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Towards Bounding Complexity of a Minimal Model

I-HSUN TSAI
Department of Mathematics, National Taiwan University, Taipei, Taiwan.
e-mail: ihtsai@math.ntu.edu.tw

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Abstract. We give some effectivity results in birational geometry. We provide an upper bound on the rational constant in Rationality Theorem in terms of certain intersection numbers, under an additional condition on the variety that it admits a divisorial contraction. One consequence is an explicit bound on the number of certain extremal rays. Our main result tries to construct from a given set of ample divisors H_j on X with their intersection numbers b_i, a certain set of ample divisors L_j on X' or X^+ where X' or X^+ arises from a contraction or a flip, such that the corresponding intersection numbers of L_j are uniformly bounded in terms of b_i and the index of X. This gives a bound on the projective degree of a minimal model in special case.

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Introduction

In birational geometry one of the main achievements during the past two decades has been the construction of minimal models in dimension three, i.e. the Minimal Model Program (MMP for short). Let X be a projective threefold with only $\mathbb{Q}$-factorial terminal singularities. The smallest integer $r = r_X$ such that rK_X is Cartier is the index of X where X is a minimal model if K_X is *numerically effective* (nef) and X has a minimal model if there is a minimal model birational to X. Suppose that K_X is not nef. Let $\rho(\cdot)$ denote the Picard number of a normal projective variety. The theory of extremal rays implies the existence of a surjective morphism $f\colon X \to Y$ (associated with an extremal ray of X) with connected fibers such that $\rho(X) = \rho(Y) + 1$, $-K_X$ is f-ample and the following holds. If $\dim X = \dim Y$, then either f is birational and contracts a divisor (i.e. *a divisorial contraction*), or f is birational and contracts no divisors (i.e. *a small contraction*); if $\dim X > \dim Y$, f is said to be a $\mathbb{Q}$-Fano fibering. If f is a divisorial contraction, Y has only $\mathbb{Q}$-factorial terminal singularities. If f is a small contraction, K_Y is no longer $\mathbb{Q}$-Cartier. In this case, according to MMP one would like to prove that there exists a birational morphism $f^+\colon X^+ \to Y$ from a projective variety X^+ with only $\mathbb{Q}$-factorial terminal singularities and $\rho(X^+) = \rho(X)$, such that f^+ contracts no divisors and K_{X^+} is f^+-ample. If f^+ exists, it is said to be the flip of f (f^+ with properties above is known to be unique). The proof of the existence for f^+ (in dimension three) if f is a small contraction was finally completed by the breakthrough of Mori in [Mo]. It turns out that if X has a minimal model, then repeated application of these two operations (divisorial contractions and flips) will lead to a minimal model of X. For more detail we refer the reader to the book [KM2] which also serves as our main reference throughout the discussion.

In this paper, we try to understand how certain numerical data changes with intermediate varieties which occur in the MMP process. The following effectivity problem seems open. Does there exist a universal, effectively computable function $N(x, y)$ (independent of varieties under consideration) with the following property? Let X and W be normal projective threefolds with only $\mathbb{Q}$-factorial terminal singularities. Assume that X has a minimal model, and that there exists a birational map $g: X \to W$ which is a composition of divisorial contractions and flips. Let H be an ample Cartier divisor on X with $(H^3) \leqslant d$. Then there exists an ample Cartier divisor L on W with $(L^3) \leqslant N(d, r_X)$, r_X the index of X.

In general dimensions, the following definition is of convenience for our purpose.

DEFINITION 0.1. Let X be a normal projective variety and Z be a $\mathbb{Q}$-Cartier Weil divisor on X. The smallest natural number b such that bZ is Cartier is said to be the index of Z. We denote by m_X the least upper bound on the set of such indices provided it is finite.

A finiteness property on m_X will be given in Section 2 under an assumption that X has only rational singularities. As far as our discussions are concerned, this assumption is always satisfied ([KM2], 5.22). For convenience, henceforth we shall freely use the finiteness.

We will provide partial information about some effectivity problems. So let X be an n-dimensional ($n \geqslant 3$) normal projective variety with $\mathbb{Q}$-factorial terminal singularities and nonnegative Kodaira dimension. Let $H_1, H_2, \dots, H_\rho$ be ample Cartier divisors whose classes constitute a basis of $N^1(X)$. Denote by ξ and d upper bounds of (H_1^{n-1}, K_X) and (H_1^{n-1}, H_j), $1 \leqslant j \leqslant \rho$. Most of our results study the effectivity in terms of d and ξ (among others). In the following, there is one more assumption on X.

MAIN THEOREM 0.2. *$N_1(x, y, z, u)$, $N_2(x, y, z, u, v)$ are computable functions with the following property. For any X as in the preceding, assume that X admits a nontrivial divisorial contraction π. Then the following holds.*

(i) *There exists a contraction morphism $\phi: X \to X'$ ($\phi \neq \pi$ in general) with a set of $\rho - 1$ ample Cartier divisors L_i on X', such that $\{L_i\}_{1 \leqslant i \leqslant \rho-1}$ constitutes a basis of $N^1(X')$ and satisfies*

$$(L_1{}^{n-1} \cdot K_{X'}),\ (L_1{}^{n-1} \cdot L_j) \leqslant N_1(n, d, \xi, m_X), \quad 1 \leqslant j \leqslant \rho - 1.$$

(ii) *Suppose ϕ in* (i) *is small and a flip $\phi^+: X^+ \to X'$ exists. Then there exists a set of ρ ample Cartier divisors H_i^+ which give a basis of $N^1(X^+)$ such that the intersection numbers of the preceding type are bounded by $N_2(n, d, \xi, m_X, r_{X^+})$.*

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Remark 0.2.1. (i) The assumption on π above is automatically satisfied in smooth, nonminimal 3-fold case. (ii) If H_1 is very ample, then the dependence on ξ can be dropped [cf. (1.1)]. Further, L_1 (resp. H_1^+) can be chosen to be very ample [cf. (7.2)].

In the flow chart of MMP for a 3-fold X, our result leads to a minimal model Z with a controlable [in the sense of (0.2)] degree d provided no intermediate variety which contains only flipping extremal rays is met in the process. By abuse of language, it may be said the complexity of Z is bounded by d (see [Ca], p. 564 for complexity in another context). The author is tempted to speculate that all minimal models of X have controlable degrees. In the case of the general type, by combining (0.2) with estimates on the number of irreducible components of Chow varieties (e.g. [Ts]), it is possible to get an effective bound (as in, e.g., [Ts]) on the number of minimal models in a special case. Instead of minimal models, there are the following problems of Severi type for which our result might find applications on effectivity (e.g. [Ts] and references therein). Let X be as in (0.2) and consider the set of biregularly inequivalent classes Y (assumed smooth for convenience) which admits a birational morphism f from X. Is this set finite? If so, does it admit a (upper) bound in terms of invariants of X, say the degree d of X? If X is of general type and f is relaxed to be generically finite morphism, similar questions can be asked. An effectivity result of E. Kani for curves ([K], p. 187 and choosing $d = 3(2g-2)$) suggests that bounds of polynomial growth for such Severi-type problems (with dimension fixed) may be impossible. In our case, computable bounds do not seem to overcome the polynomial growth until possibly Section 6 (due to an induction process there).

Our proof of the Main Theorem follows the scheme of MMP. In the first step we provide an effective upper bound on the rational number in the Rationality Theorem in terms of certain intersection numbers. The assumption on the variety admitting a nontrivial divisorial contraction is (only) used here. One direct consequence, combined with estimates on the number of irreducible components of Chow varieties, is an effective upper bound on the number of extremal rays of divisorial type. Our second step is partly based on [KM2], pp. 81–82. By this approach, one tries to cut down the dimension of an *extremal face*, denoted by F_L associated with some nonample, nef Cartier divisor L, so as to reach an *extremal ray* in the end. Our problem will be to control some numerical data of those semi-ample divisors. This can be solved by using the first step. Another problem comes with the control of the dimensional reduction just mentioned. For the purpose of effectivity, one is tempted to bound a certain constant n_0 used in [KM2]. We do not know whether this can be done. However, we show that if n_0 fails to be effectively bounded in a certain way, it will lead to an alternative (effective) method, which is a modification of the method of [KM2], for working out the reduction of dim F_L. The remaining task of producing ample or very ample divisors on targets, with controlled numerical data, is an easy consequence of Rationality Theorem for *nef and big* divisors, and effective very ampleness [cf. (1.4)].

Notation

A normal variety X is said to have *terminal* (resp. *canonical*) singularities if the following (1) and (2) are satisfied.

(1) rK_X is a Cartier divisor for some $r \in \mathbb{N}$ (the smallest $r = r_X$ is defined to be the index of X).
(2) For any smooth resolution $f: Y \to X$ with E_i exceptional divisors, we have $K_Y = f^*K_X + \sum_i a_i E_i$ with $a_i > 0$ (resp. $a_i \geqslant 0$).

$Z_{n-1}(X)$ ($n = \dim X$) and $\mathrm{Div}(X)$ denote respectively the group of Weil divisors and Cartier divisors on X; tensoring with $\mathbb{Q}$ over $\mathbb{Z}$, write $Z_{n-1}(X)_{\mathbf{Q}}$ and $\mathrm{Div}(X)_{\mathbf{Q}}$ for $\mathbb{Q}$-divisors and $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors. X is said to be $\mathbb{Q}$ *-factorial* if every Weil divisor D on X is $\mathbb{Q}$-Cartier. Let $Z_1(X)$ be the group of 1-cycles on X. We have the notion of *numerical equivalence* and *algebraic equivalence*, denoted by $\approx$ and $\equiv$ respectively, so that if X is projective, $N_1(X) = (Z_1(X)/\approx) \otimes_{\mathbb{Z}} \mathbb{R}$ and $N^1(X) = \{$Cartier divisors *on* $X\}/\equiv \otimes_{\mathbb{Z}} \mathbb{R}$ are dual to each other under the natural intersection pairing. The *Picard number* is $\rho(X) = \dim N^1(X)$. If C is a curve on X, $[C]$ denotes the class of C in $N_1(X)$. $NE(X)$ is the convex cone in $N_1(X)$ generated by classes of effective 1-cycles in $Z_1(X)$ and $\overline{NE}(X)$ the closure of $NE(X)$ in $N_1(X)$.

Throughout this article, the general notation $N_i(x, y, z, \ldots)$ $(i = 1, 2, \ldots)$ is meant to be effectively computable functions in terms of variables $x, y, z, \ldots$.

1. Preliminary Effectivity Results

Here we survey some (mostly known) effective bounds, all of which will be needed in later sections.

PROPOSITION 1.1 (*i*) *Let X be a normal projective variety and L a very ample Cartier divisor on X with $(L^n) \leqslant d$, $n = \dim X$. Then $(L^{n-1} \cdot K_X) \leqslant d^2$. (ii) Let M and D be nef, $\mathbb{Q}$-Cartier divisors on X, and D be big. Then $(M^n) \leqslant (D^{n-1} \cdot M)^n/(D^n)^{n-1}$.*

Proof. (i) If X is smooth, the assertion is proved in Lemma 3.5 of [Ts]. By an examination the same argument is applicable here, mainly because the singularities are of codimension at least two. (ii) First assume M is ample. If D is ample, the assertion follows directly from Proposition 2.2 of [LM]. If D is only big and nef, replacing D by $kD + T$ where T is ample and $k \in \mathbb{N}$, gives it as $k \to \infty$. The similar replacement $kM + T$ works if M is only nef. Hence, Proposition 1.1. $\square$

PROPOSITION 1.2. *Let X be a normal projective variety with only $\mathbb{Q}$-factorial terminal singularities and $\phi : X \to X'$ be a contraction morphism. m_X is defined in* 0.1.

(i) *If* $\dim X = 3$, *then* $r_X = m_X$.
(ii) *If* $\dim X = n \geqslant 3$ *and ϕ is divisorial, then* $m_{X'} \leqslant (2n-2)m_X^2$.
(iii) *If ϕ is small with its flip $\phi^+ : X^+ \to X'$, then $m_{X^+} \leqslant m_X r_X r_{X^+}$.*

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Proof. (i) is in ([Ka2], Corollary 5.2), (ii) in ([Ko], Proposition 6.16) and (iii) straightforward from the proof in ([KM], Proposition 3.37). Hence, Proposition 1.2.

Remark 1.2.1. For a three-dimensional flip X^+ the index r_{X^+} was explicitly studied by Kollár and Mori ([KM], Section 13) for *exceptional* cases.

As a corollary to the Rationality Theorem (cf. [KM2], Theorem 3.5) one has

COROLLARY 1.3. *Let X be a projective variety with only canonical singularities such that K_X is not nef. Suppose L is a big and nef Cartier divisor such that $mL + rK_X$ is ample for sufficiently large m with $r = r_X$. Then the $\mathbb{Q}$-divisor*

$$L + \frac{1}{r(n+1)+1} K_X$$

is ample, $n = \dim X$.

Combining the Rationality Theorem and [Ko2] gives

PROPOSITION 1.4 (Effective very ampleness). *There are explicit positive integers $a(n)$, $b(n)$ such that if Z is any n-dimensional normal projective variety with only canonical singularities and H an ample, Cartier divisor on Z, then $ar_Z K_Z + br_Z H$ is very ample.*

COROLLARY 1.4.1. *In the notation of Proposition* 1.4, *there exists a very ample divisor L on X with the property that*

$$(L^n) \leqslant (a(n)r_Z d + b(n)r_Z \xi)^n / d^n, \quad d = (H^n),\ \xi = (H^{n-1} \cdot K_X).$$

Let X be a normal projective variety with only canonical singularities, L be an ample Cartier divisor on X and l an extremal ray of X. Write $l = \mathbb{R}_{\geqslant 0}[C]$ for an irreducible (reduced) curve C. We shall take C to satisfy the property that for any irreducible (reduced) curve C_1 with $[C_1] = e[C]$ we have $e \geqslant 1$. (The existence of such a curve is easily checked.)

DEFINITION 1.5. For the sake of convenience we call C an *extremal curve* associated with l if there is no danger of confusion.

Let X be an n-dimensional normal projective variety with canonical singularities. Let l be any extremal ray of X. There exists a rational curve C' with $[C'] \in l$ such that $-2n \leqslant (K_X \cdot C')(<0)$ (cf. Theorem 3.7 of [KM2]). If $n = 3$ one has

PROPOSITION 1.6. *Assume furthermore X is $\mathbb{Q}$-factorial with nonnegative Kodaira dimension, then $-4 < (K_X \cdot C)(<0)$ if* $\dim X = 3$.

Proof. Let $\phi: X \to X'$ be the contraction morphism associated to l. If ϕ is small, one has (e.g. [CKM], (14.5.7)) $-1 \leqslant (K_X \cdot C)(<0)$. Suppose ϕ is divisorial with E the exceptional locus of ϕ. If $\phi(E)$ is a curve, let C' be a generic fiber of $\phi|_E$, which is a rational curve (cf. [CKM], (14.5.3)). The argument in (14.5.7) of [CKM] applies here (because $\phi^{-1}(p)$ is at most one-dimensional for $p \in X'$ so that the vanishing result $(*)$ in (14.5.6) of [CKM] remains true), so $-1 \leqslant (K_X \cdot C')(<0)$. If $\phi(E)$ is a point, then by Remark 6.15 of [Ko] E is covered by rational curves C' satisfying $-((E+K_X)\cdot C') \leqslant 2(n-1)$. Using $(E, C') < 0$ gives $-(K_X \cdot C') < 4$. Hence, $-(K_X \cdot C) < 4$, as desired. □

2. Finiteness Property on Indices of Divisors

Let X be a normal projective variety of dimension $n \geqslant 3$ with only rational singularities.

PROPOSITION 2.1. *Let X be as above. Then the set of indices for $\mathbb{Q}$-Cartier Weil divisors on X is finite.*

Motivated by a result of Kollár ([Ko3], (2.1.8)) one first proves

LEMMA 2.1.1. *Let X be as above, and L a line bundle on $X\backslash Z$ where Z is a subvariety of codimension at least two. Suppose $c_1(L)$ is contained in the image $H^2(X, \mathbb{Z}) \to H^2(X\backslash Z, \mathbb{Z})$. Then L extends to a line bundle on X.*

Proof. First assume $\dim Z = 0$. Let U be a Stein neighborhood of Z, so $\mathrm{Pic}(U) \cong H^2(U, \mathbb{Z})$ (e.g. [KM2], (4.13)). A preimage $\bar{c} \in H^2(X, \mathbb{Z})$, of $c_1(L)$, gives an $M \in \mathrm{Pic}(U)$. After a possible shrinking of U, $c_1 : \mathrm{Pic}(U\backslash Z) \to H^2(U\backslash Z, \mathbb{Z})$ is injective (cf. [Fl], 6.1). One concludes $M|_{U\backslash Z} \cong L|_{U\backslash Z}$; L extends. The case $\dim Z > 0$ can be done by exactly the same induction in (2.1.8) of [Ko3]. Hence, Lemma 2.1.1. □

Proof of Proposition 2.1. Let $f: Y \to X$ be a smooth resolution of singularities such that $f|_{Y\backslash E} : Y \setminus E \to X\backslash\mathrm{Sing}(X)$ is an isomorphism, where $E = \cup_{1 \leqslant j \leqslant m} E_j$ is the exceptional locus of f. The following construction is similar to that by Kawamata in Proposition 5.5 of [Ka]. Let L, M be very ample divisors on Y, X. Write $b_1 < b_2 < \cdots < b_p \leqslant n-2$ and $0 = s_0 < s_1 < s_2 < \cdots < s_p = m$ for the numbers satisfying that after a reindex of E_j, $\dim f(E_j) = b_r$ for $j \in I_r = [s_{r-1}+1, s_r]$. Fix a r with $1 \leqslant r \leqslant p$. Take general members $L_i \in |L|$ and $M_j \in |M|$ with $1 \leqslant i \leqslant n-2-b_r$ and $1 \leqslant j \leqslant b_r$, set $M_j{}'$ to be the strict transform of M_j and let $S_r = \cap_i L_i \cap_j M_j{}'$. For $k \in I_r$, write $C_k = E_k \cap S_r$. Then it can be checked that $f(C_k)$ is zero-dimensional, and the matrix (E_i, C_j) is nondegenerate by using Hodge index theorem. For that check one useful ingredient is that if $r' < r$ and $k' \in I_{r'}$, then the curve $E_{k'} \cap S_r$ is disjoint from C_k.

Given a $\mathbb{Q}$-Cartier divisor D. $f^*D = D' + \sum d_i E_i$ where D' is the proper transform of D and $d_i \in \mathbb{Q}$. The equations $(f^*D \cdot C_j) = 0$ give an integer l dependent only on

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$(E_i \cdot C_j)$ such that $ld_i \in \mathbb{Z}$ for each i. Write $\iota_X : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Q})$, and denote by $H^2(X, \mathbb{Z})_{\mathrm{tor}}$ the torsion subgroup; similar notation for Y. Define finitely generated subgroups of $H^2(Y, \mathbb{Q})$:

$$\iota_Y f^* H^2(X, \mathbb{Z}) := A, \quad f^* H^2(X, \mathbb{Q}) \cap \iota_Y H^2(Y, \mathbb{Z}) := B; \quad A \subset B.$$

It follows, for some l', $l'B \subset A$. One finds $ll'f^*D$ is Cartier and $c_1(ll'f^*D) \in \iota_Y f^* H^2(X, \mathbb{Z})$. Since by Leray spectral sequence and $R^1 f_* \mathbb{Q} = 0$ (cf. [KM], (12.1.3)), $f^* : H^2(X, \mathbb{Q}) \to H^2(Y, \mathbb{Q})$ is injective, one concludes that $c_1(ll'D) \in \iota_X H^2(X, \mathbb{Z})$, i.e. $c_1(ll'D) = \iota_X(c)$ for $c \in H^2(X, \mathbb{Z})$. Moreover, since $ll'f^*D$ is Cartier and $aH^2(Y, \mathbb{Z})_{\mathrm{tor}} = 0$ for some $a \in \mathbb{N}$, $a(f^*c - c_1(ll'f^*D)) = 0$ in $H^2(Y, \mathbb{Z})$, so $ac - c_1(all'D|_{X \setminus Z}) = 0$ in $H^2(X \setminus Z, \mathbb{Z})$ by $f : Y \setminus E \simeq X \setminus Z$, giving $all'D$ is Cartier by Lemma 2.1.1. Proposition 2.1 is proved. □

3. A Bound on Picard Number

The following bound on Picard number will be needed in later sections.

PROPOSITION 3.1. *There exists an* $N(x, y, z, w)$ *such that if* Z *is any normal projective variety with terminal singularities and nonnegative Kodaira dimension, and* H *is an ample Cartier divisor on* Z, *then* $\rho(Z) \leqslant N(n, d, \xi, r_Z)$, $n = \dim Z$, $(H^{n-1} \cdot K_Z) \leqslant \xi$ *and* $(H^n) \leqslant d$.

We use the following lemma.

LEMMA 3.1.1. *Let* X *be a smooth minimal surface of general type. Then* $h^{1,1}(X) \leqslant 4c_1^2(X) + 30$.

Proof. First $h^{1,1} = 2 + 2p_g - \frac{1}{3}c_1^2 + \frac{2}{3}c_2$ with p_g the geometric genus by [Hir] and Hodge index theorem. For a minimal surface X of general type, one has the Noether inequality $p_g \leqslant \frac{1}{2}c_1^2 + 2$ and $c_2 \leqslant 5c_1^2 + 36$ (cf. [BPV], p. 210-211). Hence, Lemma 3.1.1. □

Proof of Proposition 3.1. First assume that Z is smooth. It suffices to bound $h^{1,1}(Z)$. By the effective very ampleness (1.4) replacing H by L of the form $L = aH + bK_Z$, we assume L and $L' = (n-2)L + K_Z$ are very ample. Let S be a smooth surface cut out by $n-2$ generic members of $|L|$. By Lefchetz Hyperplane Theorem (cf. [GH]), and Lemma 3.1.1 it suffices to estimate (K_S^2). One has $(K_S^2) = (L'^2 \cdot L^{n-2}) \leqslant ((L' + L)^n)$, on which applying Proposition 1.1 yields a bound. If Z is not smooth, a small modification gives very ample divisors L, L' (with K_Z replaced by $r_Z K_Z$) and S similarly. Since Z is smooth in codimension two (e.g. [CKM], (6.13)) and $\rho(S) = h^{1,1}(S)$ (e.g. (19.3.1) and (19.1.5) of [Fu]), it suffices to show that

$$\rho(Z) \leqslant \rho(S). \tag{3.1.2}$$

If D is any Cartier divisor on Z such that its restriction to S is numerically zero, then by ([Fu], Example 19.3.3, p. 389) D is numerically zero on Z, yielding (3.1.2). The proof of Proposition 3.1 is completed.

4. An Effective Result on Rationality Theorem

Let X be a normal projective variety. Assume that X is $\mathbb{Q}$-factorial. Let $f\colon X \to Y$ be the contraction morphism associated with an extremal ray $l \in \overline{NE}(X)$. Assume the Kodaira dimension of X is non-negative. If f is divisorial (resp. small) (e.g. [KM2], Proposition 2.5), we say l is of *divisorial type* (resp. *flipping type*).

The main result of this section is the following.

THEOREM 4.1. *There exists a computable function* $N(x, y, z, w)$ *with the following property. Let* X *be as in Main Theorem* 0.2. H *an ample Cartier divisor on* X *and define*

$$r = r_H = \max\{t \in \mathbb{R} : H + tK_X \text{ nef}\}$$

Then

$$r \leqslant N(n, d, \xi, m_X).$$

where $(H^n) \leqslant d$, $(H^{n-1} \cdot K_X) \leqslant \xi$ *and* $n = \dim X$.

We need the following lemmas for the proof.

LEMMA 4.1.1. *Let* L, D *and* E *be* $\mathbb{Q}$*-Cartier divisors on a projective variety* X *of dimension* n. *Fix nonnegative integers* p *and* q *with* $p + q = n - 1$. *Suppose the intersection numbers* $(L^q \cdot (L + D)^i \cdot (L - D)^{p-i} \cdot E) \geqslant 0$ *for all* i *with* $0 \leqslant i \leqslant p$. *Then* $|(L^q \cdot D^p \cdot E)| \leqslant (L^{n-1} \cdot E)$.

Proof. Write $(L^{n-1} \cdot E) \pm (L^q \cdot D^p \cdot E) = (L^q \cdot L^p \cdot E) \pm (L^q \cdot D^p \cdot E)$, which equals

$$\sum_{i=0}^{i=p} c_i^{\pm} (L^q \cdot (L + D)^i \cdot (L - D)^{p-i} \cdot E)$$

by the simple algebraic identity:

$$a^p \pm b^p = \sum_{i=0}^{i=p} c_i^{\pm} (a + b)^i (a - b)^{p-i},$$

where

$$c_i^{\pm} = \frac{1}{2^p}\left(\binom{p}{i} \pm (-1)^{p-i}\binom{p}{i}\right) \geqslant 0.$$

Hence, Lemma 4.1.1 follows. □

LEMMA 4.1.2. *Let* $f\colon W \to V$ *be a birational morphism between* $\mathbb{Q}$*-factorial normal projective varieties with terminal singularities. Let* E *denote the exceptional locus of* f

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and N be any nef Cartier divisor on W. Suppose the Kodaira dimension of W is non-negative. Then $(N^{n-1} \cdot E) \leqslant r_V(N^{n-1} \cdot K_W)$, $n = \dim W$ *and* r_V *the index of V.*

Proof. One has (e.g. [KM2], Section 2.3)

$$r_V K_W = \phi^* r_V K_V + \sum_i q_i E_i, \quad q_i \in \mathbb{N}. \tag{4.1.2.1}$$

where $\cup_i E_i = E$. Since $|mK_W| \neq \emptyset$ and $|m\phi^* K_V| \neq \emptyset$ for m large, one has

$$(N^{n-1} \cdot K_W) \geqslant 0, \quad (N^{n-1} \cdot \phi^* K_V) \geqslant 0.$$

It follows from (4.1.2.1) that $(N^{n-1} \cdot E) \leqslant (N^{n-1} \cdot r_V K_W)$. Hence Lemma 4.1.2. □

Proof of Theorem 4.1. Let C be an extremal curve of divisorial type in X. Write $\pi : X \to Y$ for the extremal contraction with respect to C, and E the exceptional divisor of π. Let

$$r_C = -\frac{(H \cdot C)}{(K_X \cdot C)} \tag{4.1.3}$$

so that $((H + r_C K_X) \cdot C) = 0$. It follows that

$$H + r_C K_X \in \pi^*(\mathrm{Pic}(Y)) \otimes \mathbb{Q} \tag{4.1.4}$$

(e.g. [KM2, p. 76]). Since $((H + rK_X) \cdot C) \geqslant 0$ and $(K_X \cdot C) < 0$, one has

$$r \leqslant r_C. \tag{4.1.5}$$

It suffices therefore to bound r_C.

For bounding r_C our method needs to modify H first. By Corollary 1.3 $[r_X(n+1)+1]H + K_X := L'$ is ample. Let $L = L' + (r_X(n+1)+1)H$. Then L, $L + K_X$ $(= 2L')$ and $L - K_X$ $(= \text{constant} \cdot H)$ are ample, $\mathbb{Q}$-Cartier. Rewriting r_C gives

$$r_C = \frac{1}{2(r_X(n+1)+1)} \frac{(L \cdot C)}{(-K_X \cdot C)} + \frac{1}{2(r_X(n+1)+1)}. \tag{4.1.6}$$

Note that the intersection numbers $(L^n) \cdot (L^{n-1}.K_X)$ associated with L can be bounded in terms of those with H by using Proposition 1.1. In view of (4.1.6) and the preceding note, there would be no harm if we replace H by L. In the following let H be L, still denoted by H, with the assumption that both $H + K_X$ and $H - K_X$ are ample.

Since E is mapped to a subvariety of codimension at least two, by (4.1.4) it follows

$$\big((H + r_C K_X)^{n-1} \cdot E\big) = 0$$

which is a polynomial equation $P(r_C) = 0$ of degree no more than $n-1$. An explicit bound on r_C is surely obtainable if one can bound the coefficients of $P(x)$, for which one is reduced to bounding $|(H^q \cdot K_X^p \cdot E)|$, $p + q = n - 1$. By Lemmas 4.1.1, 4.1.2 and 1.2,

$$|(H^q \cdot K_X^p \cdot E)| \leqslant (H^{n-1} \cdot E), \quad p + q = n - 1,$$

and

$$(H^{n-1} \cdot E) \leqslant m_X^2(2n-2)(H^{n-1} \cdot K_X).$$

Therefore the proof of Theorem 4.1 is completed. □

Somewhat unexpectedly, (ii) of the following corollary bounds degrees of certain extremal curves which are not necessarily of divisorial type. This will be needed in (6.1.1).

COROLLARY 4.2. (i) *In the notation of Theorem* 4.1, *let* C *in* X *be any extremal curve of divisorial type. Then* $(H.C) \leqslant 2nN(n, d, \xi, m_X)$. (ii) *Let* $C' \in F_{H+r_H K_X}$ *be any extremal curve* (*not necessarily of divisorial type*). *Then* $(H \cdot C') \leqslant 2nN(n.d, \xi, m_X)$.

Proof. (i) follows from (4.1.3), (1.6) and a bound of r_C; (ii) from $(H \cdot C')/(-K_X \cdot C') = r$ and $r \leqslant r_C$ by (4.1.5). Hence, Corollary 4.2. □

COROLLARY 4.3. *There exists an* $N(x, y, z, w)$ *such that in the notation of Theorem* 4.1, *the number of extremal rays of divisorial type in* X *is bounded* (*above*) *by* $N(n, d, \xi, m_X)$.

Proof. Let $\{C_i\}$ be a set of extremal curves of divisorial type. One finds a very ample divisor L and bounds $(L \cdot C_i)$ by, say d' (cf. 1.4.1, 4.2 and 1.1). Form a Chow variety $\mathcal{C}_m$ of irreducible curves (in X) of degree m (w.r.t. L). Set $\mathcal{C} = \coprod_{m \leqslant d'} \mathcal{C}_m$. Since C_i and C_j $(i \neq j)$ fall in different irreducible components of $\mathcal{C}$ as $[C_i] \neq [C_j]$, by explicit estimates on the number of irreducible components of $\mathcal{C}_m$ in terms of m and degree of X (e.g. [Ts], Proposition 3.1), one obtains this corollary. Indeed a glance at (3.1) of [Ts] with the notation there requires, in addition to m and $\deg X = (L^n)$, the information for N and $h^0(G(N-k-1, N), \mathcal{O}_G(m))$ with $k = 1$, where $N = h^0(X, L) - 1$. Now $\deg X \geqslant N - n + 1$ (e.g. [GH], p. 173). And $h^0(G(N-k-1, N), \mathcal{O}_G(m)) \leqslant C(C(N+m), m, k+1)$ where $C(a, b)$ denotes the binomial coefficient. For this proof we refer to ([Ca], p. 578). Hence Corollary 4.3. □

The following discussion on smooth 3-folds is included for its own right. In notation above, bounding r_C is equivalent to bounding $(H.C)$. For smooth 3-folds one can bound $(H.C)$ directly. This method gives a better bound of r_H.

PROPOSITION 4.4. *Let* X *be a smooth projective threefold with nonnegative Kodaira dimension*, H *and* r_H *as in Theorem* 4.1. *Then* $r_H \leqslant 4(H^2 \cdot K_X)$.

We will use the following lemma.

LEMMA 4.4.1. *Let* $\pi : S \to C$ *be a ruled surface with an ample divisor* M *on* S. *Then* $(M \cdot f) \leqslant (M^2)$ *where* f *is a fiber* (*isomorphic to* $\mathbb{P}^1$).

Proof. Write $X \cong \mathbb{P}(E)$ for some vector bundle E of rank 2 over C. There exists a section C_0 of $\pi: X \to C$ such that $C_0 = \mathcal{O}_{\mathbb{P}(E)}(1)$ as line bundles ([Ht], p. 373).

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Put $-e = (C_0^2)$ and write $M = aC_0 + bf$ as numerical classes (f a fiber) ([Ht], p. 370). Thus $(M \cdot f) \leqslant (M^2)$ is equivalent to $a \leqslant -a^2e + 2ab$. For $e \geqslant 0$, $b \geqslant ae + 1$ by ([Ht], p. 382) and for $e < 0$, $b \geqslant \frac{1}{2}ae + \frac{1}{2}$ ([Ht], p. 382). Thus one has proved Lemma 4.4.1. □

Proof of Proposition 4.4. Let C be any extremal curve in X with $\phi: X \to X'$ the associated (divisorial) contraction morphism. It suffices to bound $(H \cdot C)$. Let E be the exceptional locus of ϕ. According to a classification (cf. [Mo2], Theorem 3.3), there are five cases for ϕ and E. For cases where $\phi(E)$ is a point and E is $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or an irreducible reduced singular quadric surface $\mathbf{Q}$ in $\mathbb{P}^3$, the divisor class group $\mathrm{Cl}(E)$ is $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$. Write

$$H|_E = \sum_i a_i C_i, \quad a_i \in \mathbb{Z},$$

where $C_i \subset E$ are (reduced) irreducible curves. Since any curve C' of X with $\phi(C')$ being a point must be numerically a multiple of C, we put $[C_i] = e_i[C]$ with $e_i \in \mathbb{R}$ and $e_i \geqslant 1$ (C being extremal). It follows the image of $\mathrm{Cl}(E)$ in $N_1(X)$ is isomorphic to $\mathbb{Z}$. Hence, $e_i \in \mathbb{N}$, $[H|_E] = q[C]$ in $N_1(X)$ and $q \in \mathbb{N}$. Thus $(H \cdot C) \leqslant (H \cdot H|_E) = (H^2.E)$. The remaining case where $\phi|_E: E \to \phi(E)$ is a ruled surface is solved by (4.4.1). By (4.1.2.1) and (1.2), the proof of Proposition 4.4 is completed.

5. Review of Cone Theorem

The Cone Theorem was first treated by Mori [Mo2] for smooth threefolds. For the general result after Mori, we refer to (9.10) of [CKM] for a detailed historical account. In Lecture 11 of [CKM] or [KM2, p. 82] a new proof of part of the Cone Theorem was given due to J. Kollár, T. Luo, K. Matsuki and S. Mori. We shall describe some basics of their method. Being included for the convenience of the reader this section is expository in nature; all of what follows is taken from [KM2] except the definition of $v_L(G)$.

Let L be any nonample, nef Cartier divisor such that $L^\perp$ does not meet $(\overline{NE}(X) \cap (K_X)_{\geqslant 0})$ except at 0. Define F_L to be $L^\perp \cap \overline{NE}(X)$. F_L is the *extremal face* of $\overline{NE}(X)$ associated with L. Note it follows that $(K_X \cdot z) < 0$ if $z \in F_L$. L is also called a *supporting function* for F_L. Given any ample divisor G set

$$r_L(v, G) = \max\{t \in \mathbb{R} : vL + G + (t/e)K_X \text{ is nef}\}, \quad e = [r_X(\dim X + 1)]!,\ v \in \mathbb{N},$$

which is an integer by Rationality Theorem.

The goal of the following is to get a supporting function for an extremal ray.

(i) $r_L(v, G)$ is nondecreasing and stabilizes to a positive integer $r_L(G)$ for $v \gg 0$. For our purpose we assume $v_0 \equiv v_L(G)$ to be the minimum among integers v_1 with the property that $r_L(v, G) = r_L(G)$ if $v \geqslant v_1$. Note that $D(vL, G) \equiv veL + eG + r_L(G)K_X$ where $v \geqslant v_0$, is a nonample, nef divisor such that $F_{D(vL,G)}$ is well-defined in the sense above.

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(ii) $F_{D(\nu L,G)}$ is contained in F_L provided that $\nu \geqslant 1+\nu_0$. Moreover $F_{D(\nu' L,G)} = F_{D(\nu L,G)}$ if $\nu' \geqslant \nu > \nu_0$, which is seen by the equality

$$D(\nu' L, G) = D(\nu L, G) + (\nu' - \nu)eL.$$

(iii) Suppose $\dim F_{D(\nu L,G)} = \dim F_L > 1$, $\nu \gg 0$. Then the set of equations, in which the ample divisors $\{H_i\}_i$ give a basis of $N^1(X)$,

$$\left\{\left[\nu L + H_i + \left(\frac{r_L(H_i)}{e}\right)K_X\right] \cdot z\right\} = 0, \quad z \in N_1(X),\ \nu \gg 0,$$

cannot all be satisfied on F_L. Thus there is an i such that $\dim F_{D(\nu L,H_i)}$ is strictly less than $\dim F_L$ if $\nu \geqslant 1 + \nu_L(H_i)$.

Repeating the argument over successfully smaller faces for ρ_0 times where $\rho_0 < \rho(X)$, one obtains an L' such that $F_{L'} \subset F_L$ and $\dim F_{L'} = 1$.

6. Construction of a Supporting Function with Effective Bounds

The goal of this section is to construct a supporting function in an effective way.

PROPOSITION 6.1. *There exists an* $N(x, y, z, w)$ *with the following property. Let the notation and assumption be as in Main Theorem. There exists a Cartier divisor*

$$D = vK_X + \sum_i u_i H_i$$

on X, *with the property that it is a supporting function for an extremal ray (not necessarily of divisorial type) in* X, *and satisfies*

$$0 \leqslant v, u_i \leqslant N(n, d, \xi, m_X), \quad \forall i.$$

Proof. We shall freely use the notation of Section 5. Set $M = \sum_{j=1}^{j=\rho} H_j$. Consider $L = eM + r(M)K_X$ such that L is nef but non-ample. Write $T = H_i$ with $1 \leqslant i \leqslant \rho$.

(6.1.1) *Bound of* $r(M)$ *and* $r_L(T)$. Theorem 4.1 (and 1.1) controls $r(M)$. For $r_L(T)$, let C be any extremal curve in $F_{D(\nu L,T)}$ so that $r_L(T) = e(T \cdot C)/(-K_X \cdot C)$. Since $(T \cdot C) \leqslant (M \cdot C)$ by definition, it reduces to $(M \cdot C)$. This follows from ii) of Corollary 4.2. Hence a bound on $r_L(T)$.

(6.1.2) The main task in the proof is to study $\nu_L(T)$, on which an effective bound remains unknown. We shall show that if $\nu_L(T)$ occurs out of control in some way, by modifying the method of Section 5 one will be led to another semi-ample divisor whose associated extremal face is of smaller dimension. Let us set up definitions. In the notation of Section 5, denote $r_L(\nu, T)$ by $r(\nu)$ for simplicity. Define

$$D'(\nu L, T) \equiv D'_\nu \equiv \nu L + T + \left(\frac{r(\nu)}{e}\right)K_X, \quad \nu \in \mathbb{N} \cup \{0\}.$$

Note the difference between eD'_ν and $D(\nu L, T)$ (which is defined in (i) of Section 5): eD'_ν equals $D(\nu L, T)$ provided $\nu \geqslant \nu_L(T)$. By the definition of $r(\nu)$, D'_ν is a nef but non-ample $\mathbb{Q}$-divisor. Define

$$A = \{t | \, t \in \mathbb{N} \cup \{0\}, \;\; F_{D'_s} \subset F_L, \forall s \geqslant t\}$$
$$t_0 = -1 + \min A$$

We shall show that t_0 can be explicitly bounded (6.1.3). To relate $v_L(T)$ to t_0, it turns out that if $v_L(T)$ exceeds a certain critical value, eD'_v can be used for the dimensional reduction of the extremal face (6.1.4).

(6.1.3) *Bound of* t_0. Pick an $l \in F_{D'_{t_0}} \backslash F_L$. Writing out $(D'_{t_0} \cdot l) = 0$ yields, by $(L \cdot l) \neq 0$ since $l \notin F_L$,

$$t_0 = -\frac{e(T \cdot l) + r_L(t_0, T)(K_X \cdot l)}{e(L \cdot l)}. \tag{6.1.3.1}$$

Since an extremal face is the convex hull of its extremal rays (cf. [KM2], p. 18), without loss of generality we assume that $\mathbb{R}_+ l \in F_{D'_{t_0}} \backslash F_L$ is an extremal ray of X. By $r_L(t_0, T) \leqslant r_L(T)$ we have from (6.1.3.1),

$$t_0 \leqslant r_L(T) \frac{-(K_X \cdot l)}{e(L \cdot l)}$$
$$\leqslant 2n \frac{r_L(T)}{e}$$

by Proposition 1.6. Thus a bound on t_0 follows from (6.1.1).

(6.1.4) *Either*

$$t_0 \leqslant v_L(T) \leqslant t_0 + 1$$

holds, or, given any $b \geqslant t_0 + 1$, *we have* $F_{D'_b} \subset F_L$ *and*

$$\dim F_{D'_b} < \dim F_L.$$

Proof of (6.1.4). Write $v_0 = v_L(T)$. In (6.1.4) the statement that

$$F_{D'_{b'}} \subset F_L \tag{6.1.4.1}$$

for any $b' > t_0$, is nothing but a restatement of the definition of t_0. We postpone the proof $v_0 \geqslant t_0$ until the end. To prove the rest of (6.1.4) one argues by contradiction. Suppose

$$v_0 > t_0 + 1; \quad \dim F_{D'_{b_1}} \geqslant \dim F_L \quad \text{for some } b_1 \geqslant t_0 + 1. \tag{6.1.4.2}$$

By (6.1.4.2) and (6.1.4.1)

$$F_{D'_{b_1}} = F_L, \tag{6.1.4.3}$$

and by (6.1.4.1) for any $b' > t_0$,

$$F_{D'_{b'}} \subset F_{D'_{b_1}}. \tag{6.1.4.4}$$

Fix a b' satisfying $b' > b_1$ and $b' \geqslant v_0$. Write

$$D'_{b'} = D'_{b_1} + (b' - b_1)L + \frac{c}{e}K_X \tag{6.1.4.5}$$

where $c \equiv r(b') - r(b_1)$.

Case $b_1 < v_0$. Fix a nonzero $l \in F_{D'_{b'}}$. By (6.1.4.3) and (6.1.4.4), one arrives at

$$(D'_{b'} \cdot l) = (D'_{b_1} \cdot l) = (L \cdot l) = 0. \tag{6.1.4.6}$$

Since $c \neq 0$ from $b' \geqslant v_0$ and the minimality of v_0, $(K_X \cdot l) = 0$ by (6.1.4.5), a contradiction.

Case $b_1 \geqslant v_0$. Then $F_{D'_{b_1}} \subset F_{D'_{v_0}}$ (with b', b_1 in (6.1.4.5) set to be b_1, v_0 respectively and with $c = r(b_1) - r(v_0) = 0$ by minimality of v_0). Hence, by (6.1.4.3),

$$F_L \subset F_{D'_{v_0}}. \tag{6.1.4.7}$$

Since $v_0 - 1 > t_0$ from (6.1.4.2), by definition of t_0,

$$F_{D'_{v_0-1}} \subset F_L. \tag{6.1.4.8}$$

With (6.1.4.7) and (6.1.4.8) we find as before that for all $l \in F_{D'_{v_0-1}}$, $(K_X \cdot l) = 0$ (using v_0, $v_0 - 1$ for (6.1.4.5) and noting $c = r(v_0) - r(v_0 - 1) > 0$ by minimality), a contradiction.

It remains to prove $v_0 \geqslant t_0$. Suppose otherwise: $t_0 > v_0$. Fixing any $s \geqslant t_0$, in (6.1.4.5) with $c = r(s) - r(v_0) = 0$, one has $F_{D'_s} \subset F_L$, a contradiction since $t_0 \notin A$. The proof of (6.1.4) is thereby completed.

Completion by induction. Start with $L = eM + r(M)K_X := L_1$, and suppose $\dim F_{L_1} > 1$. There exists an $H_i := T_1$, such that $n_1 = \dim F_{D(v_1 L_1, T_1)} < \dim F_{L_1}$ for v_1 large. By using (6.1.4) and (6.1.3), if v_1 has a bound (6.1.4) and $n_1 > 1$, set $L_2 = D(v_1 L_1, T_1)$; or, there exists a t_2 with explicit bound (6.1.3), such that with $L_2 = eD'(t_2 L_1, T_1)$, $n'_1 = \dim F_{L_2} < \dim F_{L_1}$. In either case, replacing L_1 by L_2 one checks the process can be repeated. Since it takes at most ρ times, where ρ has a bound (3.1), one finds a supporting function D as stated. The proof of Proposition 6.1 is completed.

7. Proof of Main Theorem

With D in (6.1), form $D_i := ew_i D + eH_i + r_D(H_i)K_X$ for w_i large, such that $F_{D_i} \subset F_D$. Then $F_{D_i} = F_D$ since $\dim F_D = 1$, and

$$D_i = \phi^* L_i \tag{7.1}$$

for ample Cartier divisors L_i on X' where $\phi: X \to X'$ is the contraction associated with D. In order to bound w_i one rules out the second possibility in (6.1.4) because $\dim F_D = 1$. Thus one finds an explicit w_0, such that D_i with any $w_i \geqslant w_0$ does the job. It is easily verified by linear algebra that by setting $w_i = w$ for some $w \leqslant w_0 + \rho + 1$ the linear span of D_i's (in $N^1(X)$) is of imension $\rho - 1$. Hence,

$\{L_i\}_{1\leqslant i\leqslant \rho-1}$ constitute a basis of $N^1(X')$ (with intersection numbers controlled), proving (i) of the Main Theorem.

We prove (ii) of Main Theorem. With L_j's in (7.1), by Kleiman's criterion $m\phi^{+*}L_j + r_{X^+}K_{X^+}$ is ample on X^+ for m large. One bounds m by (1.3) and denotes by H_j^+ $(1\leqslant j\leqslant \rho-1)$ the resulting ample divisors. Similarly, $H_\rho^+ := (r_{X^+})(r_{X^+}n + r_{X^+}+1)\phi^{+*}H_1^+ + r_{X^+}K_{X^+}$ is ample. These form a basis of $N^1(X^+)$. The proof of Main Theorem is completed.

Remark 7.2. Some remarks about making the divisors very ample on X, X^+ are in order. If $\phi\colon X\to X'$ is divisorial, X' is $\mathbb{Q}$-factorial. Applying (1.4) to L_i of (7.1) produces very ample divisors: $V_i = a_iL_i + b_iK_{X'}$. V_i's can be made to be linearly independent in $N^1(X')$. But if ϕ is small, X' is no longer $\mathbb{Q}$-Gorenstein. Instead, one may use effective base point freeness (cf. [Ko2]) to conclude $|mD_i|$ for D_i of (7.1) is base point free with m effectively bounded. It follows $mD_i = \phi^*(M_i)$ for very ample divisors M_i on X' (e.g. [KM2], p. 84–85). But the numerical data produced this way appears much larger than that in the previous case, in view of the work of [Ko2] and [Siu]. The case for $\phi^+\colon X^+\to X'$ is similar and omitted.

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