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A CHARACTERIZATION OF THE QUADRATIC IRRATIONALS

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ABSTRACT. Let α be a positive irrational real number, and let $f_\alpha(n) = [(n+1)\alpha] - [n\alpha] - [\alpha], n \geq 1$, where $[x]$ denotes the greatest integer not exceeding x. It is shown that the sequence f_α has a certain 'substitution property' if and only if α is the root of a quadratic equation over the rationals.

1. **Introduction.** The astronomer J. Bernoulli [1] considered the sequence $([(n+1)\alpha + 1/2] - [n\alpha + 1/2] : n \geq 1)$, for positive irrational numbers α, and gave (without proof) an explicit description of the terms of this sequence, based on the simple continued fraction expansion of α.

A. A. Markov [5] proved the validity of Bernoulli's description, and did this by first describing the terms of the sequence $f_\alpha = (f_\alpha(n) : n \geq 1)$, where $f_\alpha(n) = [(n+1)\alpha] - [n\alpha] - [\alpha]$. A beautiful exposition (about 2 pages long) of Markov's proof is given by B. A. Venkov [8].

K. B. Stolarsky [7] gave a different description of the sequence f_α, for certain values of α. A. S. Fraenkel, M. Mushkin, and U. Tassa [3] gave a very short and polished proof which extended Stolarsky's result to all positive α, including rational values. Both [7] and [3] contain extensive lists of references.

Stolarsky gave two proofs of his result. In his second proof, he used Markov's theorem to show that if $\alpha = [0, k, k, \ldots] = \frac{1}{k+}\frac{1}{k+}\frac{1}{k+}\ldots$, then the sequence f_α, which is a sequence of 0's and 1's, is invariant under the substitution $0 \longrightarrow B_1 = 0^{k-1}1, 1 \longrightarrow B_2 = 0^{k-1}10$. To say that f_α is invariant under this substitution means that if each 0 in f_α is replaced by the block B_1, and each 1 is replaced by B_2, then the resulting sequence is identical with f_α. Here 0^{k-1} indicates a block of $k-1$ consecutive 0's, and if $k = 1$ then 0^{k-1} is the empty block. (If $k = 3$, the substitution is $0 \longrightarrow 001, 1 \longrightarrow 0010$.)

In this note we give a simple proof, which does not use Markov's theorem, of a generalization of Stolarsky's result. Our main results are based on the fact that if α is any quadratic irrational, then the simple continued fraction for α is periodic.

Throughout, we use the standard notation $[a_0, a_1, a_2, \ldots]$ for the simple continued fraction $a_0 + 1/(a_1 + 1/(a_2 + \ldots))$.

We show that if $\alpha = [0, \overline{a_1, \ldots, a_m}] = [0, a_1, \ldots, a_m, a_1, \ldots, a_m, \ldots] = [0, a_1, \ldots, a_m + \alpha]$ then there are blocks B_1 and B_2 (not both of length 1) such that f_α is invariant under the substitution $0 \longrightarrow B_1, 1 \longrightarrow B_2$.

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We also show that such blocks B_1 and B_2 cannot be found for *all* quadratic irrationals α.

However, we show that it is true that for every quadratic irrational α, f_α is invariant under a substitution of the following kind. There always exist blocks s, t, C_1, C_2 of 0's and 1's (where C_1 is longer than s or C_2 is longer than t) such that f_α may be written as a sequence of s's and t's, C_1 and C_2 may be written as blocks of s's and t's, and f_α is invariant under the substitution $s \longrightarrow C_1, t \longrightarrow C_2$.

Finally, we show that if f_α is invariant under a substitution in the above sense, then α is a quadratic irrational. Thus the 'substitution property' of f_α characterizes quadratics among the irrationals.

2. **Results.** We will make use of the sequence $g_\alpha = (g_\alpha(n) : n \geq 1)$, the characteristic function of the sequence $([n\alpha] : n \geq 1)$, where $g_\alpha(n) = 1$ if $n = [k\alpha]$ for some k, and $g_\alpha(n) = 0$ otherwise. We will also use the fact (from the definition of f_α) that if j is any integer with $0 \leq j < \alpha$, then $f_\alpha = f_{\alpha-j}$.

It will be convenient to use the notation $1/b = [0, b]$, $1/(a + 1/b) = [0, a, b]$, etc., for positive real numbers a, b.

We begin with a fact which is mentioned by Fraenkel, Mushkin, and Tassa [3]:

LEMMA 1. *For any irrational $\alpha > 1, g_\alpha = f_{1/\alpha}$.*

PROOF. It is straightforward to show from the definitions of g_α and $f_{1/\alpha}$ that $g_\alpha(n) = 1 \Rightarrow f_{1/\alpha}(n) = 1$ and $g_\alpha(n) = 0 \Rightarrow f_{1/\alpha}(n) = 0$.

DEFINITION. Let $k \geq 1$ be fixed, and let w be any block of 0's and 1's or any sequence of 0's and 1's. Then $h_k(w)$ is obtained from w by applying the substitution $0 \longrightarrow 0^{k-1}1, 1 \longrightarrow 0^{k-1}10$, where 0^{k-1} is a block of $k-1$ consecutive 0's. That is, $h_k(w)$ is obtained from w by replacing each 0 in w by $0^{k-1}1$, and each 1 by $0^{k-1}10$. If $k = 1$ the substitution is $0 \longrightarrow 1, 1 \longrightarrow 10$.

LEMMA 2. *Let $k \geq 1$ and α be given, where α is irrational and $0 < \alpha < 1$. Then $h_k(f_\alpha) = g_{k+\alpha} = f_{1/(k+\alpha)}$.*

PROOF. By definition,

$$f_\alpha = f_\alpha(1)f_\alpha(2)\ldots f_\alpha(j)\ldots,$$

where

$$f_\alpha(j) = [(j+1)\alpha] - [j\alpha], j \geq 1,$$

and

$$h_k(f_\alpha) = D_1D_2\ldots D_qD_{q+1}\ldots,$$

where

$$D_j = h_k\big(f_\alpha(j)\big), j \geq 1.$$

Note that each block D_j contains exactly one "1", which is in the kth position, and has length either k or $k+1$.

Consider $h_k(f_\alpha)$ now as a sequence of 0's and 1's, and let n be the position in this sequence of the $(q+1)^{\text{st}}$ "1", that is, the 1 in the block D_{q+1}. Then

$$n = L(D_1D_2\dots D_q) + k,$$

where $L(D_1D_2\dots D_q)$ denotes the *length* of $D_1D_2\dots D_q$.

Since the block D_j has length k if $f_\alpha(j) = 0$ and has length $k+1$ if $f_\alpha(j) = 1$, it follows that

$$L(D_1D_2\dots D_q) = qk + f_\alpha(1) + \cdots + f_\alpha(q).$$

Since $f_\alpha(j) = [(j+1)\alpha] - [j\alpha]$, and $[\alpha] = 0$, the sum telescopes to

$$L(D_1D_2\dots D_q) = qk + [(q+1)\alpha].$$

Thus n, the position of the $(q+1)^{\text{st}}$ "1" in the sequence $h_k(f_\alpha)$, satisfies

$$n = qk + [(q+1)\alpha] + k = [(q+1)(k+\alpha)].$$

Thus $\left[h_k(f_\alpha)(n) = 1\right] \Leftrightarrow \left[n = [(q+1)(k+\alpha)]. \text{ for some } q \geq 0\right] \Leftrightarrow \left[g_{k+\alpha}(n) = 1\right]$. That is, $h_k(f_\alpha) = g_{k+\alpha}$.

Using Lemma 1, this gives

$$h_k(f_\alpha) = g_{k+\alpha} = f_{1/(k+\alpha)}.$$

THEOREM 1. *Let $\alpha = [0, \overline{a_1, \dots, a_m}]$. Then f_α is invariant under the substitution*

$$0 \longrightarrow B_1 = h_{a_1} \circ h_{a_2} \circ \cdots \circ h_{a_m}(0)$$
$$1 \longrightarrow B_2 = h_{a_1} \circ h_{a_2} \circ \cdots \circ h_{a_m}(1),$$

where $\circ$ denotes composition.

PROOF. By Lemma 2, we have $h_{a_m}(f_\alpha) = f_{1/(a_m+\alpha)} = f_{[0,a_m+\alpha]}$, $h_{a_{m-1}} \circ h_{a_m}(f_\alpha) = f_{1/(a_{m-1}+[0,a_m+\alpha])} = f_{[0,a_{m-1},a_m+\alpha]}, \dots, h_{a_1} \circ h_{a_2} \circ \cdots \circ h_{a_m}(f_\alpha) = f_\beta$, where $\beta = [0, a_1, a_2, \dots, a_m + \alpha] = \alpha$.

REMARK. The blocks $B_1 = h_{a_1} \circ h_{a_2} \circ \cdots \circ h_{a_m}(0)$ and $B_2 = h_{a_1} \circ h_{a_2} \circ \cdots \circ h_{a_m}(1)$ can be described as follows. Let $S_0 = 0$, $S_1 = 0^{a_1-1}1$, and for $2 \leq j \leq m$, let $S_j = (S_{j-1})^{a_j} S_{j-2}$. Then $B_1 = S_m$, and $B_2 = S_m S_{m-1}$. This can be seen by induction on m.

COROLLARY. *For any block w of 0's and 1's, let $H(w)$ be obtained from w by replacing each 0 by B_1, and each 1 by B_2. (That is, $H = h_{a_1} \circ h_{a_2} \circ \cdots \circ h_{a_m}$.) Then f_α can be generated by starting with $w = f_\alpha(1)$ and repeatedly replacing w by $H(w)$.*

PROOF. Let $E_1 = f_\alpha(1)$ and for $k \geq 1$ let $E_{k+1} = H(E_k)$. Then by the Theorem and induction, each E_k is an initial segment of f_α.

THEOREM 2. *Let $\beta > 0$ be any quadratic irrational. Since $f_\beta = f_{\beta-[\beta]}$, assume without loss of generality that $0 < \beta < 1$, so that (for suitable a_i, b_j) $\beta = [0, b_1, \ldots, b_q, \overline{a_1, \ldots, a_m}]$. Let*

$$s = h_{b_1} \circ h_{b_2} \circ \cdots \circ h_{b_q}(0)$$
$$t = h_{b_1} \circ h_{b_2} \circ \cdots \circ h_{b_q}(1),$$

and

$$C_1 = h_{b_1} \circ h_{b_2} \circ \cdots \circ h_{b_q} \circ h_{a_1} \circ h_{a_2} \circ \cdots \circ h_{a_m}(0)$$
$$C_2 = h_{b_1} \circ h_{b_2} \circ \cdots \circ h_{b_q} \circ h_{a_1} \circ h_{a_2} \circ \cdots \circ h_{a_m}(1).$$

Then f_β, C_1, C_2 can be written as sequences of s's and t's, and f_β is invariant under the substitution $s \longrightarrow C_1, t \longrightarrow C_2$.

PROOF. Let $\alpha = [0, \overline{a_1, \ldots, a_m}]$, so that $\beta = [0, b_1, b_2, \ldots, b_q + \alpha]$. Let $H_1 = h_{b_1} \circ h_{b_2} \circ \cdots \circ h_{b_q}$, $H_2 = h_{a_1} \circ h_{a_2} \circ \cdots \circ h_{a_m}$, $B_1 = H_2(0)$, $B_2 = H_2(1)$. Then we have $s = H_1(0)$, $t = H_1(1)$, and $C_1 = H_1(B_1)$, $C_2 = H_1(B_2)$, so that C_1, C_2 can be written as blocks of s's and t's.

By the proof of Theorem 1, $H_1(f_\alpha) = f_\beta$, so that f_β can be written as a sequence of s's and t's, and $H_2(f_\alpha) = f_\alpha$. Note that $H_1(f_\alpha) = f_\beta$ is obtained from f_α by applying the substitution $0 \longrightarrow s, 1 \longrightarrow t$, and $H_2(f_\alpha) = f_\alpha$ is obtained by applying the substitution $0 \longrightarrow B_1, 1 \longrightarrow B_2$.

Therefore we can transform f_β into f_β by successively applying the substitutions $\left[s \longrightarrow 0, t \longrightarrow 1\right]$, $\left[0 \longrightarrow B_1, 1 \longrightarrow B_2\right]$, and $\left[B_1 \longrightarrow C_1, B_2 \longrightarrow C_2\right]$, which transform f_β into f_α, f_α into f_α, and f_α into f_β respectively.

THEOREM 3. *Let $\beta = [0, 5, 1, 1, 1, \ldots]$. Then there do not exist non-trivial blocks B_1, B_2 of 0's and 1's such that f_β is invariant under the substitution $0 \longrightarrow B_1, 1 \longrightarrow B_2$.*

PROOF. Let $\alpha = [0, 1, 1, \ldots] = (\sqrt{5} - 1)/2$. Then $f_\alpha(1) = 1$, and by Theorem 1, f_α is invariant under $0 \longrightarrow 1, 1 \longrightarrow 10$, so $f_\alpha = 10110\ldots$. By the proof of Theorem 1, $h_5(f_\alpha) = f_\beta$, so $f_\beta = tstts\ldots$, where $s = 00001$, $t = 000010$. Thus

$$\begin{array}{rcccccccccc} f_\beta & = & 000010 & 00001 & 000010 & 000010 & 00001 & 000010 & 00001 & 000010 & \ldots \\ & = & t & s & t & t & s & t & s & t & \ldots. \end{array}$$

It was shown by Karhumaki [4] that the sequence $f_\alpha = 10110\ldots$ does not contain any 4^{th} power. That is, f_α contains no non-empty block of the form *DDDD*. (See also [2] and [6].) Thus also the sequence $f_\beta = tsttstst\ldots$, *when regarded as a sequence of s's and t's*, contains no non-empty block of the form *DDDD*, *where D is a block of s's and t's.*

Now suppose that f_β is invariant under $0 \longrightarrow B_1, 1 \longrightarrow B_2$, and for any block or sequence w of 0's and 1's, let $H(w)$ be the result of applying this substitution to w. Note that $f_\beta = H(f_\beta) = B_1B_1B_1B_1B_2\ldots$, so that B_1 is some initial segment of f_β.

Our goal is to show that B_1 can be written as a block of s's and t's, which form an initial segment of f_β, *when f_β is regarded as a sequence of s's and t's.* Since $f_\beta = B_1B_1B_1B_1\ldots$, this will contradict Karhumaki's result, and the proof will be finished.

To this end, first note that the block B_1 must contain at least one "1," since otherwise, by the Corollary to Theorem 1, f_β would be identically 0. Note also that f_β consists of blocks of either 4 or 5 consecutive 0's separated by single 1's. This implies that the block B_1 ends either with 1 or with 10, since otherwise $f_\beta = H(f_\beta) = B_1B_1\ldots$ would contain a block of more than five consecutive 0's. (For example if $B_1 = C100$, then since $C = 00001D$ (which is true since B_1 is an initial segment of $f_\beta = 000010\ldots$), we would have

$$f_\beta = H(f_\beta) = B_1B_1\ldots = C100\,C100\ldots = C100\,00001\,D100\ldots,$$

with too many consecutive 0's.)

Suppose now that B_1 *fails* to be an initial segment of s's and t's in f_β (when f_β is regarded as a sequence of s's and t's).

If B_1 ends in 0, then $B1 = Cs0$, where Cs *is* an initial segment of s's and t's *in* f_β (when f_β is regarded as a sequence of s's and t's). Note that $C = 00001D$. Then since Cs is an initial segment of f_β, we have $f_\beta = Cs\underline{0000}1\ldots$, but also we have

$$f_\beta = H(f_\beta) = B_1B_1\ldots = Cs0Cs0\ldots = Cs\underline{0\,0000}1Ds0\ldots,$$

a contradiction.

If B_1 ends in 1, then $B_1 = C00001$, where Ct is an initial segment of s's and t's in f_β (when f_β is regarded as a sequence of s's and t's). Note again that $C = 00001D$. Then since Ct is an initial segment of f_β, we have $f_\beta = Ct\ldots = C00001\underline{0\,0000}1\ldots$, but also we have

$$f_\beta = H(f_\beta) = B_1B_1\ldots = C00001C00001\ldots = C00001\underline{0000}1D00001\ldots,$$

a contradiction.

Thus we have shown that if f_β is invariant under a substitution $0 \longrightarrow B_1, 1 \longrightarrow B_2$, then B_1 can be written as a block of s's and t's, which form an initial segment of f_β, *when* f_β *is regarded as a sequence of s's and t's*. This gives the desired contradiction to Karhumaki's result, and completes the proof.

THEOREM 4. *Let α be a positive irrational real number, and let f_α be written as a sequence on s, t, where s, t are blocks of 0's and 1's. Let C_1, C_2 be blocks of s's and t's such that f_α is invariant under the non-trivial substitution $s \longrightarrow C_1, t \longrightarrow C_2$. Then α is a quadratic irrational.*

PROOF. First consider the case where $0 < \alpha < 1$ and $s = 0, t = 1$. Suppose that C_1 contains a 0's and b 1's, and that C_2 contains c 0's and d 1's.

For any block w of 0's and 1's, let $H(w)$ denote the word obtained from w by replacing each 0 by C_1 and each 1 by C_2. Let $E_1 = H(f_\alpha(1))$ and for $p \geq 1$ let $E_{p+1} = H(E_p)$. Let e_p denote the number of 1's which occur in the block E_p. Since the number of 1's which occur in the block $f_\alpha(1)f_\alpha(2)\ldots f_\alpha(n)$ is $f_\alpha(1) + f_\alpha(2) + \cdots + f_\alpha(n)$, which equals

$[(n+1)\alpha]$, we have $e_p = [(L(Ep)+1)\alpha]$. Also, $e_{p+1} = e_p d + \big(L(E_p) - e_p\big)b$, and $L(E_{p+1}) = e_p(c+d) + \big(L(E_p) - e_p\big)(a+b)$, so that

$$\frac{e_{p+1}}{L(E_{p+1})} = \frac{\frac{e_p d}{L(E_p)} + \left(1 - \frac{e_p}{L(Ep)}\right)b}{\frac{e_p(c+d)}{L(E_p)} + \left(1 - \frac{e_p}{L(E_p)}\right)(a+b)}.$$

Taking the limit as $p \longrightarrow \infty$, we obtain

$$\alpha = \frac{\alpha d + (1-\alpha)b}{\alpha(c+d) + (1-\alpha)(a+b)},$$

so that α is a quadratic irrational.

The general case can be handled similarly. We omit the details.

REFERENCES

1. J. Bernoulli III, *Sur une nouvelle espece de calcul, Recueil pour les astronomes*, Vols. 1, 2, Berlin, 1772.

2. J. Berstel, *Mots de Fibonacci*, Seminaire d'Informatique Theorique, L. I. T. P. Université Paris VI et VII, Année 1980/81, 57–58.

3. A. S. Fraenkel, M. Mushkin, and U. Tassa, *Determination of* $[n\theta]$ *by its sequence of differences*, Canad. Math. Bull. **21**(1978), 441–446.

4. J. Karhumaki, *On cube-free* ω*-words generated by binary morphisms*, Discrete Appl. Math. **5**(1983), 279–297.

5. A. A. Markoff, *Sur une question de Jean Bernoulli*, Math. Ann. **19**(1882), 27–36.

6. A. Restivo, *Permutation properties and the Fibonacci semigroup*, Semigroup Forum **38**(1989), 337–345.

7. K. B. Stolarsky, *Beatty sequences, continued fractions, and certain shift operators*, Canad. Math. Bull. **19**(1976), 473–482.

8. B. A. Venkov, *Elementary Number Theory*, Translated and edited by Helen Alderson, Wolters-Noordhoff, Groningen, 1970, 65–68.

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