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Geometric Characterizations of Hilbert Spaces

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Abstract. We study some geometric properties related to the set

$$\Pi_X := \{(x, x^*) \in \mathsf{S}_X \times \mathsf{S}_{X^*} : x^*(x) = 1\}$$

obtaining two characterizations of Hilbert spaces in the category of Banach spaces. We also compute the distance of a generic element $(h, k) \in H \oplus_2 H$ to Π_H for H a Hilbert space.

1 Introduction

Deville, Godefroy, and Zizler [2] formally introduced the set

$$\Pi_X := \{(x, x^*) \in \mathsf{S}_X \times \mathsf{S}_{X^*} : x^*(x) = 1\}$$

for X a normed space and they use it to define a modulus of the Bishop–Phelps–Bollobás property for functionals. However, the set Π_X appears implicitly in other indices or moduli such as the numerical index of a Banach space, since the numerical range of a continuous linear operator $T \in \mathcal{L}(X)$ can be rewritten as $V(T) := \{x^*(T(x)) : (x, x^*) \in \Pi_X\}$. We refer the reader to [4] for an excellent survey paper on the numerical index of a Banach space.

In this paper we study the geometric properties of the set Π_X and obtain two characterizations of Hilbert spaces in the category of Banach spaces. In our second characterization, Π_H plays a fundamental role when H is a Hilbert space, so the set Π_H must also be studied more accurately.

Recall that a normed space X is said to be smooth provided that at any vector of norm 1 there exists only one functional of norm 1 attaining its norm at the vector. If X is a smooth normed space, then the dual map of X is defined as $\mathrm{J}_X : X \to X^*$ where $\|\mathrm{J}_X(x)\| = \|x\|$ and $\mathrm{J}_X(x)(x) = \|x\|^2$ for all $x \in X$. It is obvious that if X is smooth, then $\Pi_X = \{(x, \mathrm{J}_X(x)) : x \in \mathsf{S}_X\}$. We refer the reader to [3] for a wide perspective on smooth spaces and differentiability of the norm.

It is well known that if H is a Hilbert space, then J_H is a surjective linear isometry, and so we can identify H with H^* via its dual map. After this identification, Π_H turns out to be the intersection of $\mathsf{S}_H \times \mathsf{S}_H$ with the diagonal of $H \times H$.

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2 Extremal Structure of Π_X

Given a normed space X we will define the set

$$\mathrm{E}_X := (\mathrm{ext}(\mathsf{B}_X) \times \mathsf{S}_{X^*}) \cup (\mathsf{S}_X \times \mathrm{ext}(\mathsf{B}_{X^*})).$$

Theorem 2.1 *Let X be a normed space. The following conditions are equivalent.*

(i) $\Pi_X \subseteq \mathrm{E}_X$.
(ii) $\mathsf{S}_X = \mathrm{ext}(\mathsf{B}_X) \cup \mathrm{smo}(\mathsf{B}_X)$.

Proof (i) $\Rightarrow$ (ii). Let $x \in \mathsf{S}_X \setminus \mathrm{ext}(\mathsf{B}_X)$. If $x \notin \mathrm{smo}(\mathsf{B}_X)$, then there are $x^* \neq y^* \in \mathsf{S}_{X^*}$ such that $x^*(x) = y^*(y) = 1$. Notice that $(x, \frac{x^*+y^*}{2}) \in \Pi_X$ but neither x nor $\frac{x^*+y^*}{2}$ are extreme points of their respective balls.

(ii) $\Rightarrow$ (i). Let $(x, x^*) \in \Pi_X$. Assume that $x \notin \mathrm{ext}(\mathsf{B}_X)$. By hypothesis $x \in \mathrm{smo}(\mathsf{B}_X)$. Now if $y^*, z^* \in \mathsf{S}_{X^*}$ and $x^* = \frac{y^*+z^*}{2}$, then $y^*(x) = z^*(x) = 1$ which means that $y^* = x^*$ by the smoothness of x. ■

Recall that an exposed face is the set of all vectors of norm 1 at which a given functional of norm 1 attains its norm. An edge is a maximal segment of the unit sphere which is an exposed face.

Corollary 2.2 *Let X be a normed space.*

(i) *If $\Pi_X \subseteq \mathrm{E}_X$, then every edge of B_X is a maximal face of B_X.*
(ii) *If X is real and 2-dimensional, then $\Pi_X \subseteq \mathrm{E}_X$.*

Proof (i) Let $[x, y] \subset \mathsf{S}_X$ be an edge of B_X and consider $u^* \in \mathsf{S}_{X^*}$ such that $[x, y] = (u^*)^{-1}(1) \cap \mathsf{B}_X$. Suppose to the contrary that $[x, y]$ is not a maximal face of B_X, so then it must be contained in a maximal face C. According to the Hahn–Banach separation theorem, maximal faces are exposed faces, so there exists $v^* \in \mathsf{S}_{X^*}$ such that $C = (v^*)^{-1}(1) \cap \mathsf{B}_X$. Note that $u^* \neq v^*$ since $[x, y] \subsetneq C$. Finally, $\frac{x+y}{2} \in \mathsf{S}_X$, but $\frac{x+y}{2} \notin \mathrm{ext}(\mathsf{B}_X) \cup \mathrm{smo}(\mathsf{B}_X)$.

(ii) If $x \in \mathsf{S}_X \setminus \mathrm{ext}(\mathsf{B}_X)$, then x belongs to the interior of a segment entirely contained in the unit sphere. Since X is real and has dimension 2, there is only one hyperplane supporting B_X on that segment, and hence $x \in \mathrm{smo}(\mathsf{B}_X)$. ■

The next example shows the existence of Banach spaces that can never be equivalently renormed such that $\Pi_X \subseteq \mathrm{E}_X$. For this we will need a bit of background.

Let ω_1 denote the first uncountable ordinal. The space of all bounded real-valued functions on $[0, \omega_1]$ will be denoted by $\ell_\infty(0, \omega_1)$, which becomes a Banach space endowed with the sup norm. The subspace of $\ell_\infty(0, \omega_1)$, composed of those functions with countable support, is denoted by m_0.

Theorem 2.3 *No equivalent norm on m_0 makes $\Pi_{m_0} \subseteq \mathrm{E}_{m_0}$.*

Proof We will divide the proof into two steps.

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Step 1 $\Pi_{m_0} \nsubseteq \mathrm{E}_{m_0}$ when m_0 is endowed with the sup norm. Indeed, note that in this case m_0 endowed with the sup norm isometrically contains ℓ_∞^3. Now observe that Theorem 2.1 shows that the condition $\Pi_X \subseteq \mathrm{E}_X$ is a hereditary property. Finally, it is sufficient to realize that $\Pi_{\ell_\infty^3} \nsubseteq \mathrm{E}_{\ell_\infty^3}$ by virtue of of Corollary 2.2 (i).

Step 2. Assume that m_0 is endowed with any equivalent norm. In accordance with [3, Theorem 7.12], m_0 endowed with any (non-necessarily equivalent) norm has a subspace which is linearly isometric to m_0 endowed with the sup norm. Again, the hereditariness of the condition $\Pi_X \subseteq \mathrm{E}_X$ together with 1 concludes the proof. ∎

3 A Characterization of Hilbert Spaces in Terms of Diagonals

For a topological space X the diagonal of $X \times X$ is denoted by

$$\mathrm{D}_X := \{(x, y) \in X \times X : x = y\}.$$

In case X is a topological vector space, then the anti-diagonal is defined as

$$\mathrm{D}_X^- := \{(x, y) \in X \times X : x = -y\}.$$

The following lemma helps communicate the nature and importance of diagonals in direct products of topological vector spaces.

Lemma 3.1 *Let X be a topological vector space.*

(i) *For every $(x, y) \in X \times X$ we have*

$$(x, y) = \Big(\frac{x+y}{2}, \frac{x+y}{2}\Big) + \Big(\frac{x-y}{2}, \frac{y-x}{2}\Big).$$

(ii) *D_X and D_X^- are topologically complemented in $X \times X$ and both are isomorphic to X.*

Proof (i) Immediate. (ii) It suffices to notice that the linear projection

$$\begin{aligned} P\colon X \times X &\to \mathrm{D}_X \\ (x, y) &\to P(x, y) = \Big(\frac{x+y}{2}, \frac{x+y}{2}\Big) \end{aligned}$$

is continuous and $(I - P)(x, y) = (\frac{x-y}{2}, \frac{y-x}{2})$ or all $(x, y) \in X \times X$. ∎

Theorem 3.2 *Let H be a Hilbert space and consider $H \oplus_2 H$. Then $(\mathrm{D}_H)^\perp = \mathrm{D}_H^-$.*

Proof Let $h, k \in H$. By the parallelogram law we have that

$$\begin{aligned} \|(h, k)\|_2^2 &= \|h\|^2 + \|k\|^2 \\ &= \frac{\|h+k\|^2}{2} + \frac{\|h-k\|^2}{2} \\ &= \Big\|\frac{h+k}{2}\Big\|^2 + \Big\|\frac{h+k}{2}\Big\|^2 + \Big\|\frac{h-k}{2}\Big\|^2 + \Big\|\frac{h-k}{2}\Big\|^2 \\ &= \Big\|\Big(\frac{h+k}{2}, \frac{h+k}{2}\Big)\Big\|_2^2 + \Big\|\Big(\frac{h-k}{2}, \frac{k-h}{2}\Big)\Big\|_2^2. \end{aligned}$$ ∎

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Corollary 3.3 *Let X be a Banach space. If D_X and D_X^- are L^2-complemented in $X \oplus_2 X$, that is, $X \oplus_2 X = \mathrm{D}_X \oplus_2 \mathrm{D}_X^-$, then X is a Hilbert space.*

Proof It suffices to look at the proof of Theorem 3.2 to realize that, under these assumptions, X verifies the parallelogram law and thus it is a Hilbert space. ∎

4 A Characterization of Hilbert Spaces Involving Π_X

If H denotes a Hilbert space, then it is clear that $\Pi_H = (\mathsf{S}_H \times \mathsf{S}_H) \cap \mathrm{D}_H = \sqrt{2}\mathsf{S}_{\mathrm{D}_H}$ provided that $H \times H$ is endowed with the $\|\cdot\|_2$-norm.

Theorem 4.1 *Let X be a Banach space. If there exists a vector subspace V of $X \oplus_2 X^*$ such that $\Pi_X = \sqrt{2}\mathsf{S}_V$, then X is a Hilbert space and $V = \mathrm{D}_X$.*

Proof We will divide the proof into two steps.

Step 1 We will show that X is smooth. Suppose to the contrary that X is not. Then we can find $(x, x^*), (x, y^*) \in \Pi_X$ such that $x^* \neq y^*$. Then

$$(0, x^* - y^*) = (x, x^*) - (x, y^*) \in \Pi_X - \Pi_X \subseteq V.$$

Thus

$$\sqrt{2}\frac{(0, x^* - y^*)}{\|x^* - y^*\|} \in \sqrt{2}\mathsf{S}_V = \Pi_X,$$

which is impossible.

Step 2 According to [1, Theorem 3.2], it is sufficient to show that $\mathsf{J}_X(x+y) = \mathsf{J}_X(x) + \mathsf{J}_Y(y)$ for all $x, y \in \mathsf{S}_X$. So we fix arbitrary elements $x, y \in \mathsf{S}_X$. We may assume that x and y are linearly independent. Note that

$$(x + y, \mathsf{J}_X(x) + \mathsf{J}_X(y)) = (x, \mathsf{J}_X(x)) + (y, \mathsf{J}_X(y)) \in \Pi_X + \Pi_X \subseteq V.$$

Therefore

$$\sqrt{2}\frac{(x + y, \mathsf{J}_X(x) + \mathsf{J}_X(y))}{\sqrt{\|x + y\|^2 + \|\mathsf{J}_X(x) + \mathsf{J}_X(y)\|^2}} \in \sqrt{2}\mathsf{S}_V = \Pi_X.$$

So there exists $z \in \mathsf{S}_X$ such that

$$\sqrt{2}\frac{(x + y, \mathsf{J}_X(x) + \mathsf{J}_X(y))}{\sqrt{\|x + y\|^2 + \|\mathsf{J}_X(x) + \mathsf{J}_X(y)\|^2}} = (z, \mathsf{J}_X(z)).$$

This implies that $z = \frac{x+y}{\|x+y\|}$ and

$$\mathsf{J}_X\Big(\frac{x + y}{\|x + y\|}\Big) = \sqrt{2}\frac{\mathsf{J}_X(x) + \mathsf{J}_X(y)}{\sqrt{\|x + y\|^2 + \|\mathsf{J}_X(x) + \mathsf{J}_X(y)\|^2}}. \tag{4.1}$$

Taking norms and solving for $\|\mathsf{J}_X(x) + \mathsf{J}_X(y\|$ we obtain that

$$\|\mathsf{J}_X(x) + \mathsf{J}_X(y)\| = \|x + y\|.$$

Going to back to Equation (4.1), we deduce that $\mathsf{J}_X(x + y) = \mathsf{J}_X(x) + \mathsf{J}_Y(y)$. ∎

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5 The Distance to Π_H

Our final aim is to find the distance from a generic element $(h,k) \in H \oplus_2 H$ to Π_H for H a Hilbert space. In order to accomplish this, we will make use of Lemmas 5.3 and 5.4. However, to do so, we must first study this issue in a more general context.

Proposition 5.1 *Let X be a normed space and consider Π_X in $X \oplus_2 X^*$. Let $x \in \mathsf{S}_X$ and $y^* \in \mathsf{S}_{X^*}$.*

(i) $d((x,y^*),\Pi_X) \le d(y^*, x^{-1}(1) \cap \mathsf{B}_{X^*})$.

(ii) *If y is norm-attaining, then* $d((x,y^*),\Pi_X) \le d(x,(y^*)^{-1}(1) \cap \mathsf{B}_X)$.

(iii) $|y^*(x) - 1| \le 2d((x,y^*),\Pi_X)$.

Proof (i) Let $x^* \in x^{-1}(1) \cap \mathsf{B}_{X^*}$. Then $(x,x^*) \in \Pi_X$ and so $d((x,y^*),\Pi_X) \le \|(x,y^*) - (x,x^*)\|_2 = \|y^* - x^*\|$, which means that

$$d((x,y^*),\Pi_X) \le d(y^*, x^{-1}(1) \cap \mathsf{B}_{X^*}).$$

(ii) It follows a similar proof as in (i). (iii) Let $(z,z^*) \in \Pi_X$. Note that

$$\begin{aligned} |y^*(x) - 1| &= |y^*(x) - z^*(z)| \le |y^*(x) - z^*(x)| + |z^*(x) - z^*(z)| \\ &\le \|y^* - z^*\| + \|x - z\| \le 2\|(x,y^*) - (z,z^*)\|_2, \end{aligned}$$

which implies that $|y^*(x) - 1| \le 2d((x,y^*),\Pi_X)$. ∎

Corollary 5.2 *Let X be a normed space and consider Π_X in $X \oplus_2 X^*$. If $x \in \mathsf{S}_X$ and $y^* \in \mathsf{S}_{X^*}$ is norm-attaining, then*

$$\frac{|y^*(x) - 1|}{2} \le d((x,y^*),\Pi_X) \le \min\{d(y^*, x^{-1}(1) \cap \mathsf{B}_{X^*}), d(x,(y^*)^{-1}(1) \cap \mathsf{B}_X)\}.$$

Now we can take care of computing the distance of a generic element $(h,k) \in H \oplus_2 H$ to Π_H.

Lemma 5.3 *Let X be a normed space. If $x \in X \setminus \{0\}$, then $d(x,\mathsf{S}_X) = \|x - \frac{x}{\|x\|}\| = \big|\|x\| - 1\big|$.*

Proof Indeed, $d(x,\mathsf{S}_X) \le \|x - \frac{x}{\|x\|}\| = \big|\|x\| - 1\big|$ and if $y \in \mathsf{S}_X$, then

$$\Big\|x - \frac{x}{\|x\|}\Big\| = \big|\|x\| - 1\big| = \big|\|x\| - \|y\|\big| \le \|x - y\|. \tag{5.1}$$ ∎

Lemma 5.4 *Let X be a normed space and assume that $X = M \oplus_p N$ with $1 \le p \le \infty$. Fix arbitrary elements $m \in M$ and $n \in N$.*

(i) $d(m+n, M) = \|n\|$.

(ii)

$$d(m+n,\mathsf{S}_M) = \begin{cases} \sqrt[p]{\|n\|^p + \big|\|m\| - 1\big|^p} & \text{if } p < \infty, \\ \max\{\|n\|, \big|\|m\| - 1\big|\} & \text{if } p = \infty. \end{cases}$$

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Proof (i) Indeed, $d(m+n, M) \leq \|m+n-m\| = \|n\|$ and if $m' \in M$, then

$$\|n\| \leq (\|m-m'\|^p + \|n\|^p)^{\frac{1}{p}} = \|m+n-m'\|_p \quad \text{for } p < \infty,$$
$$\|n\| \leq \max\{\|m-m'\|, \|n\|\} = \|m+n-m'\|_p \quad \text{for } p = \infty.$$

(ii) We may assume that $m \neq 0$ and recalling (5.1), we have that

$$d(m+n, \mathsf{S}_M) \leq \Big\| m + n - \frac{m}{\|m\|} \Big\|_p = \begin{cases} \sqrt[p]{\|n\|^p + \big|\|m\|-1\big|^p} & \text{if } p < \infty, \\ \max\{\|n\|, \big|\|m\|-1\big| & \text{if } p = \infty, \end{cases}$$

and if $m' \in \mathsf{S}_M$, then

$$\sqrt[p]{\|n\|^p + \big|\|m\|-1\big|^p} \leq \sqrt[p]{\|n\|^p + \|m-m'\|^p} = \|m+n-m'\|_p \quad \text{for } p < \infty,$$
$$\max\{\|n\|, \big|\|m\|-1\big|\} \leq \max\{\|n\|, \|m-m'\|\} = \|m+n-m'\|_p \quad \text{for } p = \infty. \qquad \blacksquare$$

The reader may notice that Lemma 5.4 (i) still holds if M and N are simply 1-complemented in X.

Theorem 5.5 *Let H be a Hilbert space and consider $H \oplus_2 H$. For every $h, k \in H$ we have that*

$$d((h,k), \mathrm{D}_H) = \frac{\|h-k\|}{\sqrt{2}},$$
$$d((h,k), \mathsf{S}_{\mathrm{D}_H}) = \Big(\frac{\|h-k\|^2}{2} + \Big| \frac{\|h+k\|}{\sqrt{2}} - 1 \Big|^2 \Big)^{\frac{1}{2}},$$
$$d((h,k), \sqrt{2}\mathsf{S}_{\mathrm{D}_H}) = \Big(\frac{\|h-k\|^2}{2} + \Big| \frac{\|h+k\|}{\sqrt{2}} - \sqrt{2} \Big|^2 \Big)^{\frac{1}{2}}.$$

Proof Notice that $H \oplus_2 H = \mathrm{D}_H \oplus_2 \mathrm{D}_H^-$ in virtue of Theorem 3.2. By applying Lemma 5.4 (i) we deduce that

$$d((h,k), \mathrm{D}_H) = \Big\| \Big(\frac{h-k}{2}, \frac{k-h}{2} \Big) \Big\|_2 = \frac{\|h-k\|}{\sqrt{2}}.$$

In accordance with Lemma 5.4 (ii) we have that

$$d((h,k), \mathsf{S}_{\mathrm{D}_H}) = \Big(\frac{\|h-k\|^2}{2} + \Big| \frac{\|h+k\|}{\sqrt{2}} - 1 \Big|^2 \Big)^{\frac{1}{2}}.$$

Finally,

$$\begin{aligned} d((h,k), \sqrt{2}\mathsf{S}_{\mathrm{D}_H}) &= d\Big(\sqrt{2}\Big(\frac{1}{\sqrt{2}}(h,k) \Big), \sqrt{2}\mathsf{S}_{\mathrm{D}_H} \Big) \\ &= \sqrt{2} d\Big(\Big(\frac{h}{\sqrt{2}}, \frac{k}{\sqrt{2}} \Big), \mathsf{S}_{\mathrm{D}_H} \Big) \\ &= \sqrt{2}\Big(\frac{\|h-k\|^2}{4} + \Big| \frac{\|h+k\|}{2} - 1 \Big|^2 \Big)^{\frac{1}{2}} \\ &= \Big(\frac{\|h-k\|^2}{2} + \Big| \frac{\|h+k\|}{\sqrt{2}} - \sqrt{2} \Big|^2 \Big)^{\frac{1}{2}} \end{aligned} \qquad \blacksquare$$

As we mentioned at the beginning of this section, $\Pi_H = \sqrt{2}\mathsf{S}_{\mathsf{D}_H}$, so we immediately deduce the following final corollary.

Corollary 5.6 *Let H be a Hilbert space and consider $H \oplus_2 H$. If $h, k \in H$, then*

$$d((h,k),\Pi_H) = \Big(\frac{\|h-k\|^2}{2} + \Big| \frac{\|h+k\|}{\sqrt{2}} - \sqrt{2} \Big|^2 \Big)^{\frac{1}{2}}.$$

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