

GAMES ON A COMPACT SET

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1. Introduction. Von Neumann's fundamental theorem of the theory of games has been extended by various authors and recently in two different directions by Kneser (8) and Nikaidô (9). We present here a form of the theorem, which is more general than that of both these authors. We develop some consequences of this theorem, which make it easier to decide whether certain classes of games have a value and we give several illustrative examples.

2. The fundamental theorem. Our theorem concerns the following situation: \tilde{X} is a convex subset of a real linear space, Y is any subset of a real linear space and \tilde{Y} is the convex set generated by Y . A real function f is defined on $\tilde{X} \times \tilde{Y}$ and $f(x, y)$ is a concave function of $x \in \tilde{X}$ and a convex function of $y \in \tilde{Y}$; i.e., for $0 \leq \lambda \leq 1$, we have both

$$f(\lambda x_1 + (1 - \lambda)x_2, y) \geq \lambda f(x_1, y) + (1 - \lambda) f(x_2, y)$$

and

$$f(x, \lambda y_1 + (1 - \lambda)y_2) \leq \lambda f(x, y_1) + (1 - \lambda) f(x, y_2).$$

THEOREM 1. *If \tilde{X} is compact in a topology (no separation axiom is assumed) which is such that for every $y \in Y$, $f(x, y)$ is an upper semi-continuous function of x , then*

$$\sup_{x \in \tilde{X}} \inf_{y \in Y} f(x, y) \geq \inf_{y \in \tilde{Y}} \sup_{x \in \tilde{X}} f(x, y).$$

To prove this theorem we follow the procedure of Kneser (8) and establish the following

LEMMA. *If $\{f_1, \dots, f_n\}$ is a finite set of real concave functions, which are upper semi-continuous on a compact, convex set K , and if for all $x \in K$, $\min_{1 \leq i \leq n} f_i(x) < 0$, then there is a mean*

$$g = \sum_{i=1}^n \lambda_i f_i, \quad \sum_{i=1}^n \lambda_i = 1, \quad \lambda_i \geq 0, \quad i = 1, \dots, n$$

such that for all $x \in K$, $g(x) < 0$.

We prove the lemma first for two concave upper semi-continuous functions f and g defined on the compact convex set K , such that $\min(f(x), g(x)) < 0$. Let $F = \{x : f(x) \geq 0\}$ and $G = \{x : g(x) \geq 0\}$. These subsets are compact

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and disjoint. If F , or G , is void, the proof is trivial. We thus assume that both F and G are not void.

From the hypothesis of upper semi-continuity we may find $p \in F$ and $q \in G$ such that

$$(1) \quad \max_{x \in F} f(x)/-g(x) = f(p)/-g(p) = \alpha \geq 0,$$

$$(2) \quad \max_{x \in G} g(x)/-f(x) = g(q)/-f(q) = \beta \geq 0.$$

Since $f(p) \geq 0$ and $f(q) < 0$, we may find $\rho > 0$ and $\sigma \geq 0$ so that $\rho + \sigma = 1$ and $\rho f(p) + \sigma f(q) = 0$. Because f is concave $f(\rho p + \sigma q) \geq 0$, whence by hypothesis and the concavity of g ,

$$\rho g(p) + \sigma g(q) \leq g(\rho p + \sigma q) < 0.$$

Thus using (2) and (1),

$$\rho g(p) < -\sigma g(q) = \sigma \beta f(q) = -\rho \beta f(p) = \rho \alpha \beta g(p).$$

But $\rho > 0$ and $g(p) < 0$ so that $\alpha \beta < 1$. Now choose $\gamma > \alpha$, $\delta > \beta$ so that $\gamma \delta = 1$ and put

$$\lambda = (1 + \gamma)^{-1} = \delta(1 + \delta)^{-1}, \quad \mu = \gamma(1 + \gamma)^{-1} = (1 + \delta)^{-1}.$$

Then if $x \in F$, we have $f(x) \geq 0$, $g(x) < 0$ whence

$$\lambda f(x) + \mu g(x) \leq (\mu - \lambda \alpha)g(x) = (\gamma - \alpha)(1 + \gamma)^{-1}g(x) < 0.$$

Similarly, $\lambda f(x) + \mu g(x) < 0$ if $x \in G$. If $x \notin F$ and $x \notin G$, the statement is trivial.

To extend this to n functions f_1, \dots, f_n , we put $F = \{x : f_n(x) \geq 0\}$. The set F is compact and convex. Supposing the theorem true for the functions f_1, \dots, f_{n-1} on the set F , we have

$$h = \mu_1 f_1 + \dots + \mu_{n-1} f_{n-1}, \quad \sum_{i=1}^{n-1} \mu_i = 1, \quad \mu_i \geq 0, \quad i = 1, \dots, n-1,$$

such that for all $x \in F$, $h(x) < 0$. Since a mean of concave upper semi-continuous functions is concave and upper semi-continuous, we may apply the case $n = 2$ to the functions f_n and h on K to obtain $g = \lambda h + \mu f_n$ such that for all $x \in K$, $g(x) < 0$. Putting

$$\lambda_i = \lambda \mu_i, \quad i = 1, \dots, n-1$$

and $\lambda_n = \mu$ we have

$$g = \sum_{i=1}^n \lambda_i f_i, \quad \sum_{i=1}^n \lambda_i = 1, \quad \lambda_i \geq 0, \quad i = 1, \dots, n.$$

This completes the proof of the lemma.

To prove Theorem 1, put

$$(3) \quad v = \inf_{y \in \bar{Y}} \sup_{x \in \bar{X}} f(x, y).$$

We shall establish that

$$(4) \quad \exists x \in \tilde{X} \forall y \in Y, f(x, y) \geq v.$$

Suppose (4) is not true, then

$$(5) \quad \forall x \in \tilde{X} \exists y \in Y, f(x, y) < v.$$

Put $G_y = \{x : f(x, y) < v\}$, then by (5) and the hypothesis, $\{G_y : y \in Y\}$ is an open covering of \tilde{X} . Because \tilde{X} is compact, we may select $y_1, \dots, y_n \in Y$ so that

$$\tilde{X} = G_{y_1} \cup \dots \cup G_{y_n}.$$

Thus

$$\forall x \in \tilde{X} \exists i \in \{1, \dots, n\}, f(x, y_i) < v,$$

so that the functions $f_i(x) = f(x, y_i) - v, i = 1, \dots, n$ satisfy the hypothesis of the lemma. For some

$$y = \sum_{i=1}^n \lambda_i y_i \in \tilde{Y},$$

we therefore have

$$\forall x \in \tilde{X}, f(x, y) \leq \sum_{i=1}^n \lambda_i f(x, y_i) < v.$$

However $\sum \lambda_i f(x, y_i)$, being an upper semi-continuous function of $x \in \tilde{X}$, achieves its maximum. This is a contradiction of (3) so that (4) is true. But (4) means that

$$\sup_{x \in \tilde{X}} \inf_{y \in Y} f(x, y) \geq v = \inf_{y \in \tilde{Y}} \sup_{x \in \tilde{X}} f(x, y).$$

We have immediately two corollaries. The first is a consequence of the well-known inequality

$$(6) \quad \sup_{x \in \tilde{X}} \inf_{y \in \tilde{Y}} f(x, y) \leq \inf_{y \in \tilde{Y}} \sup_{x \in \tilde{X}} f(x, y),$$

and the second is a consequence of (6) and the fact that if $f(x, y)$ is linear in $y \in \tilde{Y}$, then

$$\inf_{y \in Y} f(x, y) = \inf_{y \in \tilde{Y}} f(x, y).$$

COROLLARY 1. *If $\tilde{Y} = Y$, then*

$$\sup_{x \in \tilde{X}} \inf_{y \in \tilde{Y}} f(x, y) = \inf_{y \in \tilde{Y}} \sup_{x \in \tilde{X}} f(x, y).$$

COROLLARY 2. *If $f(x, y)$ is linear in $y \in \tilde{Y}$, then*

$$\sup_{x \in \tilde{X}} \inf_{y \in \tilde{Y}} f(x, y) = \inf_{y \in \tilde{Y}} \sup_{x \in \tilde{X}} f(x, y).$$

The first corollary generalizes both the theorems of Kneser (8) and Nikaidô (9). Our referee points out that it is the theorem of Pettis (10, Theorem 5) and was observed by Kuhn in his review of Kneser's paper for Mathematical Reviews (14, 301). In the following section we exploit Corollary 2.

We show now that both the above corollaries generalize a theorem of Karlin (7, Theorem 3). We write $f_x(y) = f(x,y)$ and $F = \{f_x : x \in \tilde{X}\}$. We define a mapping T from \tilde{X} onto F by $Tx = f_x$, and we put $(y,Tx) = f(x,y)$. We introduce on F a topology \mathfrak{F} which is generated by the sub-base

$$\{\{Tx : (y,Tx) < a\} : y \in Y, a \in \text{reals}\}.$$

We shall say that the mapping T is *weakly compact* if the range F of T is compact in the topology \mathfrak{F} . Since

$$\{Tx : (y,Tx) < a\} = \{f_x : f(x,y) < a\},$$

it follows that the set F is \mathfrak{F} -compact if and only if the convex set \tilde{X} is compact in the topology \mathfrak{X} generated by the sub-base

$$\{\{x : f(x,y) < a\} : y \in Y, a \in \text{reals}\}.$$

We thus have that \tilde{X} is \mathfrak{X} -compact if and only if T is a weakly compact mapping. However \mathfrak{X} is a topology for Theorem 1, so that *the hypothesis of Theorem 1 is equivalent to the statement that T is weakly compact*.

In this form we may see that Theorem 1 and its corollaries generalize the theorem of Karlin cited above, for Karlin assumes that $\tilde{Y} = Y$, that $f(x,y)$ is linear in y and that T is weakly compact using a topology which is finer than \mathfrak{F} and is therefore weakly compact in the above sense.

The next theorem shows that it is possible to determine whether a game has a value, by examining another associated game.

THEOREM 2. *If f is a function defined on $\tilde{X} \times Y$ and if for each $y \in Y$ there exist numbers $\phi(y) > 0$ and $\psi(y)$ such that \tilde{X} is compact in a topology in which for all $y \in Y$, $g(x,y) = \phi(y)f(x,y) + \psi(y)$ is a concave upper semi-continuous function of x , then*

$$\sup_{x \in \tilde{X}} \inf_{y \in \tilde{Y}} f(x,y) = \inf_{y \in \tilde{Y}} \sup_{x \in \tilde{X}} f(x,y),$$

where f is extended to $\tilde{X} \times \tilde{Y}$ by linearity in y .

Since the set

$$\{x : g(x,y) < a\} = \{x : f(x,y) < -\psi(y) + a/\phi(y)\},$$

\tilde{X} is also compact in a topology such that for each $y \in Y$, $f(x,y)$ is an upper semi-continuous function of x . The theorem therefore follows by Corollary 2.

If the function f is such that for each $y \in Y$,

$$(7) \quad -\infty < \inf_{x \in \tilde{X}} f(x,y) < \sup_{x \in \tilde{X}} f(x,y) < +\infty,$$

then $\phi(y)$ and $\psi(y)$ can be chosen so that for each $y \in Y$,

$$\sup_{x \in \tilde{X}} g(x, y) = 1 \text{ and } \inf_{x \in \tilde{X}} g(x, y) = 0.$$

Thus a game (\tilde{X}, Y, f) satisfying (7) has a value by Corollary 2, if and only if a certain associated normalized game (\tilde{X}, Y, g) has a value by Corollary 2.

3. Special cases. If the hypothesis of Theorem 1 is restricted further, we obtain the following, which is a generalization of the theorems of Choquet (3), Ville (11) and Wald (12).

THEOREM 3. *If X is a compact Hausdorff space, Y is any set and f is a real function on $X \times Y$, which is such that for each y , $f(x, y)$ is a continuous function of x , then*

$$\sup_{\xi} \inf_{\eta} f(\xi, \eta) = \inf_{\eta} \sup_{\xi} f(\xi, \eta)$$

where ξ ranges over the normalized measures on the Borel subsets of X , η ranges over the finite mixtures

$$\eta = \sum_{k=1}^n \eta_k y_k, \quad \sum_{k=1}^n \eta_k = 1, \quad \eta_k \geq 0, \quad k = 1, \dots, n$$

and

$$f(\xi, \eta) = \sum_{k=1}^n \eta_k \int_X f(x, y_k) d\xi(x).$$

This is a consequence of Theorem 1. Let $C(X)$ be the Banach space of all continuous real functions on X , and $C^*(X)$ its conjugate space. The space \tilde{X} of normalized measures on the Borel subsets of X is a closed convex subset (6) of the weak* compact unit sphere in $C^*(X)$. This is a topology such that if $\phi \in C(X)$ then $\phi(\xi) = \int \phi d\xi$ is a continuous function of ξ , so it follows that $f(\xi, y) = \int_X f(x, y) d\xi(x)$ is a continuous function of ξ . The hypotheses of Theorem 1 and its Corollary 2 are therefore satisfied.

The same theorem may be rephrased in the following form.

THEOREM 3'. *If F is a subset of the space $C(X)$ of real continuous functions on a compact Hausdorff space, if \tilde{F} is the convex set generated by F and \tilde{X} is the set of normalized measures on the Borel subsets of X then*

$$\sup_{\xi \in \tilde{X}} \inf_{f \in F} \int_X f d\xi = \inf_{f \in \tilde{F}} \sup_{x \in X} f(x).$$

We may weaken the hypothesis of the compactness of X if we restrict the pay-off function. Thus we obtain

THEOREM 4. *If X is a locally compact Hausdorff space, if Y is any set and if f is a function on $X \times Y$ such that for each $y \in Y$, there is a number $m(y)$ such that $f(x, y) - m(y)$ is a non-negative continuous function which vanishes at infinity, then*

$$\sup_{\xi} \inf_{\eta} f(\xi, \eta) = \inf_{\xi} \sup_{\eta} f(\xi, \eta)$$

where ξ ranges over the normalized measures on the Borel subsets (5) of X , η ranges over the finite mixtures

$$\eta = \sum_{k=1}^n \eta_k \mathcal{Y}_k, \sum_{k=1}^n \eta_k = 1, \eta_k \geq 0, \quad k = 1, \dots, n$$

and

$$f(\xi, \eta) = \sum_{k=1}^n \eta_k \int_X f(x, y_k) d\xi(x).$$

To prove this theorem we let $X_\infty = X \cup \{\infty\}$ be the one-point compactification of X and we define $f(\infty, y) = m(y)$. The hypothesis of Theorem 3 is therefore satisfied for f on $X_\infty \times Y$. We thus have

$$(8) \quad \sup_{\xi \in \tilde{X}_\infty} \inf_{\eta \in \tilde{Y}} f(\xi, \eta) = \inf_{\eta \in \tilde{Y}} \sup_{\xi \in \tilde{X}_\infty} f(\xi, \eta),$$

where \tilde{X}_∞ is the set of normalized measures on the Borel subsets of X_∞ and $f(\xi, \eta)$ is defined as above with the integral taken over X_∞ . Let \tilde{X} be the set of normalized measures on the Borel subsets of X . We shall prove that

$$(9) \quad \sup_{\xi \in \tilde{X}} \inf_{\eta} f(\xi, \eta) \geq \sup_{\xi \in \tilde{X}_\infty} \inf_{\eta} f(\xi, \eta),$$

from which the result follows by using (8) and (6).

Let $\zeta \in \tilde{X}_\infty$ be chosen arbitrarily and suppose that $\zeta(X) = \theta$. Choose any $\xi' \in \tilde{X}$ and define ξ by $\xi(E) = (1 - \theta)\xi'(E) + \zeta(E)$ for every Borel subset E of X . Then because $\theta \leq 1$ we have that $\xi \in \tilde{X}$, and that for all $E, \zeta(E) \leq \xi(E)$. Now put

$$g(x, y) = f(x, y) - m(y).$$

Since $g(x, y) \geq 0$ and $g(\infty, y) = 0$, we have that

$$\forall_{y \in Y}, \int_X g(x, y) d\xi(x) \geq \int_X g(x, y) d\zeta(x) = \int_{X_\infty} g(x, y) d\zeta(x).$$

Thus

$$\forall_{\zeta \in \tilde{X}_\infty} \exists_{\xi \in \tilde{X}} \forall_{y \in Y}, f(\xi, y) \geq f(\zeta, y),$$

and therefore

$$\forall_{\zeta \in \tilde{X}_\infty} \exists_{\xi \in \tilde{X}} \inf_{\eta \in \tilde{Y}} f(\xi, \eta) \geq \inf_{\eta \in \tilde{Y}} f(\zeta, \eta),$$

whence we may obtain the statement (9).

If in the above we suppose, for simplicity, that for all $y, m(y) = 0$, we have the following analogue of Theorem 3':

THEOREM 4'. *If $C(X)$ is the space of continuous real functions which vanish at infinity on a locally compact Hausdorff space X , if F is a subset of $C(X)$ consisting of non-negative functions, and \tilde{F} is the convex set generated by F , then*

$$\sup_{\xi \in \tilde{X}} \inf_{f \in F} \int_X f d\xi = \inf_{f \in \tilde{F}} \sup_{x \in X} f(x).$$

We note that in Theorem 4, if X is a set with its discrete topology, then we obtain the theorem of Dulmage and Peck (4).

4. Examples. We now give some examples to illustrate the results obtained above. The first example shows the usefulness of Theorem 1, Corollary 2, in determining whether a game has a value. What is significant is that, in investigating the compactness of the space of mixed strategies \tilde{X} , it is not necessary to consider the upper semi-continuity of $f(x,y)$ for *all* mixed strategies y , but only for pure strategies $y \in Y$. This means that we are examining \tilde{X} in a weaker topology. Indeed it may happen that \tilde{X} is compact in this weaker topology and not compact in the topologies considered by Kneser (8) and Nikaidô (9). The following example illustrates this situation.

Example 1. Two persons each choose a natural number, say m and n . If $m = n$ the pay-off is 2^m , otherwise it is zero. The spaces of pure strategies X and Y are thus the natural numbers and the spaces of mixed strategies \tilde{X} and \tilde{Y} are probability distributions on X and Y respectively. The pay-off function is $f(m,n) = 2^m \delta_{m,n}$, which when extended to mixed strategies is

$$f(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} 2^m \delta_{m,n} x_m y_n = \sum_{i=1}^{\infty} 2^i x_i y_i.$$

We show that \tilde{X} is not compact in the weakest topology such that for all $y \in \tilde{Y}$, $f(x,y)$ is an upper semi-continuous function of x .

Let $y^{(n)} = \{0, 0, \dots, 0, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\} \in \tilde{Y}$ where the weight on the first n coordinates is zero. Then for each $x \in \tilde{X}$ we have

$$f(x,y^{(n)}) = 2^n(x_{n+1} + x_{n+2} + x_{n+3} + \dots).$$

Let $G_n = \{x: f(x,y^{(n)}) < 2^n\}$, then each G_n is an open set in the topology considered. But since $G_n = \{x: x_{n+1} + x_{n+2} + \dots < 1\}$ the family $\{G_n: n = 1, 2, \dots\}$ is an open covering of \tilde{X} which has no finite sub-covering.

However, we must also show that \tilde{Y} is not compact in the weakest topology such that for all $x \in \tilde{X}$, $f(x,y)$ is a lower semi-continuous function of y . But this is simple, since if

$$H_m = \{y: f(m,y) > 0\} = \{y: y_m > 0\},$$

then $\{H_m: m = 1, 2, \dots\}$ is an open covering with no finite sub-covering.

The compactness of \tilde{X} (and of \tilde{Y}) thus fails in the stronger topology used by Kneser and Nikaido. However it is not difficult to show that \tilde{X} is compact in the topology of Theorem 1. As a matter of fact, this game has a value, also because of Theorem 4, or the theorem in (4). The value of the game is one. *Example 1* does not satisfy the hypothesis of the theorem of Blackwell and Girshick (2, Theorem 2.3.3).

A more interesting example, similar to the above, which does not satisfy the hypothesis in (4), but does have a value by Theorem 4 is the following.

Example 2. Two players each choose a real number, say x and y , and the pay-off is $f(x,y) = \psi(x - y)e^y$, where $\psi(s) = 1 - |s|$ for $|s| \leq 1$ and $\psi(s) = 0$ for $|s| > 1$. The value of the game is zero. If the game is played over the quadrant $x \geq a, y \geq a$, its value is positive and lies between $\frac{1}{2}e^{a-1}$ and e^{a+1} .

The third and fourth are examples of simple games, both of which satisfy the conditions of Theorem 1, (both have the value zero) but which do not satisfy the hypotheses of the theorems of Karlin (7, Theorem 3 and remark 4), Berge (1, § 25), and Wald (13).

Example 3. Both X and Y are the closed interval $[0,1]$ and $f(x,y) = 1$ if $x = y, f(x,y) = 0$ if $x \neq y$.

Example 4. Both X and Y are the sets of natural numbers and the pay-off function is the same as in Example 3.

Example 4 is the associated normalized form of Example 1 in the sense of Theorem 2.

The fifth is an example which shows that Theorem 1 still has its limitations. This is a game which does not satisfy the hypothesis of Theorem 1, but satisfies the hypotheses of the theorems of Blackwell and Girschik (2, Theorem 2.3.3), and Wald (13). Moreover the operator T is completely continuous in the sense of Karlin (7, remark 4).

Example 5. The sets of pure strategies X and Y are the natural numbers and the pay-off function is $f(m,n) = 1$ for $m \geq n$, and $f(m,n) = 1 - 2^{-n}$ for $m < n$. We show that neither \tilde{X} nor \tilde{Y} are compact in the topology of Theorem 1. Let

$$G_n = \{x : f(x,n) < 1 - 2^{-n}\}.$$

Given $x \in \tilde{X}$ we may choose n so that $\sum_{m \geq n} x_m < 2^{-n}$, whence

$$f(x,n) = \frac{1}{2}x_1 + \frac{3}{4}x_2 + \dots + (1 - 2^{1-n})x_{n-1} + \sum_{m \geq n} x_m < (1 - 2^{1-n}) + 2^{-n} = 1 - 2^{-n}.$$

Therefore $\{G_n : n = 1, 2, \dots\}$ is an open covering of \tilde{X} . However if x is the n th pure strategy $x = n$, we have for $j \leq n$ that $f(x,j) = 1 > 1 - 2^{-j}$, so that $x \notin G_1 \cup \dots \cup G_n$. Thus \tilde{X} is not compact.

Now put $H_m = \{y : f(m,y) > 1 - 2^{-m}\}$. Given $y \in \tilde{Y}$ we choose m so that y_m is the first non-zero coefficient of y . Then

$$f(m,y) = y_m + (1 - 2^{-m})(y_{m+1} + y_{m+2} + \dots) > 1 - 2^{-m},$$

so that $y \in H_m$. Therefore $\{H_m : m = 1, 2, \dots\}$ is an open covering of \tilde{Y} . However it has no finite sub-covering because if y is the $(m + 1)$ th pure strategy $y = m + 1$ we have for $i \leq m, f(i,m) = 1 - 2^{-i}$, so that $y \notin H_i$. Thus \tilde{Y} is also not compact.

We shall omit verification that the game satisfies the hypotheses of the theorems cited above.

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