the C_0 coarse structure that every subset of X with compact closure is bounded. To show the converse, let $B \subset X$ be a bounded set with respect to the C_0 structure, and suppose that B does not have compact closure. Then, in particular, there are two distinct points $p, q \in B$, and we set the distance $\varepsilon = d(p,q) > 0$. Since B is bounded, the square $B \times B$ is controlled, and hence there exists a compact set $K \subset X$ such that $d(x,y) < \varepsilon/2$ whenever $(x,y) \in B \times B \setminus K \times K$. Since the closure of B is not compact, B is not contained in K. Fix a point $r \in B \setminus K$ and observe that $(p,r), (q,r) \in B \times B \setminus K \times K$. This implies that $\varepsilon = d(p,q) \le d(p,r) + d(q,r) < \varepsilon/2 + \varepsilon/2 = \varepsilon$, which is a contradiction.

We further assume that X is separable. To prove that the C_0 structure is proper, it remains only to show that there is a controlled neighborhood of the diagonal Δ_X. Since X is locally compact and separable metrizable, we can take a countable locally finite open cover $\{U_n \mid n \in \mathbb{N}\}$ such that each U_n has compact closure. Then we can define a continuous function $f\colon X \to (0,\infty)$ by

$$f(x) = \sum_{i \in \mathbb{N}} \min\{2^{-i}, d(x, X \setminus U_i)\}.$$

Then it is easy to see that the function f vanishes at infinity; that is, for all $\varepsilon > 0$ there is a compact set $K \subset X$ such that $0 < f(x) < \varepsilon$ for every $x \notin K$. This implies that the set

$$E = \left\{ (x,y) \in X \times X \mid d(x,y) < \min\{f(x), f(y)\} \right\}$$

is a controlled neighborhood of Δ_X. ∎

Let $X = (X, \mathcal{E})$ be a coarse space. A bounded (not necessarily continuous) function $f\colon X \to \mathbb{R}$ is a *Higson function* on X if for every controlled set $E \in \mathcal{E}$ and $\varepsilon > 0$ there is a bounded set $B \subset X$ such that $|f(x) - f(y)| < \varepsilon$ whenever $(x,y) \in E \setminus B \times B$. The Higson functions on X form a unital Banach algebra that is denoted by $B_h(X)$.

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A coarse space is usually equipped with a topology, and it makes sense to speak of continuous functions on the coarse space. Let X be a locally compact Hausdorff coarse space, and let $C_h(X)$ be the Banach algebra of *continuous* Higson functions on X. Let $e : X \to \mathbb{R}^{C_h(X)}$ be an embedding into a product of lines defined by $e(x) = (f(x))_{f \in C_h(X)}$. Then the compactification $hX = \mathrm{cl}_{\mathbb{R}^{C_h(X)}}\, e(X)$ of X is homeomorphic to the maximal ideal space of $C_h(X)$. We call hX the *Higson compactification* of X, and its boundary $\nu X = hX \setminus X$ is then called the *Higson corona* of X.

Remark 2.2 The Higson compactification of X is often defined as the maximal ideal space $h_{\mathbb{C}}X$ of the C^*-algebra of complex-valued continuous Higson functions on X (*cf.* [10]). This compactification $h_{\mathbb{C}}X$ is, however, equivalent to hX. Indeed, it is easy to show that a bounded real-valued continuous function on X is a Higson function if and only if it has a real-valued continuous extension on $h_{\mathbb{C}}X$ (*cf.* [1, 3.12.22(e)]).

The next lemma connects Higson functions and coarse maps. The proof is straightforward and left to the reader.

Lemma 2.3 *Let X and Y be coarse spaces and $f: X \to Y$ a coarse map. Then for every Higson function ϕ on Y, the composition $\phi \circ f$ is a Higson function on X. Consequently, f induces a ring homomorphism $f^*: B_h(Y) \to B_h(X)$. If, moreover, f is continuous, then f induces $f^*: C_h(Y) \to C_h(X)$.* ∎

Remark 2.4 In the definition of Higson functions we used the notion of bounded set, which is a purely coarse one. In many cases a coarse space has a topology, and it is natural to assume that the bounded sets have some relation with the topology. For a locally compact Hausdorff coarse space X, we consider the following condition:

(2.1) A subset of X is bounded if and only if it has compact closure.

Hereafter we will consider the Higson corona of X only when this condition is satisfied. Condition (2.1) is satisfied by the following coarse structures: the bounded structures on proper metric spaces, the continuously controlled structures (defined below), the C_0 structures on locally compact metric spaces (Proposition 2.1), and all coarsely connected proper coarse spaces.

For a set X and subsets $E \subset X \times X$ and $K \subset X$, we define $E[K]$ to be the set of $x \in X$ such that $(x, y) \in E$ for some $y \in K$. This set is the "image" of K under E, where E is considered to be a multivalued function from the second coordinate to the first coordinate. Now assume that X has a topology. Then $E \subset X \times X$ is called *proper* if each of $E[K]$ and $E^{-1}[K]$ has compact closure for every compact subset K of X.

Let X be a locally compact Hausdorff space with a (Hausdorff) compactification $\widetilde{X}$. Denote the boundary $\widetilde{X} \setminus X$ by ∂X. Then since X is locally compact, X is open in $\widetilde{X}$ and hence ∂X is compact. A subset $E \subset X \times X$ is then defined to be *continuously controlled* by $\widetilde{X}$ if one of (hence all of) the following three equivalent conditions is satisfied:

(a) the closure of E in $\widetilde{X} \times \widetilde{X}$ intersects the complement of $X \times X$ only in the diagonal $\Delta_{\partial X} = \{(\omega, \omega) \mid \omega \in \partial X\}$;

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(b) E is proper (in the sense defined in the previous paragraph), and for every net $((x_\lambda, y_\lambda))$ in E, if (x_λ) converges to $\omega \in \partial X$, then (y_λ) also converges to ω;
(c) E is proper, and for every point $\omega \in \partial X$ and every neighborhood V of ω in $\widetilde{X}$, there is a neighborhood $U \subset V$ of ω in $\widetilde{X}$ such that $E \cap (U \times (X \setminus V)) = \varnothing$.

Then the collection of all continuously controlled subsets is shown to be a coarse structure called the *continuously controlled coarse structure* induced by $\widetilde{X}$ (see [10, Section 2.2]).

Remark 2.5 For a continuously controlled structure, it is easy to see that condition (2.1) is always satisfied, while it may happen that there is no controlled neighborhood of the diagonal, even if the space is paracompact. This means that such a structure need not be proper. (In [10, Theorem 2.27], it is asserted that every continuously controlled structure on a paracompact space is proper, but the proof given there is actually incorrect, as pointed out by Berndt Grave; see [11].) As an example, let $X = [0, \infty)$ and consider the Stone–Čech compactification βX of X. Let U be any neighborhood of Δ_X in $X \times X$. For each $n \in \mathbb{N}$, let $a_n = n$ and take b_n so that $0 < b_n - a_n < 2^{-1}$ and $(a_n, b_n) \in U$ are satisfied. Then $A = \{a_n \mid n \in \mathbb{N}\}$ and $B = \{b_n \mid n \in \mathbb{N}\}$ are disjoint closed subsets in X, and hence there exists a continuous map $f: X \to [0,1]$ with $f(A) = \{0\}$ and $f(B) = \{1\}$. This f admits a continuous extension $\widetilde{f}: \beta X \to [0,1]$, and we have $\mathrm{cl}_{\beta X} A \subset \widetilde{f}^{-1}(0)$ and $\mathrm{cl}_{\beta X} B \subset \widetilde{f}^{-1}(1)$. In particular, $\mathrm{cl}_{\beta X} A$ and $\mathrm{cl}_{\beta X} B$ are disjoint. Since A is noncompact, there exists a point $\omega \in (\mathrm{cl}_{\beta X} A) \setminus X$ and a net (a_{n_λ}) in A convergent to ω. Then the net (b_{n_λ}) has a subnet $(b_{n'_\mu})$ convergent to some point $\omega' \in \mathrm{cl}_{\beta X} B$. The corresponding subnet $(a_{n'_\mu})$ converges to ω. Then $(a_{n'_\mu}, b_{n'_\mu}) \in U$ and $(a_{n'_\mu}, b_{n'_\mu}) \to (\omega, \omega') \notin \Delta_{\beta X \setminus X}$, showing that U is not controlled.

In the rest of this section, we discuss how a noncontinuous coarse map between proper coarse spaces induces a continuous map between their Higson coronas. The results will be applied to prove our main theorem (Theorem 4.2).

For a proper coarse space X satisfying (2.1), let $B_0(X)$ denote the set of bounded, real-valued functions that vanish at infinity, in the sense that for all $\varepsilon > 0$ there exists a compact set K such that we have $|f(x)| < \varepsilon$ for all $x \in X \setminus K$. Let $C_0(X)$ denote the subalgebra of all continuous functions in $B_0(X)$. The Banach algebra $C(\nu X)$ of real-valued continuous functions of the Higson corona is then isomorphic to $C_h(X)/C_0(X)$. There is a natural isomorphism $C_h(X)/C_0(X) \cong B_h(X)/B_0(X)$ by [10, Lemma 2.40], and hence $C(\nu X) \cong B_h(X)/B_0(X)$. Now let X and Y be two proper coarse spaces satisfying (2.1) and let $f: X \to Y$ be a (not necessarily continuous) coarse map. By Lemma 2.3 there is an induced map $f^*: B_h(Y) \to B_h(X)$, and by the properness of f, we have $f^*(B_0(Y)) \subset B_0(X)$. Therefore, we have a map $f^*: C(\nu Y) \cong B_h(Y)/B_0(Y) \to B_h(X)/B_0(X) \cong C(\nu X)$. Then $\nu f: \nu X \to \nu Y$ is defined as the continuous map corresponding to the last f^* by Gel'fand-Naimark duality. This makes the operation ν a functor, called the *Higson corona functor*, from the category of proper coarse spaces to the category of compact Hausdorff spaces.

Of course, we can expect the map νf to be a "continuous extension" of f in some sense. In fact, we have the following proposition.

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Proposition 2.6 *Let $f: X \to Y$ be a coarse map between proper coarse spaces satisfying condition (2.1). Then the map $\nu f: \nu X \to \nu Y$ is characterized by the property that $f \cup \nu f: hX \to hY$ is continuous at each point of νX.*

Proof We first show that $\nu f: \nu X \to \nu Y$ satisfies this property. Since νf is continuous, we need only to show that for each net (x_λ) converging to a point $\omega \in \nu X$, the net $(f(x_\lambda))$ converges to $\nu f(\omega)$. If this is not the case, there exists a subnet (x_{λ_μ}) of (x_λ) such that $(f(x_{\lambda_\mu}))$ is convergent to $\omega' \in \nu Y \setminus \{\nu f(\omega)\}$. Then there exists a continuous function $\widetilde{\phi}: hY \to \mathbb{R}$ with $\widetilde{\phi}(\nu f(\omega)) = 0$ and $\widetilde{\phi}(\omega') = 1$, which restricts to a Higson function $\phi = \widetilde{\phi}|_Y \in C_h(Y) \subset B_h(Y)$. Then since f is coarse, we have $\phi \circ f \in B_h(X)$ by Lemma 2.3. Using Tietze's theorem, we can take a continuous extension $\psi: hX \to \mathbb{R}$ of $\widetilde{\phi} \circ (\nu f): \nu X \to \mathbb{R}$. The definition of νf yields that $\phi \circ f - (\psi|_X) \in B_0(X)$. This implies, by the continuity of ψ, that

$$\lim \phi \circ f(x_{\lambda_\mu}) = \lim \psi(x_{\lambda_\mu}) = \psi(\omega) = \widetilde{\phi} \circ (\nu f)(\omega) = 0.$$

On the other hand, by the continuity of $\widetilde{\phi}$,

$$\lim \phi \circ f(x_{\lambda_\mu}) = \widetilde{\phi}(\omega') = 1,$$

which is a contradiction.

The map νf is uniquely determined by the property we have now demonstrated, since every point of νX is a limit of some net in X. This completes the proof. ∎

Remark 2.7 The above proposition means that νf is characterized by the fact that $f \cup \nu f: (hX, \nu X) \to (hY, \nu Y)$ is *eventually continuous* in the sense of [6, Definition 1.14] and [12, Definition 2.4], or is *ultimately continuous* in the sense of [7, Section 2]. This observation is already made in the special case where both X and Y are continuously controlled by some metrizable compactifications $\widetilde{X}$ and $\widetilde{Y}$, respectively [7]. In fact, the Higson compactifications hX and hY are equivalent to $\widetilde{X}$ and $\widetilde{Y}$ in this special case [10, Proposition 2.48].

In some situation, it is also true that f must be coarse whenever f admits an extension as in the last proposition. For a precise statement we need the following notion: a map $f: X \to Y$ between coarse spaces is called *pre-bornologous* if $f(B) \subset Y$ is bounded for every bounded set $B \subset X$. Notice that every bornologous map between coarse spaces is pre-bornologous.

Proposition 2.8 *Let X and Y be proper coarse spaces satisfying (2.1) and let $f: X \to Y$ be a (not necessarily continuous) pre-bornologous map. Suppose that Y has the continuously controlled coarse structure induced by some compactification $\widetilde{Y}$ of Y. Then f is coarse if and only if there exists $\widetilde{f}: \nu X \to \nu Y$ (which is necessarily equal to νf) such that $f \cup \widetilde{f}: hX \to hY$ is continuous at each point of νX.*

Proof The "only if" part is Proposition 2.6. We prove the "if" part. Suppose that there is a map $\widetilde{f}: \nu X \to \nu Y$ as above. To see that f is proper, it is enough to show that $f^{-1}(K)$ has compact closure in X whenever $K \subset Y$ is compact, since both X and Y satisfy condition (2.1). Let K be a compact subset of Y. If $f^{-1}(K)$ does not have

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compact closure in X, then there exists a point $\omega \in \nu X \cap \mathrm{cl}_{hX} f^{-1}(K)$. Then we have $\widetilde{f}(\omega) \in \nu Y$, but the continuity of $f \cup \widetilde{f}$ at ω implies $\widetilde{f}(\omega) \in \mathrm{cl}_{hY} K = K \subset Y$. This is a contradiction, which means that $f^{-1}(K)$ has compact closure in X.

To prove that f is bornologous, let E be a controlled subset of $X \times X$ and consider the image $F = (f \times f)(E) \subset Y \times Y$. It is straightforward to show that F is proper as a subset of $Y \times Y$, using the fact that E is proper (see [10, Proposition 2.23]) and that f is a proper, pre-bornologous map. Let $\big((f(x_\lambda), f(x'_\lambda))\big)$ be a net in F with $(x_\lambda, x'_\lambda) \in E$ and $f(x_\lambda) \to \omega \in \widetilde{Y} \setminus Y$. It remains to show that $f(x'_\lambda) \to \omega$.

Suppose that this is not the case. Then there exist subnets (x_{λ_μ}) and (x'_{λ_μ}) (with the same index set) such that $f(x'_{\lambda_\mu}) \to \omega'$ for some $\omega' \neq \omega$. We write $x_{\lambda_\mu} = x_\mu$, $x'_{\lambda_\mu} = x'_\mu$ to simplify notation. Choose a continuous function $\widetilde{\phi}\colon \widetilde{Y} \to [0,1]$ such that $\widetilde{\phi}(\omega) = 0$ and $\widetilde{\phi}(\omega') = 1$, and let ϕ denote the restriction $\widetilde{\phi}|_Y\colon Y \to [0,1] \subset \mathbb{R}$. By [10, Proposition 2.45 (b)], there exists a continuous map $\pi\colon hY \to \widetilde{Y}$ that restricts to the identity on Y. Then the composition $F = \widetilde{\phi} \circ \pi \circ (f \cup \widetilde{f})\colon hX \to \mathbb{R}$ gives an extension of $\phi \circ f$ over hX that is continuous at each point in νX. By Tietze's theorem, there exists a continuous extension $G\colon hX \to \mathbb{R}$ of $\widetilde{\phi} \circ \pi \circ \widetilde{f} = F|_{\nu X}$. Then we have $G|_X \in C_h(X)$ and $(G - F)|_X \in B_0(X)$, which in turn implies

$$\phi \circ f = F|_X = G|_X - (G - F)|_X \in C_h(X) + B_0(X) = B_h(X).$$

This causes a contradiction, since it can also be shown that $\phi \circ f \notin B_h(X)$, as follows. Given a compact set $K \subset X$, we can take μ so large that $|\phi \circ f(x_\mu)| < 1/3$, $|\phi \circ f(x'_\mu) - 1| < 1/3$, and $x_\mu \notin E[K]$. Then $x'_\mu \notin K$ and it follows that $(x_\mu, x'_\mu) \in E \setminus K \times K$ and $|\phi \circ f(x_\mu) - \phi \circ f(x'_\mu)| \geq 1/3$. This shows that $\phi \circ f \notin B_h(X)$. ∎

3 C_0 and Continuously Controlled Coarse Structures

In this section, all locally compact metric spaces are assumed to have the C_0 coarse structures. Controlled sets, coarse maps and Higson functions will be with respect to the C_0 structure. For such structures, we first make clear how the notions of Higson functions and coarse maps are related to uniform continuity (Proposition 3.1, Corollary 3.5). Then we prove that the continuously controlled coarse structure induced by the Higson compactification is the original C_0 structure (Theorem 3.6).

Proposition 3.1 *Let (X, d) be a locally compact metric space. Then the continuous Higson functions on X are exactly the bounded uniformly continuous functions on X.*

Proof First assume that $f\colon X \to \mathbb{R}$ is bounded and uniformly continuous. Take any controlled set E in the C_0 structure and $\varepsilon > 0$. Then we can choose a $\delta > 0$ such that $d(x, y) < \delta$ implies $|f(x) - f(y)| < \varepsilon$, and then we can choose a compact set K such that $(x, y) \in E \setminus K \times K$ implies $d(x, y) < \delta$. Then $|f(x) - f(y)| < \varepsilon$ holds for every point $(x, y) \in E \setminus K \times K$. This proves that f is a Higson function.

To show the converse, suppose that f is continuous but not uniformly continuous. The latter condition means that there are $\varepsilon > 0$ and sequences $(x_n)_{n\in\mathbb{N}}$, $(x'_n)_{n\in\mathbb{N}}$ in X such that $d(x_n, x'_n) < 1/n$ and $|f(x_n) - f(x'_n)| \geq \varepsilon$. Then the set $\{x_n \mid n \in \mathbb{N}\}$ is not contained in any compact set. Indeed, if it were contained in a compact set, then the closure of $\{x_n, x'_n \mid n \in \mathbb{N}\}$ would be compact, where f must be uniformly continuous, contrary to the choice of (x_n) and (x'_n). To show that f is not a Higson function, we first notice that the set $E = \{(x_n, x'_n) \mid n \in \mathbb{N}\}$ is controlled, and take any compact subset K of X. As seen above, the set $\{x_n \mid n \in \mathbb{N}\}$ is not contained in K. Thus, we can find an N such that $x_N \notin K$. This means $(x_N, x'_N) \in E \setminus K \times K$, but we have also that $|f(x_N) - f(x'_N)| \geq \varepsilon$. Therefore, f is not a Higson function. ∎

The *Smirnov compactification* uX of a metric space X is defined as the maximal ideal space of the unital Banach algebra $C_u(X)$ of real-valued bounded uniformly continuous functions. Thus, a bounded continuous function $\phi: X \to \mathbb{R}$ is extendable continuously over uX if and only if it is uniformly continuous, and any compactification with this property is equivalent to uX. Here, two compactifications γX and δX of a space X are called *equivalent* if there exists a homeomorphism $h: \gamma X \to \delta X$ such that $h|_X = \text{id}$. Proposition 3.1 immediately implies the following corollary.

Corollary 3.2 *For any locally compact metric space X, the Smirnov compactification uX of X is equivalent to the Higson compactification of X with respect to the C_0 coarse structure.* ∎

In what follows, we give a characterization of coarse maps between locally compact metric spaces without assuming continuity. We recall from the last section that $f: X \to Y$ between coarse spaces is *pre-bornologous* if for every bounded $B \subset X$ the image $f(B)$ is bounded. Since locally compact metric spaces satisfy condition (2.1) in Remark 2.4 by Proposition 2.1, we obtain the following lemma.

Lemma 3.3 *Let X and Y be locally compact metric spaces and let $f: X \to Y$ be a (not necessarily continuous) map. Then f is proper if and only if $f^{-1}(K)$ has compact closure for every compact set K of Y. Similarly, f is pre-bornologous if and only if $f(K)$ has compact closure for every compact set K of X.* ∎

Proposition 3.4 *Let X and Y be locally compact metric spaces and let $f: X \to Y$ be a (not necessarily continuous) proper, pre-bornologous map. The following are equivalent:*

(i) *f is a coarse map.*
(ii) *For every $\varepsilon > 0$, there exist a compact set $K \subset X$ and a $\delta > 0$ such that $d(f(x), f(x')) < \varepsilon$ whenever $(x, x') \notin K \times K$ and $d(x, x') < \delta$.*

Proof (ii) ⇒ (i): Assume (ii) and let $f: X \to Y$ be a proper, pre-bornologous map. It is enough to show that f is bornologous. Take any controlled set $E \subset X \times X$ and put $F = (f \times f)(E)$. To show that F is controlled, take any $\varepsilon > 0$. By (ii), we can take a compact set $K \subset X$ and a $\delta > 0$ such that $d(x, x') < \delta$ and $(x, x') \notin K \times K$ imply $d(f(x), f(x')) < \varepsilon$. Since E is controlled, there is a compact set $K' \supset K$ such that $d(x, x') < \delta$ whenever $(x, x') \in E \setminus K' \times K'$. Then by Lemma 3.3, $L = \text{cl}_Y f(K')$ is compact, since f is pre-bornologous. Let $(y, y') \in F \setminus L \times L$. Then $(y, y') = (f(x), f(x'))$ for some $(x, x') \in E \setminus K' \times K'$. It follows that $d(x, x') < \delta$, and hence $d(y, y') = d(f(x), f(x')) < \varepsilon$, since $(x, x') \notin K \times K$.

(i) ⇒ (ii): Assume that $f: X \to Y$ is proper and pre-bornologous, and that (ii) is not the case. We then prove that f is not bornologous to obtain a contradiction. There exists $r > 0$ such that for each $n \in \mathbb{N}$ and each compact set $K \subset X$, we can take $x_{K,n}$ and $x'_{K,n}$, not both of which are in K, with $d(x_{K,n}, x'_{K,n}) < 1/n$ and $d(f(x_{K,n}), f(x'_{K,n})) \geq r$. We may exchange $x_{K,n}$ and $x_{K',n}$ if necessary to assume that $x_{K,n} \notin K$. Fix a locally finite cover (U_λ) of X by open sets U_λ with compact closure $D_\lambda = \mathrm{cl}_X U_\lambda$. Let $K_1 = \varnothing$ and inductively, define K_{n+1} as the union of all D_λ that intersects $K_n \cup \{x_{K_n,n}, x'_{K_n,n}\}$. Since (D_λ) is locally finite, we see by induction that K_n is compact for each n. Let us define $x_n = x_{K_n,n}$ and $x'_n = x'_{K_n,n}$. Notice that $K_n \subset K_{n+1}$, $x_n \notin K_n$ and $x_n, x'_n \in K_{n+1}$.

We show that the set $E = \{(x_n, x'_n) \mid n \in \mathbb{N}\} \subset X \times X$ is controlled. To see this, let $\varepsilon > 0$. Take $N \in \mathbb{N}$ so large that $1/N < \varepsilon$ holds, and let $K = K_N$. If $(x_n, x'_n) \notin K \times K$, then it follows that $n \geq N$, and hence $d(x_n, x'_n) < 1/n \leq 1/N < \varepsilon$. This shows that E is controlled.

Next, we claim that the set $\{x_n \mid n \in \mathbb{N}\}$ is not contained in any compact set. Indeed, if this set is contained in a compact set, then some subsequence (x_{n_k}) converges to a point $x_\infty \in X$, and D_λ is a neighborhood of x_∞ for some λ. Then for a large k, both x_{n_k} and $x_{n_{k+1}}$ are in D_λ. Since $x_{n_k} \in D_\lambda$, we have $D_\lambda \subset K_{n_k+1}$. Then $x_{n_{k+1}} \in D_\lambda \subset K_{n_k+1} \subset K_{n_{k+1}}$ (using $n_k + 1 \leq n_{k+1}$), which is contrary to $x_{n_{k+1}} \notin K_{n_{k+1}}$. Thus, $\{x_n \mid n \in \mathbb{N}\}$ is not contained in any compact set.

Finally, we show that $(f \times f)(E) = \{(f(x_n), f(x'_n)) \mid n \in \mathbb{N}\}$ is not controlled to prove that f is not bornologous (and hence not coarse). To this end, take any compact set $K \subset Y$. Then by Lemma 3.3, $f^{-1}(K)$ has compact closure, and hence there is some n such that $x_n \notin f^{-1}(K)$ by the last paragraph, which implies $(f(x_n), f(x'_n)) \notin K \times K$. However, we have

$$d\big(f(x_n), f(x'_n)\big) = d\big(f(x_{K_n,n}), f(x'_{K_n,n})\big) \geq r.$$

Notice that $r > 0$ is irrelevant to our choice of K. This means $(f \times f)(E)$ is not controlled. ∎

Since continuous maps between coarse spaces satisfying (2.1) are pre-bornologous, and are uniformly continuous on every compact set, we obtain the following corollary.

Corollary 3.5 *A continuous map between locally compact metric spaces is coarse with respect to the C_0 coarse structures if and only if it is proper and uniformly continuous.* ∎

Let us consider the Higson compactification $h_0 X$ with respect to the C_0 structure. Then in turn, $h_0 X$ induces a continuously controlled structure on X. As a generalization of [5, Proposition 6], we assert that this is the same as the original C_0 structure.

Theorem 3.6 *The C_0 coarse structure on a locally compact metric space X is equal to the continuously controlled structure induced by the Higson compactification $h_0 X$.*

To prove this theorem, the next lemma will be useful.

Lemma 3.7 *Let X be a locally compact metric space and E a subset of $X \times X$ with $E = E^{-1}$. Then E is controlled if and only if $d(x_n, x'_n) \to 0$ holds for every sequence $((x_n, x'_n))_{n \in \mathbb{N}}$ in E such that (x_n) has no convergent subsequence.*

Proof The "only if" part is clear. To show the "if" part, we use the construction in the proof of Proposition 3.4 (i) ⇒ (ii), as follows. First choose a locally finite covering $(U_\lambda)_{\lambda\in\Lambda}$ of X by open sets U_λ with compact closure $D_\lambda = \mathrm{cl}_X U_\lambda$. Assume that $E = E^{-1} \subset X \times X$ is not controlled. Then there exists $\varepsilon > 0$ such that for each compact set $K \subset X$, we have $d(x_K, x'_K) \geq \varepsilon$ for some $(x_K, x'_K) \in E \setminus K \times K$. Here we can choose (x_K, x'_K) so that $x_K \notin K$, since otherwise we can exchange x_K and x'_K using $E = E^{-1}$.

Let $K_1 = \varnothing$, and inductively, define K_{n+1} to be the union of all D_λ that intersects $K_n \cup \{x_{K_n}\}$. Since (D_λ) is locally finite, it follows by induction that K_n is compact for each n. Put $x_n = x_{K_n}$ and $x'_n = x'_{K_n}$. Then, clearly, $(x_n, x'_n) \in E$. Moreover, (x_n) does not have a convergent subsequence. To see this, assume that a subsequence (x_{n_k}) converges to a point $x_\infty \in X$. Then there exists a λ such that D_λ is a compact neighborhood of x_∞. Take a large k such that both of x_{n_k} and $x_{n_{k+1}}$ belong to D_λ. Then $x_{n_{k+1}} \in D_\lambda \subset K_{n_k+1} \subset K_{n_{k+1}}$, which contradicts the choice of $x_{n_{k+1}}$. ■

The next lemma, also needed to prove Theorem 3.6, is valid for general metric spaces.

Lemma 3.8 *Let (x_n) and (x'_n) be sequences in a metric space X and assume that $d(x_n, x'_n) \geq r$ for every $n \in \mathbb{N}$. Then there exist subsequences (x_{n_k}) and (x'_{n_k}) such that $d(A, A') \geq r/3$, where $A = \{x_{n_k} \mid k \in \mathbb{N}\}$ and $A' = \{x'_{n_k} \mid k \in \mathbb{N}\}$.*

Proof For $n \in \mathbb{N}$ define the subsets I_n, J_n of $\mathbb{N}$ as follows:

$$I_n = \{i \in \mathbb{N} \mid d(x_n, x'_i) < r/3\},$$
$$J_n = \{i \in \mathbb{N} \mid d(x_i, x'_n) < r/3\}.$$

Then for $i, j \in I_n$, we have

$$d(x_i, x'_j) \geq r/3. \tag{3.1}$$

Indeed, $d(x'_i, x'_j) \leq d(x'_i, x_n) + d(x_n, x'_j) < 2r/3$, and hence $d(x_i, x'_j) \geq d(x_i, x'_i) - d(x'_i, x'_j) \geq r - 2r/3 = r/3$, as desired. Similarly, inequality (3.1) also holds for $i, j \in J_n$. Thus if I_n (or J_n) is infinite for some n, the enumeration $I_n = \{n_k \mid k \in \mathbb{N}\}$ (or $J_n = \{n_k \mid k \in \mathbb{N}\}$) with $n_1 < n_2 < \cdots$ gives the desired subsequences (x_{n_k}) and (x'_{n_k}). We are left with the case where I_n and J_n are finite for all n.

We inductively construct a sequence (n_k) that will give the desired subsequences. Let $n_1 = 1$ and suppose that we have constructed $n_1 < \cdots < n_{k-1}$ satisfying $d(x_{n_i}, x'_{n_j}) \geq r/3$ for every $i, j < k$. Notice that the set $S = \bigcup_{i<k} I_{n_i} \cup \bigcup_{i<k} J_{n_i}$ is finite. We define $n_k \in \mathbb{N}$ so that n_k does not belong to S and is greater than n_{k-1}. Then we have $d(x_{n_k}, x'_{n_i}) \geq r/3$ and $d(x_{n_i}, x'_{n_k}) \geq r/3$ for each $i < k$. This completes the inductive construction. ■

Proof of Theorem 3.6 By [10, Proposition 2.45(a)], every C_0 controlled set is continuously controlled by $h_0 X$. To show the converse, let $E \subset X \times X$ be a subset continuously controlled by $h_0 X$. We may replace E by $E \cup E^{-1}$ to assume that $E = E^{-1}$. To apply Lemma 3.7 to E, let $((x_n, x'_n))$ be a sequence in E such that (x_n) has no convergent subsequence, and suppose that $d(x_n, x'_n) \to 0$ does not hold. Then, passing to subsequences, we can find $r > 0$ such that $d(x_n, x'_n) \geq r$ for every n. By Lemma 3.8,

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we can further pass to subsequences to obtain $d(A, A') \geq r/3$, where $A = \{x_n \mid n \in \mathbb{N}\}$ and $A' = \{x'_n \mid n \in \mathbb{N}\}$. Now define $\phi\colon X \to \mathbb{R}$ by

$$\phi(x) = \frac{d(x, A)}{d(x, A) + d(x, A')}.$$

Notice that $\phi(A) = \{0\}$ and $\phi(A') = \{1\}$. The function ϕ is uniformly continuous and bounded, and hence is a Higson function by Proposition 3.1. Thus, ϕ admits a continuous extension $\widetilde{\phi}\colon h_0 X \to \mathbb{R}$.

On the other hand, we can take a subnet (x_{n_λ}) of (x_n) such that $x_{n_\lambda} \to \omega$ for some $\omega \in h_0 X \setminus X$. Since E is continuously controlled by $h_0 X$, we have $x'_{n_\lambda} \to \omega$. However, we then obtain

$$0 = \lim \phi(x_{n_\lambda}) = \widetilde{\phi}(\omega) = \lim \phi(x'_{n_\lambda}) = 1,$$

which is a contradiction. ■

Let us make brief remarks on the controlled coarse structures with respect to metrizable compactifications. The following characterization of the Smirnov compactification is well known.

Theorem 3.9 ([13, Theorem 2.5]) *Let γX be a (Hausdorff) compactification of a metric space $X = (X, d)$. Then the following conditions are equivalent:*

(i) *γX is equivalent to the Smirnov compactification uX,*
(ii) *for all subsets $A, B \subset X$, $d(A, B) > 0$ if and only if $\mathrm{cl}_{\gamma X} A \cap \mathrm{cl}_{\gamma X} B = \varnothing$.* ■

Thus, the continuously controlled coarse structure induced by a metrizable compactification γX can be considered as the C_0 structure with respect to an admissible metric on γX.

Corollary 3.10 *For any compact metric space $X = (X, d)$ and its dense subspace Y, the space X coincides with the Smirnov compactification uY. If, moreover, Y is locally compact (or equivalently, open in X), then the C_0 structure on Y coincides with the continuously controlled structure induced from X, and X is the Higson compactification for this structure.*

Proof The first half of the statement is immediate from Theorem 3.9. If Y is locally compact, we can consider the C_0 structure on Y with respect to the metric d induced from X, as well as the continuously controlled structure on Y induced by X. Then, by Corollary 3.2, $X = uY$ is the Higson compactification of Y for the C_0 structure. Finally, it follows from Theorem 3.6 that the continuously controlled structure on Y induced by $X = uY$ is equal to the C_0 structure. ■

4 Equivalence of Categories

To state our main result, we define two categories. Let **K** be the category of compact metrizable spaces and continuous maps. We define another category **TB** as follows: the objects of **TB** are totally bounded locally compact metric spaces with the C_0 coarse structures. The set $\mathrm{Hom}_{\mathbf{TB}}(X, Y)$ of morphisms between objects X and Y consists of

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the equivalence classes of coarse maps by the equivalence relation $\sim$, where $f \sim g$ if f and g are close (that is, $\{(f(x), g(x)) \mid x \in X\}$ is a controlled set). Such a category can be defined, since the closeness relation is compatible with composition from left and right.

Remark 4.1 The category **TB** is related to continuously controlled structures. Indeed, as seen from Corollary 3.10, the category **TB** is equivalent to the following category **CC**: the objects of **CC** are the locally compact spaces with the continuously controlled structures induced by metrizable compactifications, and the morphisms between them are the coarse maps modulo closeness.

On the other hand, Cuchillo-Ibáñez, Dydak, Koyama, and Morón [5] considered the category $\mathcal{Z}$ of Z-sets in the Hilbert cube Q and continuous maps, and they have shown that $\mathcal{Z}$ is isomorphic to the category $\mathcal{C}_0(\mathcal{Z})$ of the complements of Z-sets in Q with the C_0 coarse structures and coarse maps modulo closeness (here Q is assumed to have a fixed metric). Since every compact metrizable space is homeomorphic to some Z-set in Q, the category **K** is equivalent to $\mathcal{Z}$. It follows that the categories **K**, $\mathcal{Z}$ and $\mathcal{C}_0(\mathcal{Z})$ are equivalent to each other. The next Theorem 4.2 implies that they are equivalent to **TB**, and hence to **CC**.

Let us consider the Higson corona functor ν introduced before Proposition 2.6. This functor sends close coarse maps to the same continuous map (see [10, Proposition 2.41]), and thus coarsely equivalent proper coarse spaces have homeomorphic Higson coronas. Naturally, we can ask the converse, namely whether X and Y are coarsely equivalent if νX and νY are homeomorphic. This question has a negative answer in general (see [10, Example 2.44, Proposition 2.45 (c)]), but the next theorem states that we have an affirmative answer for objects of **TB**.

If X is an object of **TB**, then the completion $\widetilde{X}$ of X is compact, since X is totally bounded. By Corollary 3.10, $\widetilde{X}$ is the Higson compactification of X and $\widetilde{X} \setminus X$ is the Higson corona. In particular, νX is compact and metrizable. Thus, we can define a functor $\nu\colon \mathbf{TB} \to \mathbf{K}$.

Theorem 4.2 *The functor $\nu\colon \mathbf{TB} \to \mathbf{K}$ is an equivalence of categories.*

Proof It is enough to show that ν is full and faithful, and that every object in **K** is isomorphic to νX for some object X in **TB**.

We shall first show that ν is full, namely that ν gives a surjective map from $\mathrm{Hom}_{\mathbf{TB}}(X, Y)$ to the set $\mathrm{Hom}_{\mathbf{K}}(\nu X, \nu Y)$ of continuous maps from νX to νY, for each X and Y in **TB**. Let $h\colon \nu X \to \nu Y$ be a continuous map. Recall that the completion $\widetilde{X}$ of X gives the Higson compactification $hX = X \cup \nu X$ of X, and the same holds for hY. Thus, we use the notation $\widetilde{X}$ and $\widetilde{Y}$ rather than hX and hY, and their metrics extended from X and Y are denoted by d when necessary.

We construct (a representative of) a morphism $f\colon X \to Y$ in **TB** such that $\nu f = h$. The basic idea here is as follows: for $x \in X$, we take a point $a \in \nu X$ close to x and define $f(x)$ to be a point of Y close to $h(a)$, to the same extent as x is close to a. We explain this construction in detail. Let us define U_n as the open $1/n$-neighborhood of νX in $\widetilde{X}$ for $n \in \mathbb{N}$, and let $U_0 = \widetilde{X}$. Using the compactness of νY, for each $n \in \mathbb{N}$, take finitely many points $y_{n,1}, y_{n,2}, \dots, y_{n,k(n)}$ in Y such that $\nu Y \subset \bigcup_{i=1}^{k(n)} B(y_{n,i}, 1/n)$. For convenience, let $k(0) = 1$ and let $y_{0,0}$ be an arbitrarily fixed point in Y.

To define $f: X \to Y$, let $x \in X$ and take the largest $n \geq 0$ such that $x \in U_n$. If $n = 0$, then we define $f(x) = y_{0,0}$. If $n \geq 1$, choose $x' \in \nu X$ such that $d(x, x') = d(x, \nu X)$. Then we can choose $i \in \{1, 2, \dots, k(n)\}$ such that $h(x') \in B(y_{n,i}, 1/n)$. We finally define $f(x) = y_{n,i} \in Y$.

We claim that $f: X \to Y$ is a coarse map and $\nu f = h$. First, notice that f is pre-bornologous, since C_0 coarse structures satisfy condition (2.1) in Remark 2.4 and $f(X \smallsetminus U_n)$ is contained in the finite set $\{y_{m,i} \mid m < n, 1 \leq i \leq k(m)\}$ for each $n \in \mathbb{N}$. By Theorem 3.6, the C_0 coarse structure on Y is the continuously controlled structure induced by $\widetilde{Y}$. Also, we easily see that $f \cup h: X \cup \nu X = \widetilde{X} \to \widetilde{Y}$ is continuous at each point in νX. Then it follows by Proposition 2.8 (and Proposition 2.6) that f is coarse and $\nu f = h$. The fullness of ν is now proved.

Next we show that $\nu: \mathbf{TB} \to \mathbf{K}$ is faithful, namely, that ν maps each $\mathrm{Hom}_{\mathbf{TB}}(X, Y)$ injectively to $\mathrm{Hom}_{\mathbf{K}}(\nu X, \nu Y)$. To see this, let $f, g: X \to Y$ be coarse maps such that $\nu f = \nu g$. We have to show that f and g are close; in other words,

$$E = \big\{ (f(x), g(x)) \mid x \in X \big\} \subset Y \times Y$$

is controlled. By Theorem 3.6, it is enough to show that E is continuously controlled by $\widetilde{Y}$. To this end, take any $(\eta, \eta') \in \overline{E} \smallsetminus Y \times Y$, where $\overline{E}$ denotes the closure of E in $\widetilde{Y} \times \widetilde{Y}$. Then there exists a net (x_λ) in X such that $(f(x_\lambda), g(x_\lambda)) \to (\eta, \eta')$. Since f is proper, we can take a subnet (x_{λ_μ}) of (x_λ) such that $x_{\lambda_\mu} \to \omega$ for some $\omega \in \nu X = \widetilde{X} \smallsetminus X$. Then by Proposition 2.6, we have $\eta = \lim f(x_{\lambda_\mu}) = \nu f(\omega) = \nu g(\omega) = \lim g(x_{\lambda_\mu}) = \eta' \in \nu Y = \widetilde{Y} \smallsetminus Y$, which shows that E is continuously controlled by $\widetilde{Y}$.

Finally, we have to show that every object in $\mathbf{K}$ is isomorphic to νX for some object X in $\mathbf{TB}$. To see this, let K be any compact metrizable space, and fix any admissible metric d on $K \times [0,1]$. Let $X = K \times (0,1]$. Then $X = (X, d)$ is an object of $\mathbf{TB}$ and $K \times [0,1]$ is its Higson compactification by Corollary 3.10. It follows that $\nu X = K \times \{0\}$ and hence K is homeomorphic to νX. The proof is completed. ■

The following corollary is an immediate consequence of Theorem 4.2 (and Corollary 3.10).

Corollary 4.3 *Suppose that M_1 and M_2 are compact metric spaces and that $Z_1 \subset M_1$ and $Z_2 \subset M_2$ are closed nowhere dense subspaces. Then $M_1 \smallsetminus Z_1$ and $M_2 \smallsetminus Z_2$ are coarsely equivalent as C_0 coarse spaces if and only if Z_1 and Z_2 are homeomorphic.* ■

Moreover, Theorem 4.2 and the above corollary translate to the language of category $\mathbf{CC}$ introduced in Remark 4.1, in view of Corollary 3.10.

Corollary 4.4 *The Higson corona functor $\nu: \mathbf{CC} \to \mathbf{K}$ is an equivalence of categories. In particular, two metrizable compactifications $\widetilde{X}_1$ and $\widetilde{X}_2$ of a locally compact space X determine coarsely equivalent continuously controlled coarse structures if and only if their remainders are homeomorphic, $\widetilde{X}_1 \smallsetminus X \approx \widetilde{X}_2 \smallsetminus X$.* ■

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Corollary 4.5 *Every object in* **CC** *is coarsely equivalent to an object in* **CC** *that is contractible and whose Higson compactification is also contractible.*

Proof For any object X in **CC**, which has the continuously controlled structure induced by a metrizable compactification $\widetilde{X}$, consider the remainder $Z = \widetilde{X} \setminus X$. Let $\widetilde{Y}$ be the cone over Z, which is compact metrizable and is a compactification of the open cone $Y = \widetilde{Y} \setminus Z$. We can then equip Y with the continuously controlled structure induced by $\widetilde{Y}$. By Corollary 4.4, the coarse space Y is an object of **CC** coarsely equivalent to X. Clearly, Y and $\widetilde{Y}$ are contractible, and $\widetilde{Y}$ is the Higson compactification of Y by Corollary 3.10. ∎

Example 4.6 Applying Corollary 4.4, we can construct three proper coarse structures $\mathcal{E}_i (i = 1, 2, 3)$ on the same topological space X with $\mathcal{E}_1 \subset \mathcal{E}_2 \subset \mathcal{E}_3$ for which $\mathcal{E}_1$ and $\mathcal{E}_3$ are coarsely equivalent, but $\mathcal{E}_2$ fails to be equivalent to $\mathcal{E}_1$ (or $\mathcal{E}_3$). Indeed, it suffices to take three metrizable compactifications $\gamma_i X$ of the same locally compact space X that admit maps $\gamma_1 X \to \gamma_2 X \to \gamma_3 X$ extending the identity, with the remainders $Z_i = \gamma_i X \setminus X$ satisfying $Z_1 \approx Z_3$ but $Z_1 \not\approx Z_2$. Then the continuously controlled structures induced by $\gamma_i X (i = 1, 2, 3)$ give an example. It is easy to construct an explicit example where $X = [0, 1] \times [0, 1)$, Z_2 is a circle and Z_1, Z_3 are arcs.

We conclude this paper with results concerning embeddings of C_0 coarse spaces, stating that there is a "universal" C_0 coarse space in which all object in **TB** can be embedded. We say that a map $f: X \to Y$ between coarse spaces is a *coarse embedding* if the map $f: X \to f(X)$ is a coarse equivalence. Here $f(X)$ is assumed to have the induced coarse structure $\{F \subset f(X) \times f(X) \mid F \text{ is controlled in } Y\}$. The proof of the next lemma is straightforward.

Lemma 4.7 *Let X be a locally compact metric space with the C_0 coarse structure and $Y \subset X$ be a closed set. Then the induced coarse structure on Y coincides with the C_0 structure for the locally compact metric space Y.* ∎

First, we consider coarse embeddings that are topological embeddings at the same time.

Proposition 4.8 *There exists a separable locally compact metric space X such that for every object Y in* **TB** *admits a map $f: Y \to X$ that is simultaneously a topological and coarse embedding.*

Proof We can take $X = Q \times [0, 1)$, where $Q = [0, 1]^{\mathbb{N}}$ is the Hilbert cube. We define a metric on X as the restriction of any compatible metric on $Q \times [0, 1]$. Let Y be any object of **TB** and let $\widetilde{Y}$ be its completion. We fix a continuous function $\phi: \widetilde{Y} \to [0, 1]$ such that $\phi^{-1}(1) = \widetilde{Y} \setminus Y$ and a topological embedding $j: \widetilde{Y} \to Q$. Then the map $i: \widetilde{Y} \to Q \times [0, 1]$ defined by $i(y) = (j(y), \phi(y))$ gives a topological embedding such that $i^{-1}(X) = i^{-1}(Q \times [0, 1)) = Y$. Let us show that $f = i|_Y: Y \to X$ is the required map. The maps $f: Y \to f(Y)$ and $f^{-1}: f(Y) \to Y$ are proper, since they are homeomorphisms and are uniformly continuous, since they are restrictions of

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continuous maps, namely i and i^{-1}, defined on compact metric spaces. We conclude from Corollary 3.5 that $f\colon Y \to X$ is a coarse embedding. ■

If we admit coarse embeddings that are not topological embeddings (and not even continuous maps), we have the following result by Theorem 4.2.

Theorem 4.9 *For every noncompact locally compact separable metrizable space X, there exists a compatible totally bounded metric d on X such that every object in* **TB** *can be coarsely embedded into (X, d) with respect to the C_0 structure.*

Corollary 4.5 turns every object in **TB** into a contractible space, which is "continuous" in nature. The next corollary of Theorem 4.9 is a result in the opposite direction, saying that every object in **TB** can be expressed as a discrete metric space. Here, a *discrete* metric space means a metric space whose topology is discrete.

Corollary 4.10 *There exists a countable discrete metric space X such that every object in* **TB** *can be coarsely embedded into X with respect to the C_0 structures. Moreover, every object in* **TB** *is coarsely equivalent to some countable discrete metric space with the C_0 structure.*

Proof The first part readily follows from Theorem 4.9. The second part follows from the first part using Lemma 4.7. ■

To prove Theorem 4.9 (and Corollary 4.10), we need some technical lemmas.

Lemma 4.11 *Let X be a locally compact separable metric space with the C_0 coarse structure and $A, B \subset X$ with the induced structures, where $\mathrm{cl}_X A = B$. Then the inclusion $A \to B$ is a coarse equivalence.*

Proof Let $i\colon A \to B$ be the inclusion, which is clearly a coarse map. By Proposition 2.1, there exists a controlled neighborhood E_0 of the diagonal Δ_X in $X \times X$. We define $h\colon B \to A$ by choosing a point $h(b) \in A$ with $(b, h(b)) \in E_0$ for each $b \in B$. It is easy to check that $h\colon B \to A$ is also a coarse map. Then $i \circ h$ is close to the identity id_B, since the set $\{(b, h(b)) \mid b \in B\}$ is contained in E_0 and hence is controlled. Similarly, the other composition $h \circ i$ is close to the identity id_A. We conclude that $i\colon A \to B$ is a coarse equivalence. ■

Remark 4.12 Clearly, this lemma is true for a coarse space X equipped with a topology for which there is a controlled neighborhood of the diagonal Δ_X in $X \times X$, in particular for all proper coarse spaces. Furthermore, if X is such a coarse space, subsets A and B of X are coarsely equivalent with respect to the induced structures whenever they have the same closure, $\mathrm{cl}_X A = \mathrm{cl}_X B$.

Lemma 4.13 *Let X, Y be spaces in* **TB** *and νX, νY be their Higson coronas with respect to the C_0 structures. Let $j\colon \nu X \to \nu Y$ be a topological embedding. Then there exists a coarse embedding $f\colon X \to Y$ such that $\nu f = j$.*

Proof Let $\widetilde{Y} = Y \cup \nu Y$ be the Higson compactification that coincides with the completion. As shown in the proof of Theorem 4.2, there exists a coarse map $f\colon X \to Y$ such that $\nu f = j$. Let $\overline{f(X)}$ be the closure of $f(X)$ in Y. By Proposition 2.6, it is easy to see that $\mathrm{cl}_{\widetilde{Y}} \overline{f(X)} = \mathrm{cl}_{\widetilde{Y}} f(X) = \overline{f(X)} \cup j(\nu X)$. Hence by Corollary 3.10, we have $j(\nu X) = \nu \overline{f(X)}$. Let $f_0\colon X \to \overline{f(X)}$ and $j_0\colon \nu X \to j(\nu X)$ be the maps that are equal to f and j respectively, with their ranges restricted. Then we have $\nu f_0 = j_0$ by Proposition 2.6. Notice that $\overline{f(X)}$ is closed in Y; hence, its C_0 structure coincides with the structure induced from Y by Lemma 4.7. Since j_0 is a homeomorphism, $f_0\colon X \to \overline{f(X)}$ is a coarse equivalence by Theorem 4.2. The coarse equivalence f_0 factors as $X \to f(X) \to \overline{f(X)}$, and the second map is a coarse equivalence by Lemma 4.11. Then it easily follows that the first map $X \to f(X)$ is a coarse equivalence, as desired. ∎

Proof of Theorem 4.9 Recall that every compact metrizable space can be embedded into $Q = [0,1]^{\mathbb{N}}$. In view of Lemma 4.13, to prove this theorem it is enough to notice that there exists a metrizable compactification γX of X with the remainder homeomorphic to Q. Then the restriction to X of any compatible metric on γX satisfies our requirement (then γX is the Higson compactification with respect to the C_0 structure by Corollary 3.10). For completeness, we explain how to construct γX. Since X is noncompact and metrizable, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of distinct points in X without convergent subsequences. Fix a countable dense subset $\{y_n \mid n \in \mathbb{N}\}$ in Q. The map $\{x_n \mid n \in \mathbb{N}\} \to Q = [0,1]^{\mathbb{N}}$ that sends each x_n to y_n can be extended to a continuous map $h\colon X \to Q$ by Tietze's theorem. Let K be the product $(X \cup \{\infty\}) \times Q$, where $X \cup \{\infty\}$ denotes the one-point compactification. The map $i\colon X \to K$ defined by $i(x) = (x, h(x))$ is a topological embedding, and the closure of its image in K is $i(X) \cup (\{\infty\} \times Q)$, which is clearly a metrizable compactification of X with the remainder homeomorphic to Q. ∎

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