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The morphic Abel–Jacobi map

Mark E. Walker

Dedicated to Jozef Steenbrink on the occasion of his 60th birthday

Abstract

The morphic Abel–Jacobi map is the analogue of the classical Abel–Jacobi map one obtains by using Lawson and morphic (co)homology in place of the usual singular (co)homology. It thus gives a map from the group of r-cycles on a complex variety that are algebraically equivalent to zero to a certain 'Jacobian' built from the Lawson homology groups viewed as inductive limits of mixed Hodge structures. In this paper, we define the morphic Abel–Jacobi map, establish its foundational properties, and then apply these results to the study of algebraic cycles. In particular, we show the classical Abel–Jacobi map (when restricted to cycles algebraically equivalent to zero) factors through the morphic version, and show that the morphic version detects cycles that cannot be detected by its classical counterpart; that is, we give examples of cycles in the kernel of the classical Abel–Jacobi map that are not in the kernel of the morphic version. We also investigate the behavior of the morphic Abel–Jacobi map on the torsion subgroup of the Chow group of cycles algebraically equivalent to zero modulo rational equivalence.

Contents

1. Introduction

The classical Abel–Jacobi map provides an important tool for investigating the structure of the group of algebraic r-cycles on a smooth, projective complex variety X. It is a continuous homomorphism of abelian groups,

$$\Phi_r : \mathcal{Z}_r(X)_{\mathrm{hom}\sim 0} \to \mathcal{J}_r(X),$$

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from the space $\mathcal{Z}_r(X)_{\mathrm{hom}\sim 0}$ of r-cycles on X that are homologically equivalent to zero (i.e., whose classes in the singular homology group $H_{2r}^{\mathrm{sing}}(X)$ vanish) to the rth intermediate Griffiths Jacobian $\mathcal{J}_r(X)$ of X. The latter is the complex torus given as the dual of the $(r+1)$-stage of the Hodge filtration on $H^{2r+1}_{\mathrm{sing}}(X, \mathbb{C})$ modulo 'periods':

$$\mathcal{J}_r(X) = (F^{r+1}H^{2r+1}_{\mathrm{sing}}(X, \mathbb{C}))^*/H_{2r+1}^{\mathrm{sing}}(X, \mathbb{Z}),$$

where $F^\bullet$ is the Hodge filtration and 'periods' are those elements of $(F^{r+1}H^{2r+1}_{\mathrm{sing}}(X, \mathbb{C}))^*$ coming from classes in $H_{2r+1}^{\mathrm{sing}}(X, \mathbb{Z})$ under the map sending a closed integral chain c to the functional

$$\eta \mapsto \int_\eta c, \quad \text{for } \eta \in F^{r+1}H^{2r+1}_{\mathrm{sing}}(X, \mathbb{C}).$$

The original definition of the Abel–Jacobi map was given by Griffiths, in terms of integration as follows. Suppose that γ is an r-cycle that is homologically equivalent to zero. Then there is a $(2r+1)$-dimensional integral chain c with $\partial(c) = \gamma$ and we define

$$\Phi_r(\gamma) = \left(\eta \mapsto \int_c \eta\right) \quad \text{mod periods}.$$

(See [Lew99, § 12] for more details.)

The aim of this paper is to define and study the 'morphic Abel–Jacobi map',

$$\Phi_r^{\mathrm{mor}} : \mathcal{Z}_r(X)_{\mathrm{alg}\sim 0} \to \mathcal{J}_r^{\mathrm{mor}}(X),$$

which is the analogue of the classical Abel–Jacobi map that one obtains by replacing the singular homology and cohomology groups of a complex variety with the Lawson homology groups $L_rH_m(-)$ and morphic cohomology groups $L^tH^n(-)$. (See § 2 for the definitions of the Lawson and morphic (co)homology groups.) Here, $\mathcal{Z}_r(X)_{\mathrm{alg}\sim 0}$ denotes the space of r-cycles on X that are algebraically equivalent to zero and $\mathcal{J}_r^{\mathrm{mor}}(X)$ is the rth 'morphic Jacobian', which is defined analogously to $\mathcal{J}_r(X)$ by using Lawson homology in place of singular cohomology. The reason that the morphic Abel–Jacobi map is defined only on $\mathcal{Z}_r(X)_{\mathrm{alg}\sim 0}$, and not on the larger group $\mathcal{Z}_r(X)_{\mathrm{hom}\sim 0}$, comes from the fact that the kernel of the cycle class map $\mathcal{Z}_r(X) \to L_rH_{2r}(X)$ is precisely $\mathcal{Z}_r(X)_{\mathrm{alg}\sim 0}$, whereas the kernel of $\mathcal{Z}_r(X) \to H_{2r}^{\mathrm{BM}}(X)$ is $\mathcal{Z}_r(X)_{\mathrm{hom}\sim 0}$.

Just as the Lawson and morphic (co)homology groups provide refinements of their singular counterparts, so too does the morphic Abel–Jacobi map refine the classical map. In particular, we show that the classical Abel–Jacobi map (when restricted to cycles algebraically equivalent to zero) factors as

$$\mathcal{Z}_r(X)_{\mathrm{alg}\sim 0} \xrightarrow{\Phi_r^{\mathrm{mor}}} \mathcal{J}_r^{\mathrm{mor}}(X) \to \mathcal{J}_r(X).$$

Both the classical and the morphic Abel–Jacobi maps annihilate cycles rationally equivalent to zero. Moreover, we prove that the morphic Abel–Jacobi map is surjective (a property not enjoyed by the classical version), and we thus have a commutative triangle as follows.

$$\begin{tikzcd} CH_r(X)_{\mathrm{alg}\sim 0} \arrow[r, two heads, "\Phi_r^{\mathrm{mor}}"] \arrow[dr, "\Phi_r"'] & \mathcal{J}_r^{\mathrm{mor}}(X) \arrow[d] \\ & \mathcal{J}_r(X) \end{tikzcd} \tag{1.1}$$

Here, $CH_r(X)_{\mathrm{alg}\sim 0}$ is the Chow group of r-cycles on X that are algebraically equivalent to zero modulo rational equivalence.

To define the morphic Abel–Jacobi map, we rely on the general technique of Jannsen [Jan90] for the construction of Abel–Jacobi-type maps in a variety of settings. The input to Jannsen's technique is a homology/cohomology theory for varieties equipped with a suitable extra structure.

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His technique can be used to define the classical Abel–Jacobi map; in this case, the extra structure on the singular (co)homology groups is that of mixed Hodge structures, as provided by Deligne [Del71, Del74]. To define the morphic Abel–Jacobi map, the extra structure comes from viewing the Lawson homology groups as inductive limits of mixed Hodge structures. Such structures were defined by Friedlander and Mazur [FM94b] in the projective case and later generalized by Lima-Filho [Lim01] to all complex varieties. Section 4 of the current paper contains a detailed construction of these inductive limits of mixed Hodge structures (IMHSs) for Lawson homology along with proofs of the required functorality and other properties needed.

Using the factorization (1.1) and other properties of the morphic Abel–Jacobi map, we deduce several properties about the structure of cycles algebraically equivalent to zero on a variety X. In particular, we show that there are examples of cycles on a smooth, projective variety that vanish under the Abel–Jacobi map but not under the morphic Abel–Jacobi map. This is accomplished by constructing examples of varieties X for which the vertical map in (1.1) is not injective.

Thus, the morphic Abel–Jacobi map detects cycles that cannot be detected by its classical counterpart. In fact, we provide two types of such examples. Those of the first type arise by building on examples originally due to Nori [Nor93] and further developed by Friedlander [Fri00b], which show that various stages of the so-called s-filtration are non-trivial. The examples of the second type arise from examples of Schoen [Sch00] showing that there can be an infinite amount of l-torsion in the kernel of the classical Abel–Jacobi map.

We also analyze the behavior of the morphic Abel–Jacobi map on torsion subgroups. The examples due to Schoen mentioned above show that the classical Abel–Jacobi map does not always induce an injection from $(CH_r(X)_{\mathrm{alg}\sim 0})_{\mathrm{tor}}$ (the torsion subgroup of $CH_r(X)_{\mathrm{alg}\sim 0}$) to $\mathcal{J}_r(X)_{\mathrm{tor}}$, as was once conjectured. (Soulé and Voisin [SV05] have also constructed such counter-examples.) It is an intriguing question whether the morphic Abel–Jacobi map induces an isomorphism of the form

$$\Phi_r^{\mathrm{mor}}|_{\mathrm{tor}} : (CH_r(X)_{\mathrm{alg}\sim 0})_{\mathrm{tor}} \xrightarrow{?\cong} \mathcal{J}_r^{\mathrm{mor}}(X)_{\mathrm{tor}}.$$

The map $\Phi_r^{\mathrm{mor}}|_{\mathrm{tor}}$ is always onto, and we describe its kernel explicitly in Theorem 8.4. Conceivably, the kernel is trivial for any smooth, projective variety, so that $\Phi_r^{\mathrm{mor}}|_{\mathrm{tor}}$ is an isomorphism for all such varieties (see Corollary 8.8). In this paper, we show that for any smooth, projective variety belonging to the class $\mathcal{C}$ defined in [FHW04] (which includes all curves, all toric varieties, all cellular varieties, and all varieties built from these via localization, blowing up, or forming vector bundles), the kernel of $\Phi_r^{\mathrm{mor}}|_{\mathrm{tor}}$ vanishes; see Example 8.9. We also prove, in Theorem 8.11, that $\Phi_r^{\mathrm{mor}}|_{\mathrm{tor}}$ is an isomorphism for any complex projective variety that can be defined over a number field, provided that one assumes a common conjecture (namely, Conjecture 8.10) pertaining to the image of the higher Chow groups in Borel–Moore homology.

As mentioned, it was once conjectured that the classical Abel–Jacobi map is injective on torsion. The recent counter-examples of this conjecture also cast doubt on the related conjecture that the classical Abel–Jacobi map is universal among all 'regular' maps from $Z_r(X)_{\mathrm{alg}\sim 0}$ to abelian varieties. (See Conjecture 7.2 for a careful formulation.) The morphic Abel–Jacobi map suggests two ways of producing counter-examples to this conjecture. First, as we prove in Corollary 5.9 by using the properties of the morphic Abel–Jacobi map, the classical Abel–Jacobi map factors as

$$\begin{tikzcd} CH_r(X)_{\mathrm{alg}\sim 0} \arrow[r, two heads] \arrow[dr, "\Phi_r"'] & \mathcal{J}(N_{r+1}H^{\mathrm{sing}}_{2r+1}(X,\mathbb{Z}(r))) \arrow[d] \\ & \mathcal{J}_r(X) \end{tikzcd} \tag{1.2}$$

where N_{r+1} refers to the subgroup of classes supported in dimension $r+1$ and the upper-right group is the 'Jacobian' of the mixed Hodge structure $N_{r+1}H^{\mathrm{sing}}_{2r+1}(X,\mathbb{Z}(r))$. A counter-example to the universality of the classical Abel–Jacobi map would arise if the vertical map in (1.2) fails to be injective, and it is conceivable that this map is not injective for certain varieties constructed by Kollár [BCC92]; see Remark 5.10. Second, the fact that the vertical arrow in (1.1) can have a kernel suggests another possible source for counter-examples to the universality of the Abel–Jacobi map. Namely, such counter-examples would arise if the kernel of $\mathcal{J}_r^{\mathrm{mor}}(X) \to \mathcal{J}_r(X)$ were to admit a finite-dimensional quotient, in an appropriate sense; see Theorem 7.4. Constructing an actual counter-example in either of these two ways, however, remains elusive.

We now describe the organization of this paper. Section 2 recalls various basic definitions, such as that of Lawson homology and morphic cohomology. In § 3 we formalize what we mean by an IMHS and establish a few basic properties. Section 4 defines the IMHS for Lawson homology, establishes various functorial properties that are needed, and proves the compatibility of the IMHS for Lawson homology with that of singular Borel–Moore homology. The results of this section build on the foundational results of Friedlander–Mazur [FM94b] and Lima-Filho [Lim01], but much of the material here is new. In § 5 we define the morphic Abel–Jacobi map and establish is foundational properties.

Starting with § 6, we apply the morphic Abel–Jacobi map to the study of algebraic cycles. In this section we establish the examples mentioned above of cycles that vanish under the classical Abel–Jacobi map but not under the morphic version. In § 7 we indicate how the morphic Abel–Jacobi map might lead to the construction of a counter-example to the conjectured universality of the classical Abel–Jacobi map. Section 8 contains a description of the behavior of the morphic Abel–Jacobi map on the subgroup of torsion cycles, as discussed above.

2. Basics of Lawson homology

All varieties in this paper are assumed to be quasi-projective complex varieties. For such a variety U and abelian group A, we write

$$H^n_{\mathrm{sing}}(U, A), \quad H_n^{\mathrm{sing}}(U, A), \quad \text{and} \quad H_n^{\mathrm{BM}}(U, A)$$

for the singular cohomology, singular homology, and singular Borel–Moore homology of U^{an} with coefficients in A. Here U^{an} refers to the complex points of U equipped with the 'analytic' (or 'strong') topology. When $A = \mathbb{Z}$, we omit it from the notation.

The Lawson homology groups of a variety are obtained by topologizing the collection of cycles on the variety and then taking homotopy groups. In detail, if X is a projective (complex) variety, define $\mathcal{C}_{r,e}(X)$ to be the Chow variety that parameterizes degree e, dimension r effective cycles on X. (Here, degree is defined in terms of a chosen embedding $X \subset \mathbb{P}^n$.) Let $\mathcal{C}_r(X) = \coprod_e \mathcal{C}_{r,e}(X)$, an abelian monoid object in the category of ind-varieties under the operation of addition of cycles. We also let $\mathcal{C}_r(X)$ denote the topological abelian monoid $\coprod_e \mathcal{C}_{r,e}(X)^{\mathrm{an}}$ and we let $\mathcal{Z}_r(X)$ denote the 'naive' topological group completion of the topological abelian monoid $\mathcal{C}_r(X)$. For an arbitrary variety U, we define the topological abelian group $\mathcal{Z}_r(U)$ by choosing a projective closure $U \subset X$ with closed complement Y, and then setting

$$\mathcal{Z}_r(U) = \mathcal{Z}_r(X)/\mathcal{Z}_r(Y).$$

It is not hard to see that the topology on $\mathcal{Z}_r(U)$ is independent of the choice of projective closure [Lim93].

Definition 2.1. The *Lawson homology* groups of a complex variety U with coefficients in an abelian group A are

$$L_rH_n(U, A) := \pi_{n-2r}(\mathcal{Z}_r(U), A).$$

We set $L_rH_n(U) = L_rH_n(U, \mathbb{Z})$.

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There is an associated cohomology theory, called *morphic cohomology*, defined by

$$L^t H^m(X, A) = \pi_{2t-m}(\mathcal{Z}_0(X, \mathbb{P}^t)/\mathcal{Z}_0(X, \mathbb{P}^{t-1}), A),$$

where $\mathcal{Z}_0(X, \mathbb{P}^t)$ denotes the collection of cycles on $X \times \mathbb{P}^t$ that are finite over X, topologized as in [Fri98]. In this paper, we focus almost entirely on Lawson homology and not on morphic cohomology (despite the fact that we refer to the main object of study as the *morphic* Abel–Jacobi map). The existence of morphic cohomology and its relation to Lawson homology is, however, useful for understanding Lawson homology. In particular, many of the formal properties of Lawson homology are summarized by the following statement.

THEOREM 2.2 (Friedlander [Fri00a]). *Lawson homology and morphic cohomology form twisted duality theory in the sense of Bloch–Ogus.*

In particular, associated to any projective morphism $p : X \to Y$, there is a pushforward map

$$p_* : L_r H_n(X) \to L_r H_n(Y),$$

associated to any flat morphism $f : X \to Y$ of relative dimension e, there is a pullback map

$$f^* : L_r H_n(Y) \to L_{r+e} H_{n+2e}(X),$$

and given a closed subvariety $Y \subset X$ with open complement U, there is a long exact 'localization' sequence

$$\cdots \to L_r H_m(Y) \to L_r H_m(X) \to L_r H_m(U) \to L_r H_{m-1}(Y) \to \cdots . \tag{2.3}$$

Moreover, Lawson homology is homotopy invariant in that pullback along a vector bundle (or a torsor of such) induces an isomorphism, and there is a Poincaré duality isomorphism

$$L_r H_n(X) \xrightarrow{\cong} L^{d-r} H^{2d-n}(X)$$

whenever X is smooth of pure dimension d.

Correspondences act on the Lawson homology groups, even on the level of cycle spaces. Namely, for projective varieties X and Y, define $Z_s(X, Y)$ to be the group completion of the (discrete) abelian monoid $\mathrm{Hom}(X, \coprod_e \mathcal{C}_{s,e}(Y))$. When X is smooth, $Z_s(X, Y)$ may be identified with the group of cycles on $X \times Y$ that are equidimensional of relative dimension s over X (see [FL92, 1.5]). Given a map $\Gamma : X \to \mathcal{C}_{s,e}(Y)$, we have an induced map $\mathcal{C}_r(X) \to \mathcal{C}_r(\mathcal{C}_{s,e}(Y))$ which can be composed with the 'trace map'

$$\mathcal{C}_r(\mathcal{C}_{s,e}(Y)) \to \mathcal{C}_{r+s}(Y)$$

of [FL92, 7.1] to obtain the continuous map

$$\mathcal{C}_r(X) \to \mathcal{C}_{r+s}(Y)$$

of topological abelian monoids. Taking group completions and then homotopy groups gives the homomorphism

$$\Gamma_* : L_r H_n(X) \to L_{r+s} H_{n+2s}(Y).$$

This definition is extended to an arbitrary $\Gamma \in Z_s(X, Y)$ by linearity.

There are maps from Bloch's higher Chow groups [Blo86] to the Lawson homology groups,

$$CH_r(X, n) \to L_r H_{2r+n}(X),$$

and these maps are natural for pushforwards along projective morphisms and pullbacks along flat morphisms [FG93]. Moreover, the composition

$$CH_r(X, n) \to L_r H_{2r+n}(X) \to H^{\mathrm{BM}}_{2r+n}(X)$$

is the usual map from Bloch's groups to Borel–Moore homology. When $n = 0$, the map $CH_r(X) \to L_rH_{2r}(X)$ is surjective and we have

$$L_rH_{2r}(X) \cong Z_r(X)/Z_r(X)_{\mathrm{alg}\sim 0}, \tag{2.4}$$

where $Z_r(X)$ denotes the discrete group of r-cycles on X and $Z_r(X)_{\mathrm{alg}\sim 0}$ denotes the subgroup of cycles algebraically equivalent to zero [Fri91].

The connection of Lawson homology with singular homology builds on the following basic result. (We reprove this result in the category of mixed Hodge structures in Theorem 4.21 below.)

Proposition 2.5 (Friedlander [Fri91]). *For any variety X and integers n and r, we have an isomorphism*

$$L_0H_n(X) \cong H_n^{\mathrm{BM}}(X)$$

that is natural with respect to pushforwards along projective morphisms, pullbacks along flat morphism, and the boundary maps in a long exact localization sequence. When X is projective, this isomorphism comes from the identification of $\mathcal{Z}_0(X)$ with the group completion of the infinite symmetric product of X^{an} and the Dold–Thom theorem.

To relate the higher Lawson homology groups with singular homology, one uses the s-map, first defined in [FM94b]. For any X, the s-map

$$s : L_rH_m(X) \to L_{r-1}H_m(X)$$

may be defined using the composition of

$$L_0H_2(\mathbb{P}^1) \times L_rH_n(X) \to L_rH_{n+2}(\mathbb{P}^1 \times X) \to L_rH_{n+2}(\mathbb{A}^1 \times X) \xleftarrow{\cong} L_{r-1}H_n(X),$$

in which the first map is given by external product of cycles, together with a choice of generator for $L_0H_2(\mathbb{P}^1) \cong H_2^{\mathrm{sing}}(\mathbb{P}^1) \cong \mathbb{Z}$. In particular, by composing the s map and using Proposition 2.5, we obtain maps

$$L_rH_n(X) \to H_n^{\mathrm{BM}}(X). \tag{2.6}$$

Moreover, the s-map and hence the map (2.6) are natural with respect to pushforwards along projective morphisms, pullbacks along flat morphism, and the boundary maps in a long exact localization sequence.

Suslin's conjecture for Lawson homology is an analogue of the Beilinson–Lichtenbaum conjecture, and it provides a conjectural framework for understanding the nature of Lawson homology. (See [FHW04] for a more detailed discussion.)

Conjecture 2.7 (Suslin's conjecture for Lawson homology). For any abelian group A and quasi-projective variety Y of dimension d, the map

$$L_sH_n(Y, A) \to H_n^{\mathrm{BM}}(Y, A)$$

is an isomorphism for $n \geqslant d + s$ and a monomorphism for $n = d + s - 1$.

From the validity of this conjecture, one would deduce that the image of

$$L_sH_n(Y, A) \to H_n^{\mathrm{BM}}(Y, A)$$

is $N_{n-s}H_n^{\mathrm{BM}}(Y, A)$ for all Y, n, and s. Here, $N_\bullet$ denotes the niveau filtration:

$$N_kH_n^{\mathrm{BM}}(Y, A) := \sum_{\substack{V \subset Y \\ \dim(V) \leqslant k}} \mathrm{im}(H_n^{\mathrm{BM}}(V, A) \to H_n^{\mathrm{BM}}(Y, A)).$$

Since we will need it later, we now state and prove an (easy) special case of Suslin's conjecture.

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PROPOSITION 2.8. *For a projective variety Y and abelian group A, the image of*

$$L_rH_n(Y,A) \to H_n^{\mathrm{sing}}(Y,A)$$

is contained in $N_{n-r}H_n^{\mathrm{sing}}(Y,A)$. When $n = 2r+1$, we have a surjection

$$L_rH_n(Y,A) \twoheadrightarrow N_{r+1}H_{2r+1}^{\mathrm{sing}}(Y,A).$$

For any quasi-projective variety V, if $\dim(V) \leqslant r+1$, then the map

$$L_rH_{2r+1}(V,A) \xrightarrow{\cong} N_{r+1}H_{2r+1}^{\mathrm{BM}}(V,A) = H_{2r+1}^{\mathrm{BM}}(V,A)$$

is an isomorphism.

Proof. The first assertion is given by [FM94a, 4.3]. The surjectivity of $L_rH_{2r+1}(Y,A) \to N_{r+1}$ $H_{2r+1}^{\mathrm{sing}}(Y,A)$ follows by naturality for pushforwards from the final assertion.

If U is a smooth variety of dimension at most $r+1$, then the map $L_rH_{2r+1}(U,A) \to H_r^{\mathrm{BM}}(U,A)$ is isomorphic, via Poincaré duality, to the map

$$L^1H^1(U,A) \to H^1_{\mathrm{sing}}(U,A),$$

which is seen to be an isomorphism by using [FL92, 9.3] and the five lemma. For an arbitrary variety V with $\dim(V) \leqslant r+1$, let $V_s \subset V$ be the singular locus and set $U = V - V_s$. Since U is smooth, the map $L_rH_{2r+1}(U,A) \to H_{2r+1}^{\mathrm{BM}}(U,A)$ is an isomorphism. Since $\dim(V_s) \leqslant r$, we have $L_rH_{2r+1}(V_s,A) = 0 = H_{2r+1}^{\mathrm{BM}}(V_s,A)$ and $L_rH_{2r}(V_s,A) = Z_r(V_s) \otimes A = H_{2r}^{\mathrm{BM}}(V_s,A)$. The result now follows by naturality for localization sequences and the five lemma. $\square$

3. Inductive limits of mixed Hodge structures

We refer the reader to [BZ90] for the definition and properties of a *mixed Hodge structure* (MHS).

The Lawson homology groups are, in general, not finitely generated, and thus cannot be enriched to MHSs (which are, by definition, finitely generated). Rather, as established by Friedlander–Mazur [FM94b] and Lima-Filho [Lim01], they are filtered inductive limits of MHSs. (We review their constructions in the next section.)

DEFINITION 3.1. An IMHS is a countable inductive limit of MHSs with bounded filtration lengths. That is, an IMHS is a system of MHSs $\{H_\alpha\}_{\alpha\in I}$ indexed by a filtered category I with countably many objects, such that there exist integers $M < N$ so that $W_M((H_\alpha)_{\mathbb{Q}}) = 0$, $W_N((H_\alpha)_{\mathbb{Q}}) = (H_\alpha)_{\mathbb{Q}}$, $F^N((H_\alpha)_{\mathbb{C}}) = 0$, and $F^M((H_\alpha)_{\mathbb{C}}) = (H_\alpha)_{\mathbb{C}}$ for all $\alpha \in I$. A morphism of IMHSs is a morphism of filtered systems of MHSs, and we write IMHS also for the abelian category of all IMHSs.

Given an IMHS $\{H_\alpha\}_{\alpha\in I}$, define the abelian group $H := \varinjlim_\alpha H_\alpha$ and define filtrations $W_n(H_{\mathbb{Q}}) = \varinjlim_\alpha W_n((H_\alpha)_{\mathbb{Q}})$ and $F^q(H_{\mathbb{C}}) = \varinjlim_\alpha F^q((H_\alpha)_{\mathbb{C}})$. Since W_m and F^q are exact functors on MHS (the abelian category of MHSs), the triple $(H, W_\bullet, F^\bullet)$ satisfies all of the axioms of an MHS except, of course, that H need not be finitely generated. Note that the image of the canonical map $H_\alpha \to H$, with the induced filtrations, coincides with $H_\alpha/(\ker(H_\alpha \to H_\beta))$ for some $\alpha \to \beta$ in I, and thus this image is an MHS. We see, then, that an IMHS is equivalent to a triple $(H, W_\bullet, F^\bullet)$, where H is a countable abelian group, $W_\bullet(H_{\mathbb{Q}})$ and $F^\bullet(H_{\mathbb{C}})$ are finite, complete filtrations satisfying

$$Gr_n^W(H_{\mathbb{C}}) = \bigoplus_{p+q=n} H^{p,q}$$

where

$$H^{p,q} = F^pGr_{p+q}^W(H_{\mathbb{C}}) \cap \overline{F}^qGr_{p+q}^W(H_{\mathbb{C}}),$$

and such that every finitely generated subgroup of H is contained in a finitely generated subgroup H' so that $(H', W_\bullet|_{H'_{\mathbb{Q}}}, F^\bullet|_{H'_{\mathbb{C}}})$ is an MHS.

We refer the reader to [Hub93] for the general properties of ind-categories associated to abelian categories. In particular, we use that

$$\mathrm{Ext}^n_{\mathrm{IMHS}}(\mathbb{Z}(0), H) = \varinjlim_{\alpha} \mathrm{Ext}^n_{\mathrm{MHS}}(\mathbb{Z}(0), H_\alpha)$$

where H_α ranges over all finitely generate sub-IMHSs of H (i.e. over all sub-IMHS that are actually MHSs). In particular, we have

$$\mathrm{Ext}^n_{\mathrm{IMHS}}(\mathbb{Z}(0), H) = 0, \quad \text{if } n \geqslant 2,$$

for any IMHS H, since this vanishing holds for MHSs by [Beĭ86].

Definition 3.2. For a IMHS H, define

$$\Gamma(H) = \mathrm{Hom}_{\mathrm{IMHS}}(\mathbb{Z}(0), H)$$

and

$$\mathcal{J}(H) = \mathrm{Ext}^1_{\mathrm{IMHS}}(\mathbb{Z}(0), H).$$

Since $\Gamma(H) = \varinjlim_\alpha \Gamma(H_\alpha)$, we have $\Gamma(H) = H \cap W_0(H_{\mathbb{Q}}) \cap F^0(H_{\mathbb{C}})$.

Proposition 3.3 (cf. [Car80, Jan90]). *For a IMHS H, we have*

$$\mathcal{J}(H) \cong \frac{W_0(H_{\mathbb{C}})}{W_0(H) + F^0 W_0(H_{\mathbb{C}})}.$$

That is, $\mathcal{J}(H)$ is the quotient of the complex vector space $W_0(H_{\mathbb{C}})/F^0 W_0(H_{\mathbb{C}})$ by the action of $W_0(H) := \ker(H \to H_{\mathbb{Q}}/W_0(H_{\mathbb{Q}}))$.

In particular, if $W_0(H_{\mathbb{Q}}) = H_{\mathbb{Q}}$, then we have an exact sequence

$$0 \to \Gamma(H) \to H \to H_{\mathbb{C}}/F^0(H_{\mathbb{C}}) \to \mathcal{J}(H) \to 0$$

and the torsion subgroup of $\mathcal{J}(H)$ is

$$\mathcal{J}(H)_{\mathrm{tor}} \cong (H/\Gamma(H)) \otimes \mathbb{Q}/\mathbb{Z} \cong \mathrm{coker}(\Gamma(H) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \to H \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}).$$

If H is pure of weight -1, then

$$\mathcal{J}(H) \cong \frac{H^{-1,0} \oplus H^{-2,1} \oplus \cdots}{H} \cong H \otimes \mathbb{R}/\mathbb{Z}$$

and

$$\mathcal{J}(H)_{\mathrm{tor}} \cong H \otimes \mathbb{Q}/\mathbb{Z}.$$

If H is pure of weight -1 and finitely generated (i.e. actually an MHS), then $\mathcal{J}(H)$ is a complex torus.

Proof. For an MHS H, Jannsen [Jan90], building on results of Carlson [Car80], established the formula

$$\mathcal{J}(H) = \frac{W_0(H_{\mathbb{C}})}{W_0(H) + F^0 W_0(H_{\mathbb{C}})}.$$

If $W_0(H_{\mathbb{Q}}) = H_{\mathbb{Q}}$, then we clearly have an exact sequence

$$0 \to \Gamma(H) \to H \to \frac{H_{\mathbb{C}}}{F^0(H_{\mathbb{C}})} \to \mathcal{J}(H) \to 0,$$

and the formula for $\mathcal{J}(H)_{\mathrm{tor}}$ follows from the long exact sequence for $\mathrm{Tor}_*(-, \mathbb{Q}/\mathbb{Z})$. If H is an MHS of pure of weight -1, then $W_0(H_{\mathbb{C}}) = H_{\mathbb{C}} = \bigoplus_p H^{p,-p-1}$ and so $H_{\mathbb{C}}/F^0(H_{\mathbb{C}}) = H^{-1,0} \oplus H^{-2,1} \oplus \cdots$. Since $\overline{H^{p,-p-1}} = H^{-p-1,p}$, we have $H_{\mathbb{C}}/F^0(H_{\mathbb{C}}) \cong H_{\mathbb{R}}$ as abelian groups.

The corresponding formulas for IMHSs follow by taking filtered inductive limits, using that W_0 and F^0 are exact functors. □

For example, letting $H = H^{\mathrm{sing}}_{2r+1}(X, \mathbb{Z}(r))$ for a projective variety X, viewed as an MHS in the usual manner, we obtain

$$\mathcal{J}_r(X) := \mathcal{J}(H) = H_{\mathbb{C}}/(H + F^0(H_{\mathbb{C}})),$$

the rth intermediate Griffiths Jacobian of X. (This definition coincides with that given in the introduction up to natural isomorphism.) If X is smooth, then H is pure of weight -1 so that $\mathcal{J}_r(X)$ is a complex torus isomorphic to $H^{\mathrm{sing}}_{2r+1}(X, \mathbb{R})/H^{\mathrm{sing}}_{2r+1}(X, \mathbb{Z})$. Letting $r = \dim(X) - 1$ and applying Poincaré duality, we have

$$\mathcal{J}_{\dim(X)-1}(X) \cong H^1_{\mathrm{sing}}(X, \mathbb{R})/H^1_{\mathrm{sing}}(X, \mathbb{Z}) = \mathrm{Pic}^0(X),$$

recovering the classical Picard variety of X.

Since $\mathrm{Ext}^n_{\mathrm{IMHS}} = 0$ for $n \geqslant 2$, associated to a short exact sequence

$$0 \to H' \to H \to H'' \to 0$$

of IMHSs, we have the six-term exact sequence

$$0 \to \Gamma(H') \to \Gamma(H) \to \Gamma(H'') \to \mathcal{J}(H') \to \mathcal{J}(H) \to \mathcal{J}(H'') \to 0$$

of abelian groups. In particular, $\mathcal{J}(-)$ is a right exact functor from IMHS to abelian groups. In fact, if we topologize $\mathcal{J}(H)$ by declaring $W_0(H_{\mathbb{C}}) \twoheadrightarrow \mathcal{J}(H)$ to be a quotient map of topological spaces, where $W_0(H_{\mathbb{C}})$ is topologized by viewing it as a filtered colimit of finite-dimensional subspaces, then $\mathcal{J}(-)$ takes IMHSs to topological abelian groups and it takes surjections of IMHSs to quotients in the category of topological abelian groups.

Abel–Jacobi maps, both the classical versions and the morphic versions we discuss in this paper, are obtained from the boundary map $\Gamma(H'') \to \mathcal{J}(H')$ in the above six-term exact sequence [Jan90, 9.2]. Since we will need it later, we show that the restriction of this boundary map to torsion subgroups admits an alternative description.

Lemma 3.4. *Suppose that*

$$0 \to H' \to H \to H'' \to 0 \tag{3.5}$$

is a short exact sequence of IMHSs such that $W_0(H'_{\mathbb{Q}}) = H'_{\mathbb{Q}}$, $W_0(H_{\mathbb{Q}}) = H_{\mathbb{Q}}$, *and* $W_0(H''_{\mathbb{Q}}) = H''_{\mathbb{Q}}$. *Then the restriction of the boundary map coming from the long exact sequence for* $\mathrm{Ext}^*_{\mathrm{IMHS}}(\mathbb{Z}(0), -)$ *to torsion subgroups,*

$$\Gamma(H'')_{\mathrm{tor}} \to \mathcal{J}(H')_{\mathrm{tor}} \cong (H'/\Gamma(H')) \otimes \mathbb{Q}/\mathbb{Z},$$

coincides with the composition of

$$\Gamma(H'')_{\mathrm{tor}} \rightarrowtail H''_{\mathrm{tor}} \to H' \otimes \mathbb{Q}/\mathbb{Z} \twoheadrightarrow (H'/\Gamma(H')) \otimes \mathbb{Q}/\mathbb{Z},$$

where the second map comes from the long exact sequence for $\mathrm{Tor}_*(-, \mathbb{Q}/\mathbb{Z})$ *applied to (3.5) regarded as a short exact sequence of abelian groups.*

Proof. Observe that we have a diagram of abelian groups with exact rows

$$\begin{tikzcd} 0 \arrow[r] & H' \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & H'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H'_{\mathbb{C}}/F^0(H'_{\mathbb{C}}) \arrow[r] & H_{\mathbb{C}}/F^0(H_{\mathbb{C}}) \arrow[r] & H''_{\mathbb{C}}/F^0(H''_{\mathbb{C}}) \arrow[r] & 0 \end{tikzcd} \tag{3.6}$$

and that the kernels of the vertical maps are $\Gamma(H')$, $\Gamma(H)$, and $\Gamma(H'')$ and the cokernels of these maps are $\mathcal{J}(H')$, $\mathcal{J}(H)$, and $\mathcal{J}(H'')$. An examination of the proof of Proposition 3.3 found in

[Car80, Jan90] reveals that the boundary map

$$\Gamma(H'') \to \mathcal{J}(H') \cong H'_{\mathbb{C}}/(H' + F^0(H'_{\mathbb{C}}))$$

appearing in the long exact sequence for $\mathrm{Ext}^*_{\mathrm{IMHS}}(\mathbb{Z}(0), -)$ applied to (3.5) coincides with the map given by the snake lemma applied to (3.6). A diagram chase shows that restriction of the latter map to torsion subgroups coincides with the boundary map in the long exact sequence for $\mathrm{Tor}_*(-, \mathbb{Q}/\mathbb{Z})$ applied to (3.5). □

4. IMHSs for Lawson homology

The material of this section comprises the technical heart of this paper. We establish that the Lawson homology groups are IMHSs in a sufficiently functorial manner so as to allow us to define the morphic Abel–Jacobi map and to establish the properties we seek. We also show that the maps from Lawson homology to Borel–Moore homology are morphisms of IMHSs, which will allow us to compare the morphic Abel–Jacobi map with its classical counterpart.

Some of these goals have already been achieved by Friedlander–Mazur [FM94b], who showed how to endow the Lawson homology groups of a projective variety with IMHSs, and by Lima-Filho [Lim01], who extended Friedlander–Mazur's construction to quasi-projective varieties. The results of Friedlander–Mazur and Lima-Filho do not, however, provide all of the naturality properties enjoyed by these IMHSs that we need. They also do not establish the compatibility of the IMHSs for Lawson and Borel–Moore homology in full generality. (Friedlander–Mazur do establish this for smooth, projective varieties [FM94b, 4.5], but pose such compatibility in the general case as an open question [FM94b, § 3.5].) This rather lengthy and technical section is therefore required.

The basic idea, due to Friedlander–Mazur, for showing the Lawson homology groups of a projective variety X are IMHSs is to use that $\mathcal{C}_r(X)$ is a countable disjoint union of projective varieties and that the homology groups of a projective variety are MHSs by the work of Deligne [Del74]. Moreover, the Hurewicz map $L_rH_{2r+q}(X) = \pi_q(\mathcal{C}_r(X)^+) \to H_q^{\mathrm{sing}}(\mathcal{C}_r(X)^+)$ is injective and, in fact, the Milnor–Moore theorem shows that one may identify $L_rH_{2r+q}(X, \mathbb{Q})$ with the kernel of a morphism of (rational) IMHSs. Hence, $L_rH_{2r+q}(X)$ acquires the structure of an IMHS. Lima-Filho [Lim01] extended this idea by using that the Lawson homology groups of a quasi-projective variety U may be realized using a 'triple bar construction' involving $\mathcal{C}_r(Y)$ and $\mathcal{C}_r(X)$, where X is a projective closure of U and $Y = X - U$.

We more-or-less follow Lima-Filho's construction in this section. In particular, if X is projective, then we have an isomorphism

$$L_rH_{2r+n}(X) \cong \pi_{n+1}(B\mathcal{C}_r(X))$$

(see [FG93] and [Lim93]), where BM denotes the usual bar construction of an abelian monoid M (see below for a precise definition). It is easy to see $B\mathcal{C}_r(X)$ is a countable inductive limit of simplicial projective varieties (cf. [Lim01, § 5]). We thus may employ the following fundamental result of Deligne.

Theorem 4.1 (Deligne). *The homology groups of a simplicial variety $V_\bullet$*

$$H_q(V_\bullet, \mathbb{Z}) = H_q(|V_\bullet|, \mathbb{Z})$$

are naturally MHSs (i.e. morphisms of simplicial varieties induce morphisms of MHSs) in such a way that if $V_\bullet = V$ is the constant simplicial variety associated to a smooth, projective variety V, then the MHS coincides with the classical one.

Consequently, the homology groups of a countable inductive limit of simplicial varieties are naturally IMHSs.

Proof. See [Del74]. In the case of a countable inductive limit of simplicial varieties, the condition that the filtrations be of bounded length holds since $H_q^{\mathrm{sing}}(V_\bullet)$ depends only on $H_q(d \mapsto H_p^{\mathrm{sing}}(V_q))$ for finitely many p and q by [Del74, 8.3.5] and there are uniform bounds for the weights and Hodge types of $H_p^{\mathrm{sing}}(V_q)$ by [Del74, 8.2.4]. □

We will need the following lemma. The rather technical hypotheses are proved to hold in the cases that we will need them by the results of [Lim93].

LEMMA 4.2. *Suppose that $M_\bullet$ is a countable filtered inductive limit of simplicial projective varieties and that there is a pairing $M_\bullet \times M_\bullet \to M_\bullet$ (of simplicial ind-varieties) making $M_\bullet$ into an abelian monoid. Assume that the map $M_\bullet \to M_\bullet^+$, where M_n^+ denotes the naive topological group completion of the topological abelian monoid M_n^{an}, is a homotopy equivalence. Then the Hurewicz map $\pi_q(M_\bullet) \to H_q^{\mathrm{sing}}(M_\bullet)$ is naturally a split injection.*

If $M_\bullet$ is connected, then upon tensoring with $\mathbb{Q}$ the image of the Hurewicz map is the kernel of the map

$$H_q^{\mathrm{sing}}(M_\bullet, \mathbb{Q}) \xrightarrow{\phi_2 - 2} H_q^{\mathrm{sing}}(M_\bullet, \mathbb{Q}),$$

where ϕ_2 is the map induced by multiplication by 2 on $M_\bullet$. In particular, $\pi_q(M_\bullet)$ is an IMHS in such a way that the Hurewicz map is an injection of IMHSs.

Proof. The space $|M_\bullet|$ is a topological abelian monoid, since geometric realizations commute with products. The first assertion is thus well known (cf. [Lim01, 5.9]), and holds more generally, but we describe a natural splitting of the Hurewicz map here because it will be used again. The hypothesis shows that it suffices to give a spitting of

$$\pi_q(|M_\bullet^+|) \to H_q^{\mathrm{sing}}(|M_\bullet^+|).$$

This map is given by applying π_q to the map of multi-simplicial sets

$$\mathrm{Sing}_\bullet(M_\bullet^+) \to \mathbb{Z}\,\mathrm{Sing}_\bullet(\mathcal{M}_\bullet^+),$$

which admits a natural splitting since $\mathrm{Sing}_\bullet(M_\bullet^+)$ is a multi-simplicial abelian group.

The Milnor–Moore theorem [MM65] gives us that $H_*^{\mathrm{sing}}(M_\bullet, \mathbb{Q})$ is isomorphic as a Hopf algebra to $S_{\mathbb{Q}}(\pi_*(M_\bullet, \mathbb{Q}))$, where $S_{\mathbb{Q}}(V)$ denotes the symmetric algebra of a graded vector space V. In general, the multiplication and comultiplication maps making $S_{\mathbb{Q}}(V)$ into a Hopf algebra are induced by the addition map $V \oplus V \to V$ and diagonal map $V \rightarrowtail V \oplus V$, respectively. The map ϕ_2 is induced by the composition of $V \rightarrowtail V \oplus V \to V$ and hence we see that ϕ_2 acts as multiplication by 2^n on the summand $S_{\mathbb{Q}}^n(\pi_*(M_\bullet, \mathbb{Q}))$. In particular, the kernel of $\phi_2 - 2$ is $S_{\mathbb{Q}}^1(\pi_*(M_\bullet, \mathbb{Q})) = \pi_*(M_\bullet, \mathbb{Q})$. Finally, multiplication by 2 is trivially a morphism of IMHS and ϕ_2 is also, since it arises from a countable inductive limit of morphisms of simplicial varieties [Del74, § 8.3]. □

DEFINITION 4.3. For a projective variety X, we endow the Lawson homology groups of X with IMHSs by using Lemma 4.2, Theorem 4.1, and the identification

$$L_r H_{2r+n}(X) \cong \pi_{n+1}|B(\mathcal{C}_r(X))|.$$

To extend this definition to all quasi-projective varieties, we use the basic idea due to Lima-Filho of choosing projective closures and using a 'triple bar construction', but we cast the triple bar construction in terms of mapping cones. The reason that this is a useful perspective is that Deligne uses mapping cones of simplicial varieties to endow relative cohomology groups with MHSs, and thus the use of mapping cones will enable us to compare the IMHS of Lawson homology with that of Borel–Moore homology.

Suppose that $\mathcal{C}$ is any category having finite coproducts $\amalg$ and a final object $*$. We will use what follows when $\mathcal{C}$ consists of abelian monoids (perhaps with additional structure), when $\mathcal{C}$ consists of varieties, or when $\mathcal{C}$ consists of pointed varieties (i.e. the category consisting of varieties with a specified $\mathbb{C}$-point and in which morphisms preserve the basepoint). In the first case, $\amalg = \oplus$ and $* = 0$, in the second case, $\amalg$ is disjoint union and $* = \operatorname{Spec}\mathbb{C}$, and in the final case, $\amalg$ is given by wedge product $\vee$ and $* = \operatorname{Spec}\mathbb{C}$. The use of the last case is really a matter of bookkeeping, and in all of the applications the pointed varieties we consider are given as $Y_+ := Y \coprod *$ for some (unpointed) variety Y.

For a general category $\mathcal{C}$ of this type, if $u : Y_\bullet \to X_\bullet$ is a morphism of simplicial objects in $\mathcal{C}$, define $\operatorname{cone}_\amalg(u) = \operatorname{cone}(u)$ to be the simplicial object in $\mathcal{C}$ with

$$\operatorname{cone}(u)_n = * \amalg Y_0 \amalg \cdots \amalg Y_{n-1} \amalg X_n.$$

The face and degeneracy maps are defined so that $X_\bullet \subset \operatorname{cone}(u)_\bullet$ and $* \subset \operatorname{cone}(u)_\bullet$ are inclusions of simplicial objects, and for $y_j \in Y_j \subset \operatorname{cone}(u)_n$ and $0 \leqslant i \leqslant n$, we have

$$d_i(y_j) = \begin{cases} d_i^Y(y_j), & \text{if } i \leqslant j, \\ y_j, & \text{if } i > j \text{ and } j < n-1, \\ u(y_n), & \text{if } i = n \text{ and } j = n-1. \end{cases} \tag{4.4}$$

If $i = j = 0$, we interpret $d_0^Y(y_0)$ to mean $*$. By way of illustration, if $\mathcal{C}$ is the category of abelian groups (or any abelian category), then we have

$$\mathcal{N}(\operatorname{cone}_\oplus(Y_\bullet \to X_\bullet)) \cong \operatorname{cone}(\mathcal{N}(Y_\bullet) \to \mathcal{N}(X_\bullet)),$$

where $\mathcal{N}$ denotes the equivalence of categories taking a simplicial abelian group to its normalized chain complex and the cone on the right is the usual mapping cone for a map of complexes.

If $\mathcal{C}$ is the category of abelian monoids, one may readily verify that

$$\operatorname{cone}_\oplus(u) = B(0, Y_\bullet, X_\bullet), \tag{4.5}$$

where the right-hand side is May's 'triple bar construction' [May75] with $Y_\bullet$ acting on $X_\bullet$ in the obvious manner via u. In particular, we have

$$\operatorname{cone}_\oplus(\mathcal{C}_r(X) \to 0) = B(0, \mathcal{C}_r(X), 0) = B(\mathcal{C}_r(X)). \tag{4.6}$$

Note that, by convention, if Y is an object of a category $\mathcal{C}$, we regard Y as a constant simplicial object in $\mathcal{C}$ as needed.

Clearly the process of taking cones may be iterated so that one can talk about the cone of a commutative square of simplicial objects. For example, if X is a projective variety and $Y \subset X$ is a closed subvariety, we have

$$\operatorname{cone}_\oplus \left(\begin{array}{ccc} \mathcal{C}_r(Y) & \xrightarrow{\subset} & \mathcal{C}_r(X) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{array} \right) = B(B(0, \mathcal{C}_r(Y), \mathcal{C}_r(X))). \tag{4.7}$$

Proposition 4.8 (Lima-Filho [Lim93]). *If Y is a closed subvariety of a projective variety X, we have*

$$L_r H_{2r+n}(X - Y) \cong \pi_{n+1}(B(B(0, \mathcal{C}_r(Y), \mathcal{C}_r(X)))).$$

Clearly, $B(B(0, \mathcal{C}_r(Y), \mathcal{C}_r(X)))$ is a connected, simplicial monoid and it is not difficult to see that it is a countable inductive limit of simplicial varieties (cf. [Lim01, § 5]). By [Lim93, 4.5] the remaining technical hypothesis required in Lemma 4.2 is also satisfied, so that we can make the following definition.

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Definition 4.9. For a quasi-projective variety U, we endow its Lawson homology groups with IMHSs as follows. Choose a projective closure X of U and let $Y = X - U$ be the reduced closed complement. Using Proposition 4.8 and Lemma 4.2, let $L_rH_{2r+n}(U)$ be an IMHS so that it is a sub-IMHS of $H^{\mathrm{sing}}_{n+1}(B(B(0, \mathcal{C}_r(Y), \mathcal{C}_r(X))))$.

Remark 4.10. The construction we used in this definition is similar, but not identical, to that given by Lima-Filho [Lim01]. The two constructions do, in fact, yield the same IMHSs for Lawson homology, but we do not prove that here.

Proposition 4.11. *The IMHS for Lawson homology is independent of the choice of projective closure. Pushforwards along projective morphisms, pullbacks along flat morphisms, and the boundary maps in long exact localization sequences such as (2.3) are all morphisms of IMHSs.*

Proof. To prove independence of choice, it suffices to consider the situation in which $U \subset X$ and $U \subset X'$ are two such closures, with closed complements Y and Y', and such that there is a morphism $p : X' \to X$ inducing the identity on U and mapping Y' to Y. Then the maps $p_* : \mathcal{C}_r(X') \to \mathcal{C}_r(X)$ and $p_* : \mathcal{C}_r(Y') \to \mathcal{C}_r(Y)$, given by the pushforward of cycles, induce a map

$$p_* : B(B(0, \mathcal{C}_r(Y'), \mathcal{C}_r(X'))) \to B(B(0, \mathcal{C}_r(Y), \mathcal{C}_r(X)))$$

of inductive limits of simplicial varieties. By Deligne's theorem (Theorem 4.1), we see that the induced map on homology groups is a morphism (in fact, an isomorphism) of IMHSs which thus restricts to an isomorphism of IMHSs on homotopy groups.

Functorality for pushforwards is proved by a slight generalization of this argument. Namely, suppose that $p : U \to V$ is a projective morphism. Then we can construct projective compactifications $U \subset \overline{U}$ and $V \subset \overline{V}$ with closed complements U_∞ and V_∞ such that p extends to a morphism $\overline{U} \to \overline{V}$ that sends U_∞ to V_∞. Then there is an induced map

$$p_* : B(B(0, \mathcal{C}_r(\overline{U_\infty}), \mathcal{C}_r(U))) \to B(B(0, \mathcal{C}_r(\overline{V_\infty}), \mathcal{C}_r(V)))$$

of inductive limits of simplicial varieties. The map on homotopy groups coincides with the usual pushforward map $p_* : L_rH_n(U) \to L_rH_n(V)$ in Lawson homology, which is thus a morphism of IMHSs.

To prove functorality for flat morphisms, if suffices to consider the case where U and V are connected and that $\pi : U \to V$ is flat of relative dimension e. We can construct projective closures $U \subset \overline{U}$ and $V \subset \overline{V}$, with closed complements U_∞ and V_∞, such that π extends to a morphism $\overline{\pi} : \overline{U} \to \overline{V}$ such that $\overline{\pi}^{-1}(U) = V$. By the platification par eclatement theorem [RG71], we can take blow-ups and proper transforms, without affecting U, V, or π, so that $\overline{\pi}$ becomes flat, also of relative dimension e. The flat pullback map $\pi^* : L_rH_n(V) \to L_{r+e}H_{n+2e}(U)$ in Lawson homology is given by taking homotopy groups of

$$B(B(0, \mathcal{C}_r(\overline{U}), \mathcal{C}_r(U_\infty))) \xrightarrow{\pi^*} B(B(0, \mathcal{C}_r(\overline{V}), \mathcal{C}_r(V_\infty))),$$

which is a morphism of inductive limits of simplicial varieties.

To show that the boundary map in the long exact localization sequence is a morphism of IMHSs, by replacing X with a projective closure $\overline{X}$ and Y with $\overline{X} - U$, and using naturality for pullbacks along open immersions, we may assume without loss of generality that X is projective. The map of commutative squares

$$\left(\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathcal{C}_r(Y) \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd}\right) \to \left(\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathcal{C}_r(X) \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd}\right) \to \left(\begin{tikzcd} \mathcal{C}_r(Y) \arrow[r, "\subset"] \arrow[d] & \mathcal{C}_r(X) \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd}\right) \to \left(\begin{tikzcd} \mathcal{C}_r(Y) \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd}\right)$$

gives a fibration sequence upon taking mapping cones and hence, using (4.6) and (4.7), we have a fibration sequence

$$B(\mathcal{C}_r(Y)) \to B(\mathcal{C}_r(X)) \to B(B(0, \mathcal{C}_r(Y), \mathcal{C}_r(X))) \to B(B(\mathcal{C}_r(Y)))$$

that gives the long exact localization sequence in Lawson homology. In particular, the map $L_rH_n(U) \to L_rH_{n-1}(Y)$ is induced by a map of inductive limits of simplicial varieties and is thus a morphism of IMHSs. ☐

Note, in particular, that the Lawson homology groups $L_0H_n(U)$ are IMHSs. On the other hand, we know $L_0H_n(U) \cong H_n^{\mathrm{BM}}(U)$ (so that, in particular, $L_0H_n(U)$ is finitely generated and thus actually an MHS) and Deligne endows the groups $H_n^{\mathrm{BM}}(U)$ with MHSs [Del74]. Obviously, one would expect these MHSs to coincide, and this is the content of Theorem 4.21 below. This result is proved for U smooth and projective by Friedlander–Mazur [FM94b]. To set up this theorem, we need to review and reinterpret the Dold–Thom isomorphism.

For a pointed CW complex $T = (T, t)$, let $SP^n(T) = T^{\times n}/\Sigma_n$ be the n-fold symmetric product of T. Let $SP^n(T) \rightarrowtail SP^{n+1}(T)$ be the map given by 'addition by t' and define $SP^\infty(T) = \varinjlim_n SP^n(T)$, using these transition maps. Then $SP^\infty(T)$ is a topological abelian monoid and we let $SP^\infty(T)^+$ denote its 'naive' topological group completion. Note that $SP^\infty(T_+) \cong \coprod_n SP^n(T)$ and if X is a projective variety, we have

$$SP^\infty(X_+^{\mathrm{an}}) \cong \mathcal{C}_0(X).$$

By [Lim01, 4.5], $SP^\infty(T)^+$ is homotopy equivalent to $\Omega BSP^\infty(T)$, the homotopy theoretic group completion of $SP^\infty(T)$. The Dold–Thom theorem [DT58] asserts that there is a natural isomorphism

$$\Psi_n^T : \pi_n(SP^\infty(T)^+) \xrightarrow{\cong} H_n^{\mathrm{sing}}(T, t)$$

such that the composition of

$$\pi_n(T) \xrightarrow{i_*} \pi_n(SP^\infty(T)^+) \xrightarrow{\cong} H_n^{\mathrm{sing}}(T, t),$$

where $i : T \subset SP^\infty(T)$ is the evident inclusion, is the Hurewicz map. More generally, the Dold–Thom theorem provides a natural isomorphism

$$\Psi_n^{(T,S)} : \pi_n(SP^\infty(T)^+/SP^\infty(S)^+) \xrightarrow{\cong} H_n^{\mathrm{sing}}(T, S)$$

where $S \subset T$ is a CW-subcomplex. (See [FM94b, Appendix B] for an explicit description of the Dold–Thom map.) We need the following description of the inverse of the Dold–Thom map.

Proposition 4.12. *For a finite, pointed CW complex T, the composition of*

$$H_q^{\mathrm{sing}}(T, t) \xrightarrow{i_*} H_q^{\mathrm{sing}}(SP^\infty(T)^+, t) \overset{p}{\twoheadrightarrow} \pi_q(SP^\infty(T)^+),$$

where p denotes the splitting of the Hurewicz map $h_{SP(T)^+} : \pi_q(SP^\infty(T)^+) \rightarrowtail H_q^{\mathrm{sing}}(SP^\infty(T)^+, t)$ given as in the proof of Lemma 4.2, is the inverse of the Dold–Thom isomorphism. Moreover, we have $h_{SP(T)^+} \circ p \circ i_* = i_*$.

Proof. We prove, more generally, that if $S \subset T$ is a subcomplex, then the composition of

$$H_q^{\mathrm{sing}}(T, S) \xrightarrow{i_*} H_q^{\mathrm{sing}}(SP^\infty(T)^+/SP^\infty(S)^+) \overset{p}{\twoheadrightarrow} \pi_q(SP(T)^+/SP(S)^+) \tag{4.13}$$

is the inverse of the Dold–Thom isomorphism. One may readily verify that this composition is given by applying π_q to the map of simplicial abelian groups

$$\mathbb{Z}\operatorname{Sing}_\bullet(T)/\mathbb{Z}\operatorname{Sing}_\bullet(S) \xrightarrow{i_*} \operatorname{Sing}_\bullet(SP^\infty(T)^+)/\operatorname{Sing}_\bullet(SP^\infty(S)^+) \tag{4.14}$$

induced by the inclusion simplicial sets $i_* : \mathrm{Sing}_\bullet(T) \subset \mathrm{Sing}_\bullet(SP^\infty(T)^+)$ and using that $\mathrm{Sing}_\bullet(SP^\infty(T)^+)$ is a simplicial abelian group. In particular, the composition of (4.13) is natural for pairs. Since the Dold–Thom isomorphism is natural for pairs too, we may reduce to the case where $T = S^n$ and $S = *$. In this case, the Hurewicz map $h_{S^n} : \pi_q(S^n) \xrightarrow{\cong} H_q^{\mathrm{sing}}(S^n, *)$ is an isomorphism and we have $\Psi_q^{S^n} \circ p \circ i_* \circ h_{S^n} = \Psi_q^{S^n} \circ p \circ h_{SP(S^n)^+} \circ i_* = \Psi_q^{S^n} \circ i_* = h_{S^n}$, which shows that $p \circ i_*$ is the inverse of $\Psi_q^{S^n}$.

The equation $h_{SP(T)^+} \circ p \circ i_* = i_*$ follows from the fact that $p \circ i_*$ is induced by the map (4.14). □

The role of cones of maps of simplicial varieties arises from the following result, essentially due to Deligne.

Theorem 4.15 (Deligne). *If $Y_\bullet \xrightarrow{u} X_\bullet$ is a morphism of simplicial, pointed varieties, then we have a long exact sequence in the category of MHSs*

$$\cdots \to H_q^{\mathrm{sing}}(Y_\bullet, *) \to H_q^{\mathrm{sing}}(X_\bullet, *) \to H_q^{\mathrm{sing}}(\mathrm{cone}_\vee(u), *) \to H_{q-1}^{\mathrm{sing}}(Y_\bullet, *) \to \cdots .$$

Proof. This is essentially the content of [Del74, 8.3.9], except that Deligne considers unpointed simplicial varieties and uses cohomology rather than homology. The former difference is superficial (although, it should be noted that Deligne's result is slightly in error since he does not use reduced homology for the relative term as he ought to) and one deduces that the maps above are morphisms of MHSs by taking $\mathbb{Q}$-linear duals of the corresponding sequence in cohomology. □

For a simplicial pointed variety $W_\bullet$, we define its *suspension* to be

$$\Sigma_*(W_\bullet) := \mathrm{cone}_\vee(W_\bullet \to *).$$

If $Y_\bullet$ is a simplicial (unpointed) variety, define

$$\Sigma(Y_\bullet) := \Sigma_*((Y_\bullet)_+) = \mathrm{cone}_\vee((Y_\bullet)_+ \to *).$$

In detail, $\Sigma(Y_\bullet)_n = Y_0 \amalg \cdots \amalg Y_{n-1} \amalg *$ with face maps as in (4.4) but with $u(y_n) = *$. It follows from Theorem 4.15 that we have an isomorphism of MHSs

$$\tilde{H}_q^{\mathrm{sing}}(\Sigma(Y_\bullet), \mathbb{Z}) \cong H_q^{\mathrm{sing}}(Y_\bullet, \mathbb{Z}). \tag{4.16}$$

For pointed varieties V and W, we have $SP^\infty(V \vee W) \cong SP^\infty(V) \oplus SP^\infty(W)$ and $SP^\infty(*) = 0$, so that SP^∞ defines a functor from the category of pointed varieties to the category of abelian topological monoids that preserves coproduct and final objects. Moreover, for a morphism $W_\bullet \xrightarrow{u} Y_\bullet$ of simplicial, pointed varieties, we have a natural isomorphism

$$SP^\infty(\mathrm{cone}_\vee(W_\bullet \to V_\bullet)) \cong \mathrm{cone}_\oplus(SP^\infty(W_\bullet) \to SP^\infty(V_\bullet)), \tag{4.17}$$

(where SP^∞ is defined on a simplicial, pointed variety by applying it degreewise), and hence a canonical map

$$\mathrm{cone}_\vee(W_\bullet \to V_\bullet) \to \mathrm{cone}_\oplus(SP^\infty(W_\bullet) \to SP^\infty(V_\bullet)), \tag{4.18}$$

induced by the inclusions of the form $Y \subset SP^\infty(Y)$. In particular, we have an isomorphism

$$SP^\infty(\Sigma_*(W_\bullet)) \cong B(SP^\infty(W_\bullet)).$$

and a canonical simplicial map

$$\Sigma_*(W_\bullet) \to B(SP^\infty(W_\bullet)). \tag{4.19}$$

The following corollary is a version of the Dold–Thom theorem for simplicial projective varieties.

Corollary 4.20. *For a simplicial, pointed, projective variety $W_\bullet$, let $\theta_{W_\bullet}$ denote the composition of*

$$H_q^{\mathrm{sing}}(\Sigma_*(W_\bullet), *) \to H_q^{\mathrm{sing}}(B(SP^\infty(W))) \twoheadrightarrow \pi_q(B(SP^\infty(W)))$$

where the first map is induced by (4.19) and the second map is the splitting of the Hurewicz map given by Lemma 4.2. Then $\theta_{W_\bullet}$ is a natural isomorphism of MHSs.

Proof. We can replace $SP^\infty(W_\bullet)$ with $SP^\infty(W_\bullet)^+$ by [Lim01]. The map (4.19) induces a map of spectral sequences from

$$E_1^{p,q} = \tilde{H}_p^{\mathrm{sing}}(W_0 \vee \cdots \vee W_{q-1}) \Longrightarrow \tilde{H}_{p+q}^{\mathrm{sing}}(\Sigma_*(W_\bullet))$$

to

$$E_1^{p,q} = \pi_p(SP(W_0)^+ \oplus \cdots \oplus SP(W_{q-1})^+) \Longrightarrow \pi_{p+q}(B(SP(W_\bullet)^+)),$$

which is an isomorphism on E_1-terms (Proposition 4.12), and thus $\theta_{W_\bullet}$ is an isomorphism. This proposition also gives us that the composition of

$$H_q^{\mathrm{sing}}(\Sigma_*(W_\bullet), *) \xrightarrow{\theta_{W_\bullet}} \pi_q(B(SP^\infty(W_\bullet))) \xrightarrow{h} H_q^{\mathrm{sing}}(B(SP^\infty(W_\bullet)))$$

coincides with the map

$$H_q^{\mathrm{sing}}(\Sigma_*(W_\bullet)) \to H_q^{\mathrm{sing}}(B(SP^\infty(W_\bullet)))$$

induced by (4.19). The latter map is a morphism of MHSs since it comes from a map of simplicial varieties and the map h is a morphism of MHSs by construction. Since h is injective, it follows that $\theta_{W_\bullet}$ is a morphism of MHSs as well. □

Theorem 4.21. *For a quasi-projective variety U, the isomorphism*

$$H_q^{\mathrm{BM}}(U, \mathbb{Z}) \cong L_q H_q(U)$$

coming from the Dold–Thom theorem is an isomorphism of MHSs. Moreover, it is natural for pushforwards and pullbacks, and it commutes with the boundary maps in a localization long exact sequence.

Proof. Choose a projective closure X of U with reduced closed complement Y, and let $W_\bullet = \mathrm{cone}_\vee(Y_+ \rightarrowtail X_+) = \mathrm{cone}_\amalg(Y \rightarrowtail X)$. We have isomorphisms

$$H_{q+1}^{\mathrm{sing}}(\Sigma_*(W_\bullet), *) \cong H_q^{\mathrm{sing}}(W_\bullet, *) \cong H_q^{\mathrm{BM}}(U),$$

which serve to define the MHS for $H_q^{\mathrm{BM}}(U)$ (see [Del74]). Corollary 4.20 gives an isomorphism of MHS

$$H_{q+1}^{\mathrm{sing}}(\Sigma_*(W_\bullet), *) \xrightarrow{\cong} \pi_q(B(SP^\infty(W_\bullet)))$$

and we have an isomorphisms of MHSs

$$\pi_q(B(SP^\infty(W_\bullet))) \cong \pi_q(B(B(0, \mathcal{C}_0(Y), \mathcal{C}_0(X))))$$

coming from the isomorphisms of simplicial varieties (4.5) and (4.17). Finally, the isomorphism

$$\pi_q(B(B(0, \mathcal{C}_0(Y), \mathcal{C}_0(X)))) \cong L_0 H_q(U)$$

of Proposition 4.8 serves to define the MHS for $L_0 H_q(U)$.

The naturality assertions are proven just as in the proof of Proposition 4.11, using the natural map (4.18) and Deligne's theorem (Theorem 4.15). □

The following theorem summarizes all the properties of the IMHS for Lawson homology that will be needed in the rest of this paper.

THEOREM 4.22 (cf. [FM94b, Lim01]). *The Lawson homology groups of any quasi-projective complex variety may be equipped with IMHSs such that the following properties hold.*

(1) *Pushforward along projective morphisms, pullbacks along flat morphisms, and the boundary maps in a long exact localization sequence are morphisms of IMHSs.*

(2) *The non-trivial weights w of $L_rH_n(U)$ satisfy $2r-n \leqslant w \leqslant 0$ and the non-trivial Hodge numbers (p,q) satisfy $2r-n \leqslant p,q \leqslant 0$.*

(3) *For a quasi-projective variety U, the s map*

$$s : L_rH_n(U) \to L_{r-1}H_n(U)(-1)$$

is a morphism of IMHSs. That is, the s-map induces maps

$$s : W_m(L_rH_n(U,\mathbb{Q})) \to W_{m+2}(L_{r-1}H_n(U,\mathbb{Q}))$$

and

$$s : F^p(L_rH_n(U,\mathbb{C})) \to F^{p+1}(L_{r-1}H_n(U,\mathbb{C})).$$

(4) *For any U, there is a morphism of IMHSs*

$$L_rH_n(U) \to H_n^{\mathrm{BM}}(U,\mathbb{Z}(r))$$

that is compatible with pushfoward along a projective morphism, pullback along a flat morphism, and the boundary map in a long exact localization sequence.

(5) *For projective varieties X and W, recall that $Z_e(X,W)$ is the group completion of the (discrete) abelian monoid $\operatorname{Hom}(X,\mathcal{C}_e(W))$ and that associated to any $\Gamma \in Z_e(X,W)$, there is a continuous map $\Gamma_* : \mathcal{Z}_r(X) \to \mathcal{Z}_{r+e}(W)$ and hence an induced map*

$$\Gamma_* : L_rH_n(X) \to L_{r+e}H_{n+2e}(W)$$

on Lawson homology groups. This map is a morphism of IMHSs.
More generally, if $Y \subset X$ and $T \subset W$ are closed subschemes, with open complements U and V, such that the image of $\mathcal{Z}_r(Y)$ under Γ_* is contained in $\mathcal{Z}_r(T)$, then there is an induced continuous map $\Gamma_* : \mathcal{Z}_r(U) \to \mathcal{Z}_r(V)$ of topological abelian groups and hence an induced map on homotopy groups $\Gamma_* : L_rH_n(U) \to L_rH_n(V)$. All of the maps in the infinite ladder

$$\begin{tikzcd}
\cdots \arrow[r] & L_rH_{2r+n}(Y) \arrow[r] \arrow[d, "\Gamma_*"] & L_rH_{2r+n}(X) \arrow[r] \arrow[d, "\Gamma_*"] & L_rH_{2r+n}(U) \arrow[r] \arrow[d, "\Gamma_*"] & L_rH_{2r+n-1}(Y) \arrow[r] \arrow[d, "\Gamma_*"] & \cdots \\
\cdots \arrow[r] & L_rH_{2r+n}(T) \arrow[r] & L_rH_{2r+n}(W) \arrow[r] & L_rH_{2r+n}(V) \arrow[r] & L_rH_{2r+n-1}(T) \arrow[r] & \cdots
\end{tikzcd}$$

are morphisms of IMHSs.

Proof. Property (1) is the content of Proposition 4.11.

By definition $L_rH_n(U)$ is a sub-IMHS of $H^{\mathrm{sing}}_{2n-r+1}(Y_\bullet)$, where $Y_\bullet$ is an inductive limit of simplicial projective varieties and $Y_0 = \operatorname{Spec}\mathbb{C}$. To prove (2), it therefore suffices to establish that these bounds on the weights and Hodge type hold for $H^{\mathrm{sing}}_{2n-r-1}(W_\bullet)$, where $W_\bullet$ is any simplicial projective variety with $W_0 = \operatorname{Spec}\mathbb{C}$. These bounds follow from [Del74, 8.2.4 and 8.3.5].

Property (3) is established by Lima-Filho [Lim01], but since he uses a different construction for the IMHSs for Lawson homology, we reproduce a version of his proof in our context. Choose a projective closure X of U with closed complement Y. Recall that the s map in Lawson homology comes from the pairing

$$L_0H_2(\mathbb{P}^1) \times L_rH_n(U) \to L_rH_{n+2}(\mathbb{P}^1 \times U) \to L_rH_{n+2}(\mathbb{A}^1 \times U) \xleftarrow{\cong} L_{r-1}H_n(U),$$

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by choosing a generator s of $L_0H_2(\mathbb{P}^1) \cong \mathbb{Z}(1)$. It thus suffices to prove this pairing is a pairing of IMHSs. Since flat pullback is a morphism of IMHSs, it suffices to show that the external product pairing

$$L_sH_m(W) \otimes L_rH_n(U) \to L_{s+r}H_{m+n}(V \times U)$$

is a morphism of IMHSs for an arbitrary projective variety W. (In fact, the hypothesis that W be projective is not necessary.) This pairing is induced by a pairing of simplicial objects comprised of maps of the form

$$\mathcal{C}_s(W) \times \mathcal{C}_r(T) \to \mathcal{C}_{s+r}(W \times T),$$

with $T = X$ or $T = Y$, along with pairings of the form

$$\pi_i(B(\mathcal{C}_s(W))) \otimes \pi_j(B(\mathcal{C}_r(T))) \to \pi_{i+j}(B(\mathcal{C}_s(W)) \times B(\mathcal{C}_r(T))).$$

The latter pairing injects (in the category IMHS) into

$$H_i^{\mathrm{sing}}(B(\mathcal{C}_s(W))) \otimes H_j^{\mathrm{sing}}(B(\mathcal{C}_r(T))) \to H_{i+j}^{\mathrm{sing}}(B(\mathcal{C}_s(W)) \times B(\mathcal{C}_r(T))),$$

which is a morphism of IMHSs by [Del74, 8.2.10].

Property (4) follows from property (3) and Theorem 4.21.

We establish property (5) by assuming, as we may, that X is connected and that Γ is given by a single morphism $X \to \mathcal{C}_{e,d}(W)$. The map on Lawson homology is then induced by the collection of morphisms of varieties

$$\mathcal{C}_{r,d'}(X) \to \mathcal{C}_{r+e,d'+d}(X),$$

as described in § 2, giving a morphism $\mathcal{C}_r(X) \to \mathcal{C}_{r+e}(W)$. Our assumption about Y and T gives a commutative square

$$\begin{tikzcd}
\mathcal{C}_r(Y) \arrow[r] \arrow[d] & \mathcal{C}_r(T) \arrow[d] \\
\mathcal{C}_r(X) \arrow[r] & \mathcal{C}_r(W)
\end{tikzcd}$$

and this square induces the commutative diagram of simplicial objects

$$\begin{tikzcd}
B(\mathcal{C}_r(Y)) \arrow[r] \arrow[d] & B(\mathcal{C}_r(X)) \arrow[r] \arrow[d] & B(B(0,\mathcal{C}_r(X),\mathcal{C}_r(Y))) \arrow[r] \arrow[d] & B(B(\mathcal{C}_r(Y))) \arrow[d] \\
B(\mathcal{C}_r(T)) \arrow[r] & B(\mathcal{C}_r(W)) \arrow[r] & B(B(0,\mathcal{C}_r(W),\mathcal{C}_r(T))) \arrow[r] & B(B(\mathcal{C}_r(Y)))
\end{tikzcd}$$

and hence the induced commutative ladder obtained by taking homotopy groups consists entirely of morphisms of IMHSs. □

5. The morphic Abel–Jacobi map

Jannsen [Jan90] has established a technique for constructing Abel–Jacobi type maps in a quite general setting. In particular, his technique applies to our situation, leading to the definition of the morphic Abel–Jacobi map.

In detail, suppose that X is a quasi-projective variety and fix an integer $r \geqslant 0$. For each closed subvariety $Y \subset X$ of dimension r with open complement U, localization for Lawson homology gives the exact sequence

$$L_rH_{2r+1}(Y) \to L_rH_{2r+1}(X) \to L_rH_{2r+1}(U) \to L_rH_{2r}(Y) \to L_rH_{2r}(X)$$

of IMHSs by Theorem 4.22. Since $L_rH_{2r+1}(Y) = 0$, $L_rH_{2r}(Y)$ is the free abelian group of integral components of Y of dimension r, and $L_rH_{2r}(X) = CH_r(X)/CH_r(X)_{\mathrm{alg}\sim 0}$, we obtain the short exact sequence

$$0 \to L_r H_{2r+1}(X) \to L_r H_{2r+1}(U) \to Z_r^Y(X)_{\mathrm{alg}\sim 0} \to 0, \tag{5.1}$$

where $Z_r^Y(X)$ denotes the group of r-cycles on X supported on Y and $Z_r^Y(X)_{\mathrm{alg}\sim 0}$ denotes the subgroup of such cycles that are algebraically equivalent to zero on X. This sequence is a short exact sequence of IMHSs and $Z_r^Y(X)_{\mathrm{alg}\sim 0}$ has the trivial Hodge structure. Thus, the boundary map in the long exact sequence for $\mathrm{Ext}^*_{\mathrm{IMHS}}(\mathbb{Z}(0), -)$ determines a map

$$Z_r^Y(X)_{\mathrm{alg}\sim 0} \to \mathrm{Ext}^1_{\mathrm{IMHS}}(\mathbb{Z}(0), L_r H_{2r+1}(X)) = \mathcal{J}(L_r H_{2r+1}(X)).$$

Explicitly, an element in $Z_r^Y(X)_{\mathrm{alg}\sim 0}$ coincides with a morphism $\mathbb{Z}(0) \to Z_r^Y(X)_{\mathrm{alg}\sim 0}$ of IMHSs and pulling back (5.1) along this morphism give an extension of IMHSs of the form

$$0 \to L_r H_{2r+1}(X) \to E \to \mathbb{Z}(0) \to 0.$$

Definition 5.2. For a quasi-projective complex variety X and integer $r \geqslant 0$, define the rth *morphic Jacobian* to be

$$\mathcal{J}_r^{\mathrm{mor}}(X) = \mathcal{J}(L_r H_{2r+1}(X)) = \mathrm{Ext}^1_{\mathrm{IMHS}}(\mathbb{Z}(0), L_r H_{2r+1}(X)),$$

and define the rth *morphic Abel–Jacobi map*

$$\Phi_r^{\mathrm{mor}} : \mathcal{Z}_r(X)_{\mathrm{alg}\sim 0} \to \mathcal{J}_r^{\mathrm{mor}}(X)$$

to be the map induced from the maps $Z_r^Y(X)_{\mathrm{alg}\sim 0} \to \mathrm{Ext}^1_{\mathrm{IMHS}}(\mathbb{Z}(0), L_r H_{2r+1}(X))$ above by taking the filtered colimit over all dimension r closed subschemes $Y \subset X$.

If X and W are projective varieties, an equidimensional correspondence $\Gamma \in Z_s(X, W)$ induces a morphism of IMHSs

$$\Gamma_* : L_r H_{2r+1}(X) \to L_{r+e} H_{2(r+e)+1}(W),$$

by Theorem 4.22(5). Since $\mathcal{J}(-)$ is a functor on the category of IMHSs, we thus obtain the map

$$\Gamma_* : \mathcal{J}_r^{\mathrm{mor}}(X) \to \mathcal{J}_{r+e}^{\mathrm{mor}}(W).$$

Theorem 5.3. *The morphic Abel–Jacobi mapping is functorial for equidimensional correspondences between projective varieties. That is, if X and W are projective varieties and $\Gamma \in Z_e(X, W)$, then the diagram*

$$\begin{tikzcd} Z_r(X)_{\mathrm{alg}\sim 0} \arrow[r] \arrow[d, "\Gamma_*"'] & \mathcal{J}_r^{\mathrm{mor}}(X) \arrow[d, "\Gamma_*"'] \\ Z_{r+e}(W)_{\mathrm{alg}\sim 0} \arrow[r] & \mathcal{J}_{r+e}^{\mathrm{mor}}(W) \end{tikzcd}$$

commutes.

Proof. Fix a subvariety $Y \subset X$ of dimension at most r. Then the image of the subgroup $Z_r(Y) \subset Z_r(X)$ under $\Gamma_* : Z_r(X) \to Z_{r+e}(W)$ is contained in $Z_{r+e}(T)$ for some subvariety $T \subset W$ of dimension at most $r + e$. Thus, we have a commutative diagram

$$\begin{tikzcd} \mathcal{Z}_r(Y) \arrow[r, "\Gamma_*"] \arrow[d, "\subset"'] & \mathcal{Z}_{r+e}(T) \arrow[d, "\subset"] \\ \mathcal{Z}_r(X) \arrow[r, "\Gamma_*"] \arrow[d] & \mathcal{Z}_{r+e}(W) \arrow[d] \\ \mathcal{Z}_r(U) \arrow[r, "\Gamma_*"] & \mathcal{Z}_{r+e}(V) \end{tikzcd}$$

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where $U = X - Y$, $V = W - T$, and the map on the bottom is defined by the identities $\mathcal{Z}_r(U) = \mathcal{Z}_r(X)/\mathcal{Z}_r(Y)$ and $\mathcal{Z}_r(V) = \mathcal{Z}_r(W)/\mathcal{Z}_r(T)$.

Taking homotopy groups for this diagram gives a commutative ladder of IMHSs by Theorem 4.22, and upon taking limits over pairs Y, T with $\Gamma_*(Z_r(Y)) \subset Z_r(T)$, we arrive at the following commutative diagram of IMHSs.

$$\begin{tikzcd}
0 \arrow[r] & L_rH_{2r+1}(X) \arrow[r] \arrow[d, "\Gamma_*"'] & \varinjlim_Y L_rH_{2r+1}(X-Y) \arrow[r] \arrow[d] & Z_r(X)_{\mathrm{alg}\sim 0} \arrow[r] \arrow[d, "\Gamma_*"] & 0 \\
0 \arrow[r] & L_{r+e}H_{2r+2e+1}(W) \arrow[r] & \varinjlim_T L_{r+e}H_{2r+2e+1}(W-T) \arrow[r] & Z_{r+e}(W)_{\mathrm{alg}\sim 0} \arrow[r] & 0
\end{tikzcd}$$

The result is now evident by the functorality of the long exact sequence for $\mathrm{Ext}^*_{\mathrm{IMHS}}(\mathbb{Z}(0), -)$. $\square$

Recall that the rth intermediate Griffiths Jacobian of a projective variety X is

$$\mathcal{J}_r(X) := \mathcal{J}(H^{\mathrm{sing}}_{2r+1}(X, \mathbb{Z}(r))),$$

and it is the target of the (classical) Abel–Jacobi map:

$$\Phi_r : Z_r(X)_{\mathrm{hom}\sim 0} \to \mathcal{J}_r(X).$$

If one repeats the above construction of the morphic Abel–Jacobi map Φ_r^{mor} using instead the groups $H^{\mathrm{BM}}_*(-, \mathbb{Z})$, the singular Borel–Moore homology groups of complex varieties viewed as taking values in MHSs, then one obtains the classical Abel–Jacobi map Φ_r (cf. [Car80] and [Jan90, 9.2]). By Theorem 4.22(4), the natural map $L_rH_n(U) \to H^{\mathrm{BM}}_n(U, \mathbb{Z}(r))$ is a morphisms of IMHSs, and so we have an induced map $\mathcal{J}^{\mathrm{mor}}_r(U) \to \mathcal{J}_r(U)$.

Proposition 5.4. *For any quasi-projective variety X, the diagram*

$$\begin{tikzcd}
Z_r(X)_{\mathrm{alg}\sim 0} \arrow[r, "\Phi_r^{\mathrm{mor}}"] \arrow[d, hook] & \mathcal{J}^{\mathrm{mor}}_r(X) \arrow[d] \\
Z_r(X)_{\mathrm{hom}} \arrow[r, "\Phi_r"] & \mathcal{J}_r(X)
\end{tikzcd}$$

commutes.

Proof. This follows immediately from Jannsen's construction, using that we have a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & L_rH_n(Y) \arrow[r] \arrow[d] & L_rH_n(X) \arrow[r] \arrow[d] & L_rH_n(U) \arrow[r] \arrow[d] & L_rH_{n-1}(Y) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^{\mathrm{BM}}_n(Y, \mathbb{Z}(r)) \arrow[r] & H^{\mathrm{BM}}_n(X, \mathbb{Z}(r)) \arrow[r] & H^{\mathrm{BM}}_n(U, \mathbb{Z}(r)) \arrow[r] & H^{\mathrm{BM}}_{n-1}(Y, \mathbb{Z}(r)) \arrow[r] & \cdots
\end{tikzcd}$$

of IMHSs by Theorem 4.22, for any closed subscheme $Y \subset X$ with open complement U. $\square$

Example 5.5. The morphic Abel–Jacobi map coincides with its classical counter-part for zero cycles and codimension one cycles. This is conjecturally true also for codimension two cycles.

In detail, since $L_0H_n(X) \cong H^{\mathrm{BM}}_n(X, \mathbb{Z}(0))$ as MHSs for all quasi-projective varieties U, we have

$$\Phi_0^{\mathrm{mor}} = \Phi_0.$$

When X is smooth and projective, the map $\Phi_0^{\mathrm{mor}} = \Phi_0$ coincides under Poincare duality with the Albanese map $X \to \mathrm{Alb}(X) = \mathcal{J}(H^{2d-1}_{\mathrm{sing}}(X, \mathbb{Z}(d)))$ where $d = \dim(X)$. See [Lew99, 12.11(3)] for a description of this map in terms of integrals.

Likewise, if U is a variety of dimension at most d and we take $r = d-1$, then Proposition 2.8 gives the isomorphism

$$L_{d-1}H_{2d-1}(U) \cong N_{n-d+1}H^{\mathrm{BM}}_{2d-1}(U,\mathbb{Z}(d-1)).$$

(Recall that $N_k H^{\mathrm{BM}}_n$ denotes the subgroup of H^{BM}_n consisting of classes supported in dimension k.) From this, we deduce $\Phi^{\mathrm{mor}}_{d-1} = \Phi_{d-1}$ on a variety X of dimension d. If X is smooth and projective, then using Poincare duality, we have

$$\mathcal{J}^{\mathrm{mor}}_{d-1}(X) = \mathcal{J}_{d-1}(X) = \mathcal{J}(H^1(X,\mathbb{Z}(1))) = \mathrm{Pic}^0(X),$$

and the map

$$\Phi_{d-1} = \Phi^{\mathrm{mor}}_{d-1} : Z^1(X)_{\mathrm{alg}\sim 0} \to \mathrm{Pic}^0(X)$$

sends a codimension one cycle algebraically (equivalently, homologically) equivalent to zero to the corresponding point on the Picard variety of X.

A special case of Suslin's conjecture (Conjecture 2.7) predicts that the onto map

$$L_{d-2}H_{2d-3}(X) \twoheadrightarrow N_{d-1}H^{\mathrm{BM}}_{2d-3}(X,\mathbb{Z}(d-2))$$

is actually an isomorphism, for any quasi-projective variety X of dimension d. The validity of this formula would give that $\Phi^{\mathrm{mor}}_{d-2}$ and Φ_{d-2} coincide on $Z_{d-2}(X)_{\mathrm{alg}\sim 0}$, for all such X.

We see, therefore, that the cases of most interest for the morphic Abel–Jacobi map concern dimension r cycles where $0 < r < d-2$.

Example 5.6. In particular, if we take $X = C$ to be a possibly singular projective curve and $r = 0$, then the morphic Abel–Jacobi map

$$\Phi^{\mathrm{mor}}_0 : \mathcal{Z}_0(C)_{\mathrm{alg}\sim 0} \to \mathcal{J}^{\mathrm{mor}}_0(C)$$

coincides up to isomorphism with the classical Albanese map

$$\mathcal{Z}_0(C)_{\mathrm{deg}=0} \twoheadrightarrow \mathcal{J}_0(C) = \mathrm{Alb}(C).$$

If C is smooth, then $\mathrm{Alb}(C) \cong \mathrm{Pic}^0(C)$, the Jacobian variety of C, and this map is the canonical one identifying the Jacobian variety as degree 0 zero cycles on C modulo rational equivalence. More generally, if $\tilde{C} \twoheadrightarrow C$ is the normalization of a singular curve C, then $\mathcal{J}_0(C)$ is a quotient of $\mathrm{Pic}^0(\tilde{C})$ by a free abelian subgroup. The map $\Phi^{\mathrm{mor}}_0 = \Phi_0$ is uniquely determined by the existence of the commutative square of continuous surjections

$$\begin{tikzcd}
\mathcal{Z}_0(\tilde{C})_{\mathrm{deg}=0} \arrow[r, two heads] \arrow[d, two heads] & \mathrm{Pic}^0(\tilde{C}) \arrow[d, two heads] \\
\mathcal{Z}_0(C)_{\mathrm{deg}=0} \arrow[r, two heads] & \mathcal{J}_0(C)
\end{tikzcd}$$

Using the facts above, we derive numerous good properties of the morphic Abel–Jacobi map. For a smooth, projective variety X, the image of $Z_r(X)_{\mathrm{alg}\sim 0}$ under the classical Abel–Jacobi map is known (cf. [Lew99, § 12] and [Lie68]) to be a abelian variety (not merely a complex torus) and is called the *Lieberman Jacobian*, written

$$\mathcal{J}^a_r(X) := \Phi_r(Z_r(X)_{\mathrm{alg}\sim 0}).$$

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Theorem 5.7. *Let X be a (possibly singular) projective variety.*

(1) *Give $Z_r(X)_{\mathrm{alg}\sim 0}$ the structure of a topological abelian group by identifying it with the connected component of the identity in $\mathcal{Z}_r(X)$. The morphic Abel–Jacobi map*

$$\Phi_r^{\mathrm{mor}} : Z_r(X)_{\mathrm{alg}\sim 0} \twoheadrightarrow \mathcal{J}_r^{\mathrm{mor}}(X)$$

is a surjective, continuous homomorphism of topological abelian groups. Here, $\mathcal{J}_r^{\mathrm{mor}}(X)$ is topologized by viewing it as a quotient of the complex vector space $L_r H_{2r+1}(X)_{\mathbb{C}}$, which itself is topologized as a direct limit of its finite-dimensional subspaces.

(2) *The morphic Abel–Jacobi map factors through cycles modulo rational equivalence, i.e. there is a commutative diagram of the following form.*

$$\begin{tikzcd} Z_r(X)_{\mathrm{alg}\sim 0} \arrow[r, two heads] \arrow[dr, two heads, "\Phi_r^{\mathrm{mor}}"'] & CH_r(X)_{\mathrm{alg}\sim 0} \arrow[d, two heads] \\ & \mathcal{J}_r^{\mathrm{mor}}(X) \end{tikzcd}$$

We also write Φ_r^{mor} for the vertical map in this diagram.

(3) *We have a commutative diagram of continuous, surjective homomorphisms of topological abelian groups*

$$\begin{tikzcd} Z_r(X)_{\mathrm{alg}\sim 0} \arrow[r, two heads, "\Phi_r^{\mathrm{mor}}"] \arrow[dr, two heads, "\Phi_r"'] & \mathcal{J}_r^{\mathrm{mor}}(X) \arrow[d, two heads] \\ & \mathcal{J}_r^{a}(X) \end{tikzcd}$$

and the following commutative diagram of abelian groups

$$\begin{tikzcd} CH_r(X)_{\mathrm{alg}\sim 0} \arrow[r, two heads, "\Phi_r^{\mathrm{mor}}"] \arrow[dd, tail] & \mathcal{J}_r^{\mathrm{mor}}(X) \arrow[d, two heads] \\ & \mathcal{J}^{a}(X) \arrow[d, tail] \\ CH_r(X)_{hom\sim 0} \arrow[r, "\Phi_r"'] & \mathcal{J}_r(X) \end{tikzcd}$$

(4) *The morphic Abel–Jacobi map on $CH.(-)_{\mathrm{alg}\sim 0}$ is functorial for all correspondences $\gamma \in CH_{\dim(Y)+r}(Y \times X)$, for smooth, projective varieties X and Y and integers $r \geqslant 0$, i.e. the diagram*

$$\begin{tikzcd} CH_r(Y)_{\mathrm{alg}\sim 0} \arrow[r] \arrow[d, "\gamma_*"] & \mathcal{J}_r^{\mathrm{mor}}(Y) \arrow[d, "\gamma_*"] \\ CH_{r+t}(X)_{\mathrm{alg}\sim 0} \arrow[r] & \mathcal{J}_{r+t}^{\mathrm{mor}}(X) \end{tikzcd}$$

commutes, for all r.

Proof. To prove part (1), we use that the topology on $\mathcal{Z}_r(X)_0$ is 'generated by curves'. That is, letting C range over all projective curves and Γ over all equidimensional correspondences $\Gamma \in Z_r(C, X)$, we define

$$\theta : \bigoplus_{C,\Gamma} \mathcal{Z}_0(C) \to \mathcal{Z}_r(X)$$

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to be the map that on the summand indexed by C, Γ sends $c \in \mathcal{Z}_0(C)$ to $\Gamma_*(c) \in \mathcal{Z}_r(X)$. Then both θ and the restriction of θ to $\bigoplus \mathcal{Z}_0(C)_0$ are quotient maps in the category of topological abelian groups. To see this, recall that $\mathcal{Z}_0(C)$ and $\mathcal{Z}_r(X)$ are topologized as quotients of the abelian monoids $\coprod_{e,e'} \mathcal{C}_{0,e}(C) \times \mathcal{C}_{0,e'}(C)$ and $\coprod_{e,e'} \mathcal{C}_{r,e}(X) \times \mathcal{C}_{r,e'}(X)$ and that θ is induced by the collection of morphisms of the form

$$\mathcal{C}_{0,e}(C) \times \mathcal{C}_{0,e'}(C) \to \mathcal{C}_{r,e+f}(X) \times \mathcal{C}_{r,e'+f'}(X)$$

induced from pairs of algebraic morphisms $C \to \mathcal{C}_{r,f}(X)$, $C \to \mathcal{C}_{r,f'}(X)$. It follows that θ is continuous, closed, and surjective. To see that θ induces a surjection from $\bigoplus \mathcal{Z}_0(C)_0$ onto $\mathcal{Z}_r(X)_0$ as well, note that an element in the target has the form $Z_1 - Z_2$ for a pair of effective cycles Z_1, Z_2 lying in the same connected component of $\mathcal{C}_{r,e}(X)$ for some e. Thus, there is a curve C, closed points $c_0, c_1 \in C$, and a morphism $C \to \mathcal{C}_{r,e}(X)$ such that $c_i \in C$ maps to Z_i, for $i = 1, 2$, so that $\theta(c_1 - c_2) = Z_1 - Z_2$.

Likewise, the IMHS $L_r H_{2r+1}(X)$ is generated by curves in the sense that

$$\bigoplus_{C,\Gamma} H_1^{\text{sing}}(C, \mathbb{Z}(0)) \cong \bigoplus_{C,\Gamma} L_0 H_1(C) \to L_r H_{2r+1}(X) \tag{5.8}$$

is surjective. To see this, recall that we have

$$L_r H_{2r+1}(X) = \pi_1(\mathcal{Z}_r(X)) \cong H_1^{\text{sing}}(\mathcal{Z}_r(X)) \cong \varinjlim_e H_1^{\text{sing}}(\mathcal{C}_{r,e}(X)).$$

By the Lefschetz theorem of Andreotti and Frankel [AF59], the map $H_1^{\text{sing}}(C) \to H_1^{\text{sing}}(\mathcal{C}_{r,e}(X))$ is surjective for some (possibly singular) curve $C \subset \mathcal{C}_{r,e}(X)$, and the inclusion $C \rightarrowtail \mathcal{C}_{r,e}(X)$ determines a correspondence. This shows that the image of $H_1^{\text{sing}}(\mathcal{C}_{r,e}(X)) \to L_r H_{2r+1}(X)$ is contained in the image of (5.8), which establishes the claim.

Since $\mathcal{J}(-)$ is a left-exact functor, the induced map

$$\bigoplus_{C,\Gamma} \mathcal{J}_0(C) \twoheadrightarrow \mathcal{J}_r^{\text{mor}}(X)$$

is a quotient map of topological abelian groups. Since the morphic Abel–Jacobi map is natural for equidimensional correspondences, we have the following commutative square.

$$\begin{tikzcd}
\bigoplus_{C,\Gamma} \mathcal{Z}_0(C)_0 \arrow[r, two heads] \arrow[d, two heads] & \bigoplus_{C,\Gamma} \mathcal{J}_0(C) \arrow[d, two heads] \\
\mathcal{Z}_r(X)_0 \arrow[r] & \mathcal{J}_r^{\text{mor}}(X)
\end{tikzcd}$$

We know that the vertical arrows are quotients maps of topological abelian groups, and the top arrow is surjective by Example 5.6. It follows that the bottom arrow is also a continuous surjection.

To prove part (2), we need to show that $\Phi_r^{\text{mor}}(\gamma) = 0$ if $\gamma \in Z_r(X)$ is rationally equivalent to 0. For such a γ, writing $\gamma = \gamma^+ - \gamma^-$ for effective cycles γ^+, γ^-, we have that γ^+ can be joined to γ^- by a sequence of 'elementary' rational equivalences, i.e. those given by morphisms from $\mathbb{P}^1$ to $\mathcal{C}_{r,e}(X)$. In other words, γ will lie in the image of $\Gamma_* : Z_0(\mathbb{P}^1)_0 \to Z_r(X)_0$ for some correspondence $\Gamma \in Z_r(\mathbb{P}^1, X)$. The result now follows from Theorem 5.3 since $\mathcal{J}_0^{\text{mor}}(\mathbb{P}^1) = \mathcal{J}_0(\mathbb{P}^1) = 0$.

Result (3) follows from Proposition 5.4 and part (2).

Every cycle class in $CH_{\dim(Y)+r}(Y \times X)$ is linearly equivalent to an equidimensional correspondence [Sus00, 3.2], and thus part (4) follows from Theorem 5.3 and part (2). $\square$

For a projective variety X, we have from Proposition 2.8 that the image of the morphism of IMHSs

$$L_r H_{2r+1}(X) \to H_{2r+1}^{\text{sing}}(X, \mathbb{Z}(r))$$

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is $N_{r+1}H^{\text{sing}}_{2r+1}(X,\mathbb{Z}(r))$. Recall that for any abelian group A, the group $N_kH^{\text{sing}}_n(X,A)$ consists of those classes supported on subschemes of dimension k. Note that we have $N_kH^{\text{sing}}_n(X,\mathbb{Q}) = N_kH^{\text{sing}}_n(X,\mathbb{Q})\otimes_{\mathbb{Z}}\mathbb{Q}$. Since $H^{\text{sing}}_n(X,\mathbb{Z})$ is finitely generated, we have that $N_kH^{\text{sing}}_n(X,\mathbb{Z})$ is the kernel of $H^{\text{sing}}_n(X,\mathbb{Z}) \to H^{\text{BM}}_n(X-Y,\mathbb{Z})$ for a suitable closed subvariety Y. It follows that $N_kH^{\text{sing}}_n(X,\mathbb{Z})$ is a sub-MHS of $H^{\text{sing}}_n(X,\mathbb{Z})$. Define $N_kH^{\text{sing}}_n(X,\mathbb{Z}(r)) = N_rH^{\text{sing}}_n(X,\mathbb{Z}(0))\otimes\mathbb{Z}(r)$, a sub-MHS of $H^{\text{sing}}_n(X,\mathbb{Z}(r))$.

We obtain, in particular, a factorization

$$\mathcal{J}^{\text{mor}}_r(X) \twoheadrightarrow \mathcal{J}(N_{r+1}H^{\text{sing}}_{2r+1}(X,\mathbb{Z}(r))) \twoheadrightarrow \mathcal{J}^a_r(X)$$

of the canonical map. In light of Theorem 5.7, we thus obtain the following new result about the classical Abel–Jacobi map.

Corollary 5.9. *For a projective variety X, the restriction of the classical Abel–Jacobi map to cycles algebraically equivalent to zero factors as*

$$CH_r(X)_{\text{alg}\sim 0} \twoheadrightarrow \mathcal{J}(N_{r+1}H^{\text{sing}}_{2r+1}(X,\mathbb{Z}(r))) \twoheadrightarrow \mathcal{J}^a_r(X).$$

If X is also smooth, we have an exact sequence

$$\begin{aligned} H_{2r+1}(X,\mathbb{Z}(r))_{\text{tor}} &\to (H_{2r+1}(X,\mathbb{Z}(r))/N_{r+1}H_{2r+1}(X,\mathbb{Z}(r)))_{\text{tor}} \\ &\to \mathcal{J}(N_{r+1}H^{\text{sing}}_{2r+1}(X,\mathbb{Z}(r))) \to \mathcal{J}^a_r(X) \to 0.\end{aligned}$$

Proof. The exact sequence comes from the long exact sequence for $\text{Ext}^*_{\text{MHS}}(\mathbb{Z}(0),-)$, using the fact that

$$\Gamma(H) = \text{Hom}_{\text{MHS}}(\mathbb{Z}(0),H) = H_{\text{tor}}$$

for an MHS H of pure weight -1. □

Remark 5.10. If X is smooth and projective, the surjection

$$\mathcal{J}(N_{r+1}H^{\text{sing}}_{2r+1}(X,\mathbb{Z}(r))) \twoheadrightarrow \mathcal{J}^a_r(X)$$

is, in fact, a morphism of abelian varieties. This follows from Lemma 7.3 in § 7 below. As discussed in § 7, the corollary thus indicates a possible counter-example to the conjectured universality of the classical Abel–Jacobi map on $CH_r(X)_{\text{alg}\sim 0}$. Namely, if the map

$$H_{2r+1}(X,\mathbb{Z}(r))_{\text{tor}} \to (H_{2r+1}(X,\mathbb{Z}(r))/N_{r+1}H_{2r+1}(X,\mathbb{Z}(r)))_{\text{tor}}$$

fails to be surjective for some smooth, projective variety X, then such universality fails. We know of no such examples, but see § 7 for further discussion.

6. Using the morphic Abel–Jacobi map to detect cycles

Recall that a cycle is 'Abel–Jacobi equivalent' to zero if it is homologically equivalent to zero and lies in the kernel of the classical Abel–Jacobi map (equivalently, if it lies in the kernel of the Deligne cycle class map). Theorem 5.7 suggests that we can find cycles that are algebraically equivalent to zero, Abel–Jacobi equivalent to zero, and yet not 'morphic Abel–Jacobi' equivalent to zero, meaning not in the kernel of Φ^{mor}_r. Indeed, such cycles will exist precisely when the map

$$\mathcal{J}^{\text{mor}}_r(X) \twoheadrightarrow \mathcal{J}^a_r(X)$$

has a non-trivial kernel. In this section we give examples of varieties X and integers r for which this holds by building upon examples constructed by Friedlander. Additional examples, arising from the work of Schoen [Sch00], where the above map has a non-trivial kernel are discussed in § 8.

In [Nor93], Nori developed a new and subsequently widely used technique that, among other things, constructs non-trivial elements in the Griffiths groups of certain general complete intersections. Friedlander used Nori's technique in establishing the following result.

THEOREM 6.1 (Friedlander [Fri00b, 4.5]). *Fix an integer $r \geqslant 1$. Suppose that W is a smooth complete intersection in projective space, $\dim(W) = 2r+2$, and the image of $CH_{r+1}(W)_{\mathbb{Q}} \to H^{\text{sing}}_{2r+2}(W,\mathbb{Q})$ has rank at least 2. Let Y be a sufficiently general member of a Lefschetz pencil of codimension one complete intersections in W of sufficiently large degree. Then there exists an r-cycle γ on Y such that*

$$\gamma \in \ker(L_rH_{2r}(Y) \xrightarrow{s^r} L_0H_{2r}(Y) \cong H^{\text{sing}}_{2r}(Y))$$

but

$$\gamma \notin \ker(L_rH_{2r}(Y) \xrightarrow{s^{r-1}} L_1H_{2r}(Y)) \otimes \mathbb{Q}.$$

That is, there is a $\gamma \in L_rH_{2r}(Y)$ that is homologically equivalent to 0 but that remains non-zero, even modulo torsion, in the penultimate stage of the sequence of maps

$$L_rH_{2r}(Y) \xrightarrow{s} L_{r-1}H_{2r}(Y) \xrightarrow{s} \cdots \xrightarrow{s} L_1H_{2r}(Y) \xrightarrow{s} L_0H_{2r}(Y) \cong H^{\text{sing}}_{2r}(Y).$$

We construct the examples we seek by building on those of Friedlander's theorem.

THEOREM 6.2. *Pick an integer $r \geqslant 2$ and a variety Y, smooth of dimension $2r+1$, as in Friedlander's theorem (Theorem 6.1). Let $X = Y \times C$ where C is a smooth, projective curve of genus at least one. Then the kernel of*

$$\mathcal{J}^{\text{mor}}_r(X) \twoheadrightarrow \mathcal{J}^a_r(X)$$

contains the quotient of a non-zero complex vector space by a countable subgroup; in particular, it contains uncountable many non-torsion elements.

Proof. Recall that

$$\text{Griff}_r(Y) = \ker(L_rH_{2r}(Y) \to L_0H_{2r}(Y)),$$

is the Griffiths group of r-cycles homologically equivalent to 0 modulo algebraic equivalence, and note $\text{Griff}_r(Y)_{\mathbb{Q}} \neq 0$. The exterior product gives a map

$$\times : \text{Griff}_r(Y) \otimes L_0H_1(C) \to L_rH_{2r+1}(X)$$

and the composition of this map with $L_rH_{2r+1}(X) \to H^{\text{sing}}_{2r+1}(X)$ is zero, since it coincides with the composition of

$$\text{Griff}_r(Y) \otimes L_0H_1(C) \to H^{\text{sing}}_{2r}(Y) \otimes H^{\text{sing}}_1(C) \to H^{\text{sing}}_{2r+1}(X).$$

On the other hand, the basic formulas satisfied by the cup and cap product imply that the composition of

$$L_rH_{2r}(Y) \otimes L_0H_1(C) \otimes L^1H^1(C) \xrightarrow{\times \otimes \pi_2^*} L_rH_{2r+1}(X) \otimes L^1H^1(X) \xrightarrow{\cap} L_{r-1}H_{2r}(X) \xrightarrow{(\pi_1)_*} L_{r-1}H_{2r}(Y)$$

coincides with the composition of

$$\begin{aligned} L_rH_{2r}(Y) \otimes L_0H_1(C) \otimes L^1H^1(C) &\xrightarrow{\cong} L_rH_{2r}(Y) \otimes L^1H^1(C) \otimes L^1H^1(C) \\ &\xrightarrow{id \otimes \cup} L_rH_{2r}(Y) \otimes L^2H^2(C) \xrightarrow{id \otimes \pi_*} L_rH_{2r}(Y) \otimes L^1H^0(pt) \xrightarrow{\cap} L_{r-1}H_{2r}(Y). \end{aligned}$$

The map $L^1H^1(C) \otimes L^1H^1(C) \xrightarrow{\cup} L^2H^2(C)$ is onto and the map $\pi_* : L^2H^2(C) \to L^1H^0(pt) \cong \mathbb{Z}$ is an isomorphism, since C is a smooth projective curve of genus at least one. Moreover, the map

$$L_rH_{2r}(Y) \cong L_rH_{2r}(Y) \otimes L^1H^0(pt) \xrightarrow{\cap} L_{r-1}H_{2r}(Y)$$

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is the s-map. In Friedlander's example, there are elements of $\mathrm{Griff}_r(Y) \subset L_rH_{2r}(Y)$ that are not in the kernel of the s-map (even tensor $\mathbb{Q}$).

It follows that the image of

$$\mathrm{Griff}_r(Y,\mathbb{Q}) \otimes L_0H_1(C,\mathbb{Q}) \to L_rH_{2r+1}(X,\mathbb{Q})$$

is a non-zero $\mathbb{Q}$-IMHS of pure weight -1 that maps to zero in $H^{\mathrm{sing}}_{2r+1}(X,\mathbb{Q}(r))$. Letting

$$K = \ker(L_rH_{2r+1}(X) \to H^{\mathrm{sing}}_{2r+1}(X,\mathbb{Z}(r))),$$

we have that K is an IMHS having weights $-1 \leqslant w \leqslant 0$ and $W_{-1}(K_{\mathbb{Q}}) \neq 0$. Applying the long exact sequence for $\mathrm{Ext}^*_{\mathrm{IMHS}}(\mathbb{Z}(0),-)$ to the short exact sequence of IMHSs

$$0 \to K \to L_rH_{2r+1}(X) \to N_{r+1}H^{\mathrm{sing}}_{2r+1}(X,\mathbb{Z}(r)) \to 0$$

gives the exact sequence

$$N_{r+1}H^{\mathrm{sing}}_{2r+1}(X)_{\mathrm{tor}} \to \mathcal{J}(K) \to \mathcal{J}^{\mathrm{mor}}_r(X) \to \mathcal{J}(N_{r+1}H^{\mathrm{sing}}_{2r+1}(X,\mathbb{Z}(r))) \to 0.$$

Here, $\Gamma(N_{r+1}H^{\mathrm{sing}}_{2r+1}(X,\mathbb{Z}(r))) = N_{r+1}H^{\mathrm{sing}}_{2r+1}(X)_{\mathrm{tor}}$ since $N_{r+1}H^{\mathrm{sing}}_{2r+1}(X,\mathbb{Z}(r))$ is pure of weight -1.

Since the map $\mathcal{J}^{\mathrm{mor}}_r(X) \twoheadrightarrow \mathcal{J}^a_r(X)$ factors through $\mathcal{J}(N_{r+1}H^{\mathrm{sing}}_{2r+1}(X,\mathbb{Z}(r)))$, we have that $\mathcal{J}(K)/(N_{r+1}H^{\mathrm{sing}}_{2r+1}(X)_{\mathrm{tor}})$ injects into the kernel of $\mathcal{J}^{\mathrm{mor}}_r(X) \twoheadrightarrow \mathcal{J}^a_r(X)$. Finally, Proposition 3.3 shows that $\mathcal{J}(K)$ is the quotient of a non-trivial complex vector space (namely, $K_{\mathbb{C}}/F^0(K_{\mathbb{C}})$, which is non-zero since $W_{-1}(K_{\mathbb{Q}}) \neq 0$) by a countable group (namely, the image of K), and thus $\mathcal{J}(K)/(N_{r+1}H^{\mathrm{sing}}_{2r+1}(X)_{\mathrm{tor}})$ is also the quotient of a non-zero complex vector space by a countable subgroup. $\square$

Corollary 6.3. *For Y and r as in the theorem, there exist uncountably many non-torsion elements in $CH_r(Y)$ that are algebraically equivalent to 0, lie in the kernel of the Abel–Jacobi map, but do not lie in the kernel of the morphic Abel–Jacobi map. That is,*

$$\ker(\Phi_r) \cap CH_r(X)_{\mathrm{alg}\sim 0}/\ker(\Phi^{\mathrm{mor}}_r)$$

has uncountable rank.

7. On universality

Definition 7.1. For a smooth, projective complex variety X and an abelian variety A (respectively, a complex torus A), a function

$$f : CH_r(X)_{\mathrm{alg}\sim 0} \to A(\mathbb{C})$$

is *regular* (respectively, *analytic*) if given any smooth, connected, projective variety T of dimension d, base point $t_0 \in T$, and correspondence $\Gamma \in CH_{r+d}(T \times X)$, the map

$$T(\mathbb{C}) \xrightarrow{t \mapsto t - t_0} CH_0(T)_{\mathrm{alg}\sim 0} \xrightarrow{\Gamma_*} CH_r(X)_{\mathrm{alg}\sim 0} \xrightarrow{f} A(\mathbb{C})$$

is induced by a morphism of varieties (respectively, is holomorphic).

Conjecture 7.2 (Universality of the Abel–Jacobi map; cf. [Lew99]). For a smooth, projective complex variety X, the classical Abel–Jacobi map

$$\Phi_r : CH_r(X)_{\mathrm{alg}\sim 0} \twoheadrightarrow \mathcal{J}^a_r(X)$$

is universal among regular functions. That is, given an abelian variety A and a regular function

$$f : CH_r(X)_{\mathrm{alg}\sim 0} \to A,$$

the diagram

$$\begin{tikzcd} CH_r(X)_{\mathrm{alg}\sim 0} \arrow[rr, two heads, "\Phi_r"] \arrow[dr, "f"'] & & \mathcal{J}_r^a(X) \arrow[dl, dashed, "\overline{f}"] \\ & A & \end{tikzcd}$$

can be completed to a commutative diagram by some (necessarily unique) morphism of abelian varieties $\overline{f}$.

In this section, we indicate two possible ways in which the morphic Abel–Jacobi map might lead to a counter-example of the above conjecture. The first way has already been mentioned: namely, as shown in Corollary 5.9, the morphic Abel–Jacobi map implies the existence of a factorization

$$CH_r(X)_{\mathrm{alg}\sim 0} \twoheadrightarrow \mathcal{J}(N_{r+1}H_{2r+1}^{\mathrm{sing}}(X,\mathbb{Z}(r))) \twoheadrightarrow \mathcal{J}_r^a(X).$$

Now, the map $\mathcal{J}(N_{r+1}H_{2r+1}^{\mathrm{sing}}(X,\mathbb{Z}(r))) \to \mathcal{J}_r^a(X)$ will have a kernel if and only if there is a non-torsion class $\alpha \in H_{2r+1}(X,\mathbb{Z})$ such that some non-zero multiple of α belongs to $N_{r+1}H_{2r+1}(X,\mathbb{Z})$, but α itself does not belong to $N_{r+1}H_{2r+1}(X,\mathbb{Z})$. In other words, the universality conjecture will fail for X provided that such an α exists.

We do not know of any examples of such elements, but Kollár [BCC92] (see also [SV05, § 2]) has constructed a smooth, projective three-fold Y and a non-torsion class $b \in H_2^{\mathrm{sing}}(Y,\mathbb{Z})$ such that b is not algebraic (i.e. $b \notin N_1H_2^{\mathrm{sing}}(Y),\mathbb{Z})$) but some non-zero multiple of b is algebraic. In other words, Kollár provides a non-torsion counter-example to the 'integral Hodge conjecture'. (The original counter-examples to this conjecture were torsion.) Now, letting C be a smooth, projective curve of positive genus, setting $X = Y \times C$, and choosing $c \in H_1(C,\mathbb{Z})$ to be a generator, we can form the external product

$$a = b \times c \in H_3^{\mathrm{sing}}(X,\mathbb{Z}).$$

Then a is non-zero, a non-zero multiple of a belongs to $N_2H_3^{\mathrm{sing}}(X)$, and yet there is no readily apparent reason why a should belong to $N_2H_3^{\mathrm{sing}}(X)$.

Concerning the second kind of potential counter-example of universality, we recall the commutative diagram

$$\begin{tikzcd} Z_r(X)_{\mathrm{alg}\sim 0} \arrow[r, two heads, "\Phi_r^{\mathrm{mor}}"] \arrow[dr, two heads, "\Phi_r"'] & \mathcal{J}_r^{\mathrm{mor}}(X) \arrow[d] \\ & \mathcal{J}_r^a(X) \end{tikzcd}$$

and the examples of Theorem 6.2 which show that $\mathcal{J}_r^{\mathrm{mor}}(X) \twoheadrightarrow \mathcal{J}_a^r(X)$ can have a kernel. This does not immediately lead to a counter-example, because the group $L_rH_{2r+1}(X)$ may well fail to be finitely generated or of pure weight -1 and hence $\mathcal{J}_r^{\mathrm{mor}}(X) = \mathcal{J}(L_rH_{2r+1}(X))$ is typically not an abelian variety. The universality conjecture does, however, imply a very strong condition on the IMHS given as the kernel of $L_rH_{2r+1}(X) \to H_{2r+1}^{\mathrm{sing}}(X,\mathbb{Z}(r))$, as we now show.

LEMMA 7.3. *Let X be a smooth, projective variety and suppose that*

$$L_rH_{2r+1}(X) \twoheadrightarrow H$$

is a quotient of IMHSs such that H is finitely generated (i.e. is an MHS) and has pure weight -1. Then $\mathcal{J}(H)$ admits the structure of an abelian variety so that the composition of

$$CH_r(X)_{\mathrm{alg}\sim 0} \xrightarrow{\Phi_r^{\mathrm{mor}}} \mathcal{J}_r^{\mathrm{mor}}(X) \twoheadrightarrow \mathcal{J}(H)$$

is regular.

Proof. According to [Lew99, 12.23], it suffices to show that the onto map

$$\pi : CH_r(X)_{\mathrm{alg}\sim 0} \twoheadrightarrow \mathcal{J}(H)$$

is analytic, where $\mathcal{J}(H)$ acquires the structure of a complex torus via the surjection $H_{\mathbb{C}}/F^0(H_{\mathbb{C}}) \twoheadrightarrow H_{\mathbb{C}}/(H + F^0(H_{\mathbb{C}})) = \mathcal{J}(H)$. Let T and Γ be as in Definition 7.1. Since the morphic Abel–Jacobi map is natural for correspondences the composition of

$$T \to CH_0(T)_{\mathrm{alg}\sim 0} \xrightarrow{\Gamma_*} CH_r(X)_{\mathrm{alg}\sim 0} \to \mathcal{J}(H),$$

whose holomorphicity we seek to establish, coincides with the composition of

$$T \to CH_0(T)_{\mathrm{alg}\sim 0} \xrightarrow{\Phi_0^{\mathrm{mor}}} \mathcal{J}_0^{\mathrm{mor}}(T) \xrightarrow{\Gamma_*} \mathcal{J}_r^{\mathrm{mor}}(X) \twoheadrightarrow \mathcal{J}(H).$$

Note that $\mathcal{J}_0^{\mathrm{mor}}(T) = \mathcal{J}_0(T)$, $\Phi_0^{\mathrm{mor}} = \Phi_0$, and the map $T \to \mathcal{J}_0(T)$ coincides with the classical Albanese map by [Lew99, 12.11(3)], and thus is holomorphic. The composition of

$$\mathcal{J}_0^{\mathrm{mor}}(T) \xrightarrow{\Gamma_*} \mathcal{J}_r^{\mathrm{mor}}(X) \twoheadrightarrow \mathcal{J}_r(H)$$

is holomorphic since it is induced from the composition of

$$L_0H_1(T) \xrightarrow{\Gamma_*} L_rH_{2r+1}(X) \twoheadrightarrow H$$

which is a morphism of MHSs. □

Theorem 7.4. *If, for a smooth, projective variety X, the IMHS*

$$K_rH_{2r+1}(X) := \ker(L_rH_{2r+1}(X) \to H_{2r+1}^{\mathrm{sing}}(X, \mathbb{Z}(r)))$$

admits a quotient IMHS that is finitely generated (i.e. is an MHS), non-torsion, and of pure weight -1, then the universality of the Abel–Jacobi map fails for X.

Proof. The existence of such a quotient $K_rH_{2r+1}(X) \twoheadrightarrow H''$ implies that the surjective morphism of IMHSs

$$L_rH_{2r+1}(X) \twoheadrightarrow N_{r+1}H_{2r+1}^{\mathrm{sing}}(X, \mathbb{Z}(r))$$

factors as

$$L_rH_{2r+1}(X) \twoheadrightarrow H \twoheadrightarrow N_{r+1}H_{2r+1}^{\mathrm{sing}}(X, \mathbb{Z}(r))$$

for some MHS H such that $H \twoheadrightarrow N_{r+1}H_{2r+1}^{\mathrm{sing}}(X, \mathbb{Z}(r))$ has kernel H''. Then

$$N_{r+1}H_{2r+1}^{\mathrm{sing}}(X, \mathbb{Z}(r))_{\mathrm{tor}} \to \mathcal{J}(H'') \to \mathcal{J}(H) \to \mathcal{J}(N_{r+1}H_{2r+1}^{\mathrm{sing}}(X, \mathbb{Z}(r))) \to 0$$

is exact and $\mathcal{J}(H'') \to \mathcal{J}(H)$ is not the zero map since $\mathcal{J}(H'')$ is uncountable. The result now follows from the lemma. □

8. Torsion cycles

In this section, we describe the restriction of the morphic Abel–Jacobi map to torsion subgroups,

$$\Phi_r^{\mathrm{mor}}|_{\mathrm{tor}} : (CH_r(X)_{\mathrm{alg}\sim 0})_{\mathrm{tor}} \to \mathcal{J}_r^{\mathrm{mor}}(X)_{\mathrm{tor}}.$$

In particular, we show this map is surjective and give an explicit description of its kernel. In some simple cases, we prove unconditionally that this map is, in fact, an isomorphism. We show it is an isomorphism for projective varieties definable over a number field, assuming a common conjecture. We also show in this section how the examples due to Schoen [Sch00] of varieties X for which the group $CH_r(X)_{\mathrm{alg}\sim 0}$ contains an infinite amount of l-torsion lead to additional examples where the map $\mathcal{J}_r(X)^{\mathrm{mor}} \to \mathcal{J}_r(X)$ has a non-trivial kernel.

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Recall that there is a natural map

$$CH_r(X,1) \to L_rH_{2r+1}(X). \tag{8.1}$$

The description of the kernel of $\Phi_r^{\mathrm{mor}}|_{\mathrm{tor}}$ depends on the image of this map, which is not well understood. Theorem 8.4 below characterizes the morphic Abel–Jacobi map on torsion subgroups in terms of the image of (8.1), using the following lemma.

LEMMA 8.2. *For a quasi-projective variety U, the image of*

$$CH_r(U,1) \to L_rH_{2r+1}(U)$$

is contained in $\Gamma(L_rH_{2r+1}(U))$.

Proof. Every class in $CH_r(U,1)$ is supported on a closed subscheme of dimension at most $r+1$. By naturality, it thus suffices to assume $\dim(U) \leqslant r+1$. In this case, we have that the map

$$L_rH_{2r+1}(U) \to H_{2r+1}^{\mathrm{BM}}(U,\mathbb{Z}(r))$$

is an isomorphism of IMHSs by Proposition 2.8 and Theorem 4.22(4), and thus it suffices to prove that the natural map

$$CH_r(U,1) \to H_{2r+1}^{\mathrm{BM}}(U,\mathbb{Z}(r))$$

lands in $\Gamma(H_{2r+1}^{\mathrm{BM}}(U,\mathbb{Z}(r)))$. (In fact, this holds in all degrees.)

It suffices to prove this upon tensoring with $\mathbb{Q}$. Let $U \rightarrowtail V$ be a closed embedding with V smooth of dimension d. Then the map in question is isomorphic to

$$H^n_{\mathcal{M},U}(V,\mathbb{Q}(t)) \to H^n_{\mathrm{sing},U}(V,\mathbb{Q}(t)),$$

where $n = 2d-2r-1$ and $t = d-r$ and the subscripts U denote cohomology with supports. The image of this map is contained in

$$\Gamma(H^n_{\mathrm{sing},U}(V,\mathbb{Q}(t))) \cong \Gamma(H_{2r+1}^{\mathrm{BM}}(U,\mathbb{Q}(r)))$$

by [Jan90, § 8]. □

Remark 8.3. We guess that the assertion of Lemma 8.2 holds in all degrees, i.e. that the image of

$$CH_r(U,n) \to L_rH_{2r+n}(U)$$

is contained in $\Gamma(L_rH_{2r+n}(U))$, for all $r,n \geqslant 0$. We know of a proof of this statement in the case when U is projective, but omit it since it is not needed for the rest of this paper.

For a quasi-projective variety V, define

$$L_rH_{2r+1}(V)_{\mathcal{M}} := \mathrm{im}(CH_r(V,1) \to L_rH_{2r+1}(V)).$$

(The subscript $\mathcal{M}$ stands for 'motivic'.) By Lemma 8.2, $L_rH_{2r+1}(V)_{\mathcal{M}}$ is a sub-IMHS of $L_rH_{2r+1}(V)$ having trivial Hodge structure, and hence the short exact sequence of IMHSs

$$0 \to L_rH_{2r+1}(V)_{\mathcal{M}} \to L_rH_{2r+1}(V) \to L_rH_{2r+1}(V)/L_rH_{2r+1}(V)_{\mathcal{M}} \to 0$$

induces a short exact sequence of abelian groups

$$0 \to L_rH_{2r+1}(V)_{\mathcal{M}} \to \Gamma(L_rH_{2r+1}(X)) \to \Gamma(L_rH_{2r+1}(V)/L_rH_{2r+1}(V)_{\mathcal{M}}) \to 0$$

and an isomorphism

$$\mathcal{J}_r^{\mathrm{mor}}(V) = \mathcal{J}(L_rH_{2r+1}(V)) \cong \mathcal{J}(L_rH_{2r+1}(V)/L_rH_{2r+1}(V)_{\mathcal{M}}).$$

Theorem 8.4. *Let X be a projective variety and r any integer. There is a natural exact sequence*

$$\Gamma(L_rH_{2r+1}(X)/L_rH_{2r+1}(X)_{\mathcal{M}})\otimes\mathbb{Q}/\mathbb{Z}\to(CH_r(X)_{\mathrm{alg}\sim 0})_{\mathrm{tor}}\xrightarrow{\Phi_r^{\mathrm{mor}}}\mathcal{J}_r^{\mathrm{mor}}(X)_{\mathrm{tor}}\to 0,$$

where

$$\Gamma(L_rH_{2r+1}(V)/L_rH_{2r+1}(V)_{\mathcal{M}})\cong\operatorname{coker}(CH_r(X,1)\to\Gamma(L_rH_{2r+1}(X))).$$

Proof. To simplify the notation, we set

$$H(V):=L_rH_{2r+1}(V)/L_rH_{2r+1}(V)_{\mathcal{M}},$$

for a quasi-projective variety V. We use a theorem of Suslin and Voevodsky [SV96, 9.1], which implies that the natural map

$$CH_r(V,n;\mathbb{Q}/\mathbb{Z})\to L_rH_{2r+n}(V,\mathbb{Q}/\mathbb{Z})$$

is an isomorphism for all n and V. The Suslin–Voevodsky result, together with the long exact sequences in Chow groups and Lawson homology obtained from $0\to\mathbb{Z}\to\mathbb{Q}\to\mathbb{Q}/\mathbb{Z}\to 0$, yields a short exact sequence

$$0\to H(V)\to H(V)_{\mathbb{Q}}\to(CH_r(V)_{\mathrm{alg}\sim 0})_{\mathrm{tor}}\to 0 \tag{8.5}$$

and hence an isomorphism

$$H(V)\otimes\mathbb{Q}/\mathbb{Z}\cong(CH_r(V)_{\mathrm{alg}\sim 0})_{\mathrm{tor}}.$$

Let Y be a closed subvariety of X of dimension r. Using $CH_r(Y,1)=0=L_rH_{2r+1}(Y)$, $Z_r^Y(X)=CH_r(Y)\cong L_rH_{2r}(Y)$, and the localization sequences for Chow groups and Lawson homology, one obtains the exact sequence of IMHSs

$$0\to H(X)\to H(X-Y)\to CH_r^Y(X)_{\mathrm{alg}\sim 0}\to 0. \tag{8.6}$$

(Here $CH_r^Y(X)_{\mathrm{alg}\sim 0}$ denotes the subgroup of $CH_r(X)$ of cycles supported on Y and algebraically equivalent to 0 on X.) The boundary map in the associated six-term exact sequence for $\mathrm{Ext}^*_{\mathrm{IMHS}}(\mathbb{Z}(0),-)$ has the form

$$CH_r^Y(X)_{\mathrm{alg}\sim 0}\to\mathcal{J}(H(X))\cong\mathcal{J}_r^{\mathrm{mor}}(X). \tag{8.7}$$

Using that

$$0\to L_rH_{2r+1}(X)\to L_rH_{2r+1}(X-Y)\to Z_r^Y(X)_{\mathrm{alg}\sim 0}\to 0$$

surjects onto (8.6), it follows directly from the definition of Φ_r^{mor} that (8.7) is the restriction of Φ_r^{mor} to $CH_r^Y(X)_{\mathrm{alg}\sim 0}\subset CH_r(X)_{\mathrm{alg}\sim 0}$. Using now Lemma 3.4, we see that the restriction of Φ_r^{mor} to $(CH_r^Y(X)_{\mathrm{alg}\sim 0})_{\mathrm{tor}}$ coincides with the composition of

$$(CH_r^Y(X)_{\mathrm{alg}\sim 0})_{\mathrm{tor}}\to H(X)\otimes\mathbb{Q}/\mathbb{Z}\twoheadrightarrow(H(X)/\Gamma(H(X)))\otimes\mathbb{Q}/\mathbb{Z}\cong\mathcal{J}_r^{\mathrm{mor}}(X)_{\mathrm{tor}},$$

where the first map is the boundary map in the long exact sequence for $\mathrm{Tor}_*(-,\mathbb{Q}/\mathbb{Z})$ applied to (8.6).

We now take inductive limits over all dimension r subvarieties Y of X. We have $\varinjlim_Y CH_r(X-Y)=0$, and hence, using (8.5), we see that $\varinjlim_Y H(X-Y)$ is uniquely divisible. As a consequence, the boundary map $\varinjlim_Y(CH_r^Y(X)_{\mathrm{alg}\sim 0})_{\mathrm{tor}}\to H(X)\otimes\mathbb{Q}/\mathbb{Z}$ is an isomorphism.

We have shown that the restriction of Φ_r^{mor} to $(CH_r(X)_{\mathrm{alg}\sim 0})_{\mathrm{tor}}$ factors as

$$(CH_r(X)_{\mathrm{alg}\sim 0})_{\mathrm{tor}}\xrightarrow{\cong}H(X)\otimes\mathbb{Q}/\mathbb{Z}\twoheadrightarrow(H(X)/\Gamma(H(X)))\otimes\mathbb{Q}/\mathbb{Z}\cong\mathcal{J}_r^{\mathrm{mor}}(X)_{\mathrm{tor}},$$

and the theorem follows. ∎

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COROLLARY 8.8. *If, for a given projective variety X and integer r, the cokernel of the map*

$$CH_r(X,1) \to \Gamma(L_r H_{2r+1}(X))$$

is divisible modulo torsion, for example, if

$$CH_r(X,1;\mathbb{Q}) \to \Gamma(L_r H_{2r+1}(X,\mathbb{Q}))$$

is onto, then the morphic Abel–Jacobi map induces an isomorphism on torsion subgroups:

$$\Phi_r^{\mathrm{mor}}|_{\mathrm{tor}} : (CH_r(X)_{\mathrm{alg}\sim 0})_{\mathrm{tor}} \xrightarrow{\cong} \mathcal{J}_r(X)_{\mathrm{tor}}.$$

Example 8.9. Recall that $L_r H_{2r+1}(X)$ has weights $-1 \leqslant w \leqslant 0$, and thus the hypothesis of Corollary 8.8 is met if $Gr_0^W(L_r H_{2r+1}(X,\mathbb{Q})) = 0$. This holds, for example, if X is smooth and projective and the map

$$L_r H_{2r+1}(X,\mathbb{Q}) \to H_{2r+1}^{\mathrm{sing}}(X,\mathbb{Q}(r))$$

is injective. Using the results of [FHW04, § 6], we see that we have an isomorphism

$$\Phi_r^{\mathrm{mor}}|_{\mathrm{tor}} : (CH_r(X)_{\mathrm{alg}\sim 0})_{\mathrm{tor}} \xrightarrow{\cong} \mathcal{J}_r^{\mathrm{mor}}(X)_{\mathrm{tor}}$$

if X is smooth, projective, and belongs to the class $\mathcal{C}$ defined in [FHW04]. Moreover, in this case, we have $\mathcal{J}_r^{\mathrm{mor}}(X) = \mathcal{J}_r(X)$ and $\Phi_r^{\mathrm{mor}} = \Phi_r$, so that the classical Abel–Jacobi map induces an isomorphism

$$\Phi_r|_{\mathrm{tor}} : (CH_r(X)_{\mathrm{alg}\sim 0})_{\mathrm{tor}} \xrightarrow{\cong} \mathcal{J}_r^{\mathrm{mor}}(X)_{\mathrm{tor}}$$

for such X.

The class $\mathcal{C}$ includes all curves, all linear varieties (for example, all toric varieties), and all cellular varieties, and it is closed under localization, blow-ups, and the formation of bundles, in the sense made precise in [FHW04, 6.9].

To obtain more interesting examples of varieties satisfying the hypothesis of Corollary 8.8, we need to assume a well-known conjecture.

CONJECTURE 8.10 (See, e.g., [Jan90, 5.20]). If U is a smooth, quasi-projective complex variety that can be defined over $\mathbb{Q}$, then the canonical map

$$CH_r(U,n;\mathbb{Q}) \to \Gamma(H_{2r+n}^{\mathrm{BM}}(U,\mathbb{Q}(r)))$$

is a surjection for all n, r.

THEOREM 8.11. *Let X be a projective complex variety that can be defined over a number field. Assume that Conjecture 8.10 holds when $r=0$ and $n=2$ for all smooth U definable over $\mathbb{Q}$. Then the morphic Abel–Jacobi map induces an isomorphism on torsion subgroups:*

$$\Phi_r^{\mathrm{mor}}|_{\mathrm{tor}} : (CH_r(X)_{\mathrm{alg}\sim 0})_{\mathrm{tor}} \xrightarrow{\cong} \mathcal{J}_r(X)_{\mathrm{tor}}.$$

Proof. Suppose that X is definable over a number field k. By [Jan90, 5.21], if Conjecture 8.10 holds (for a fixed r and n), then the same statement holds for any smooth U definable over k (for the same r and n). Let S be a (possibly singular) complex projective surface that can be defined over k and choose a closed embedding $S \rightarrowtail \mathbb{P}^N$, defined over k, with open complement U. Then one easily sees [Jan90, 8.9] that the surjectivity of $CH_0(U,2;\mathbb{Q}) \to \Gamma(H_2^{\mathrm{BM}}(U,\mathbb{Q}(0)))$ (which we are assuming, since U is definable over k) implies the surjectivity of

$$CH_0(S,1;\mathbb{Q}) \to \Gamma(H_1^{\mathrm{sing}}(S,\mathbb{Q}(0))).$$

We claim now that $\Gamma(L_r H_{2r+1}(X,\mathbb{Q}))$ is generated by classes coming via equidimensional correspondences $\Gamma \in Z_r(S,X)$ from classes in $\Gamma(L_0 H_1(S,\mathbb{Q})) \cong \Gamma(H_1^{\mathrm{sing}}(S,\mathbb{Q}(0)))$, where S ranges over all projective surfaces definable over k. This will prove the theorem, since the diagram

$$\begin{tikzcd} CH_0(S,1;\mathbb{Q}) \arrow[r, "\Gamma_*"] \arrow[d] & CH_r(X,1;\mathbb{Q}) \arrow[d] \\ L_0H_1(S,\mathbb{Q}) \arrow[r, "\Gamma_*"] & L_rH_{2r+1}(X,\mathbb{Q}) \end{tikzcd}$$

commutes. (The fact that equidimensional cycles induce maps on Chow groups and the commutativity of this square follow from the description of Chow groups in terms of equidimensional cycles given by Suslin; see [Sus00, 2.1] and [Voe00, 4.2.9].)

To establish the claim, recall that $L_rH_{2r+1}(X) = \pi_1(\mathcal{Z}_r(X)) = H_1^{\mathrm{sing}}(\mathcal{Z}_r(X))$ and that $\mathcal{Z}_r(X)$ is the homotopy-theoretic group completion of $\mathcal{C}_r(X)$. Hence, we have

$$L_rH_{2r+1}(X) \cong H_1^{\mathrm{sing}}(\mathcal{C}_r(X)) \otimes_{H_0^{\mathrm{sing}}(\mathcal{C}_r(X))} H_0^{\mathrm{sing}}(\mathcal{Z}_r(X)).$$

In particular, $L_rH_{2r+1}(X)$ is a filtered colimit of copies of the homology groups $H_1^{\mathrm{sing}}(\mathcal{C}_{r,e}(X))$, where $\mathcal{C}_{r,e}(X)$ is the Chow variety of dimension r, degree e cycles on X. Moreover, for each e, the map $H_1^{\mathrm{sing}}(\mathcal{C}_{r,e}(X)) \cong L_0H_1(\mathcal{C}_{r,e}(X)) \to L_rH_{2r+1}(X)$ is induced by a 'universal correspondence' $\Gamma : Z_e(\mathcal{C}_{r,e}(X), X)$ (given by the identity map on $\mathcal{C}_{r,e}(X)$) and hence, by Theorem 4.22(5), is a morphism of IMHSs. Since $\Gamma(-)$ commutes with filtered colimits, we have

$$\Gamma(L_rH_{2r+1}(X,\mathbb{Q})) = \varinjlim \Gamma(L_rH_{2r+1}(\mathcal{C}_{r,e}(X),\mathbb{Q})).$$

The Andreotti–Frankel theorem [AF59] shows that, for each e, there is a surface $S_e \subset \mathcal{C}_{r,e}$, so that the map

$$H_1(S_e,\mathbb{Q}) \xrightarrow{\cong} H_1(\mathcal{C}_{r,e}(X),\mathbb{Q})$$

is an isomorphism. This shows that $\Gamma(L_rH_{2r+1}(X,\mathbb{Q}))$ is generated by classes coming via correspondence from classes in $\Gamma(H_1^{\mathrm{sing}}(S_e,\mathbb{Q}(0)))$. In fact, we claim that the Andreotti–Frankel theorem allows us to take S_e to be definable over k, and hence the result follows. Indeed, let Y be any projective variety definable over k, say of dimension n. The Andreotti–Frankel theorem amounts to the assertion that a smooth, affine complex variety of dimension d has the homotopy type of a d-dimensional CW complex. Thus, if we take Y_{n-1} to a subvariety of Y that is definable over k, that contains the singular locus of Y and all components of Y having dimension less than two, and for which $U = Y - Y_{n-1}$ is affine, then we have $H_2^{\mathrm{BM}}(U,\mathbb{Q}) = H_1^{\mathrm{BM}}(U,\mathbb{Q}) = 0$ by Poincaré duality, as long as $n > 2$. Thus, $H_1^{\mathrm{sing}}(Y_{n-1},\mathbb{Q}) \cong H_1^{\mathrm{sing}}(Y,\mathbb{Q})$, for $n > 2$, and the claim is established by induction on $n \geqslant 2$. ∎

At one time, it was conjectured that the classical Abel–Jacobi map itself ought to induce an isomorphism on torsion subgroups:

$$\Phi_r|_{\mathrm{alg}\sim 0,\mathrm{tor}} : (CH_r(X)_{\mathrm{alg}\sim 0})_{\mathrm{tor}} \xrightarrow{?\cong} \mathcal{J}_r^a(X)_{\mathrm{tor}}, \tag{8.12}$$

for X smooth and projective. More generally, it was conjectured that the map

$$\Phi_r|_{\mathrm{tor}} : (CH_r(X)_{\mathrm{hom}\sim 0})_{\mathrm{tor}} \to \mathcal{J}_r(X)_{\mathrm{tor}} \tag{8.13}$$

ought to be injective.

The map (8.12) is always surjective. In codimension one (i.e. $r = \dim(X) - 1$), the injectivity of (8.12) and (8.13) are easily seen to hold; in codimension two ($r = \dim(X) - 2$), the injectivity of these maps is a consequence of the Merkur'ev–Suslin theorem [MS82] (see [Mur85, 10.3]); and for zero-cycles ($r = 0$), the injectivity of these maps is a theorem of Rojtman [Roj80] (see also [Blo79]). The injectivity of (8.12), however, is now known to fail in general. Schoen [Sch00] has constructed examples of a smooth, projective complex variety X of dimension d, definable over a field of transcendence degree one over $\mathbb{Q}$, for which $CH_r(X)_{\text{alg}\sim 0}$ has infinite l-torsion for all $0 < r < d-2$ and some prime l. Since $\mathcal{J}_r(X)_{\text{tor}} \cong (\mathbb{Q}/\mathbb{Z})^\nu$ for some integer ν, the Abel–Jacobi map cannot be injective on $(CH_r(X)_{\text{alg}\sim 0})_{\text{tor}}$ for such varieties. Soulé and Voisin [SV05] have also shown that (8.12) can fail to be injective for four-folds with $r = 1$. In earlier work, Totaro [Tot97] constructed examples of a smooth, projective variety X, definable over a number field, for which (8.13) fails to be injective. In light of Theorem 8.11, it is interesting to note that there are no known examples of the failure of the injectivity of (8.12) for varieties definable over number fields.

We now show how Schoen's examples give rise to additional examples of varieties X for which $\mathcal{J}_r^{\text{mor}}(X) \twoheadrightarrow \mathcal{J}_r^a(X)$ has a kernel. We first recall the details of Schoen's construction.

Theorem 8.14 (Schoen [Sch00]). *Suppose that k is an algebraically closed subfield of $\mathbb{C}$, W is a smooth, projective k-variety, and E is an elliptic curve over $\mathbb{C}$ whose j-invariant does not belong to k. Then for any integer $r \geqslant 0$, the map*

$$CH_r(W) \otimes CH_0(E)_{\text{tor}} \to CH_r(W \times_k E)$$

given by external product of cycles is injective.

Remark 8.15. Schoen's theorem is actually more general than this: one may replace $k \subset \mathbb{C}$ with any extension of algebraically closed fields, even those of positive characteristic.

Schoen also proves [Sch02] that for any algebraically closed field k of characteristic zero, there exist a smooth, projective three-fold W (in fact, W can be taken to be an abelian variety) such that $CH_1(W) \otimes \mathbb{Q}_l/\mathbb{Z}_l$ has infinite corank for some prime l. Taking $k = \overline{\mathbb{Q}}$ and picking E to be an elliptic curve over $\mathbb{C}$ whose j-invariant does not belong to k, then since $CH_0(E)_{\text{tor}} \cong (\mathbb{Q}/\mathbb{Z})^2 \cong \bigoplus_l (\mathbb{Q}_l/\mathbb{Z}_l)^2$, we see that $CH_r(W) \otimes CH_0(E)_{\text{tor}}$ has infinite l-torsion for some prime l.

Theorem 8.16. *Suppose that k, W, and E are as in Theorem 8.14 and that $CH_r(W) \otimes \mathbb{Q}_l/\mathbb{Z}_l$ has infinite corank for some prime l. Then the kernel of*

$$\mathcal{J}_r^{\text{mor}}(W \times_k E) \to \mathcal{J}_r(W \times_k E)$$

contains the quotient of a non-zero complex vector space by a countable subgroup. In particular, there exists a complex abelian four-fold V, definable over an algebraically closed field of transcendence degree one over $\mathbb{Q}$, such that the kernel of

$$\mathcal{J}_1^{\text{mor}}(V) \to \mathcal{J}_1(V)$$

is uncountable.

Proof. Observe that elements of $CH_0(E)_{\text{tor}}$ are algebraically equivalent to 0 and hence we have an isomorphism

$$CH_0(E)_{\text{tor}} \cong L_0 H_1(E) \otimes \mathbb{Q}/\mathbb{Z}.$$

Moreover, the kernel of $CH_r(W) \to CH_r(W_\mathbb{C}) \to L_r H_{2r}(W_\mathbb{C})$ is divisible (since $CH_r(W)/\text{alg} \sim 0 \cong L_r H_{2r}(W_\mathbb{C})$) and thus

$$CH_r(W) \otimes CH_0(E)_{\text{tor}} \cong L_r H_{2r}(W_\mathbb{C}) \otimes L_0 H_1(E) \otimes \mathbb{Q}/\mathbb{Z}.$$

The image of the injection in Schoen's theorem is contained in $(CH_r(W_\mathbb{C} \times_\mathbb{C} E)_{\text{alg}\sim 0})_{\text{tor}}$ and there is a natural surjection

$$L_r H_{2r+1}(W_\mathbb{C} \times_\mathbb{C} E) \otimes \mathbb{Q}/\mathbb{Z} \twoheadrightarrow (CH_r(W_\mathbb{C} \times_\mathbb{C} E)_{\text{alg}\sim 0})_{\text{tor}},$$

coming from (8.5) in the proof of Theorem 8.4. Moreover, the injection in Schoen's theorem coincides with the composition of

$$\begin{aligned} CH_r(W) \otimes CH_0(E)_{\mathrm{tor}} &\cong L_rH_{2r}(W_{\mathbb{C}}) \otimes L_0H_1(E) \otimes \mathbb{Q}/\mathbb{Z} \\ &\to L_rH_{2r+1}(W_{\mathbb{C}} \times_{\mathbb{C}} E) \otimes \mathbb{Q}/\mathbb{Z} \twoheadrightarrow (CH_r(W_{\mathbb{C}} \times_{\mathbb{C}} E)_{\mathrm{alg}\sim 0})_{\mathrm{tor}}, \end{aligned}$$

where the first map is induced by external product in Lawson homology, and thus the map

$$L_rH_{2r}(W_{\mathbb{C}}) \otimes L_0H_1(E) \otimes \mathbb{Q}/\mathbb{Z} \to L_rH_{2r+1}(W_{\mathbb{C}} \times_{\mathbb{C}} E) \otimes \mathbb{Q}/\mathbb{Z}$$

is injective.

We claim that the map

$$\mathrm{Griff}_r(W_{\mathbb{C}})_{\mathbb{Q}} \otimes_{\mathbb{Q}} L_0H_1(E, \mathbb{Q}) \to L_rH_{2r+1}(W_{\mathbb{C}} \times_{\mathbb{C}} E, \mathbb{Q})$$

is not the zero map. As in the proof of Theorem 6.2, this will suffice to complete the proof. Note that $CH_r(W) \otimes \mathbb{Q}_l/\mathbb{Z}_l \cong L_rH_{2r}(W_{\mathbb{C}}) \otimes \mathbb{Q}_l/\mathbb{Z}_l$. The hypotheses imply that the map

$$\mathrm{Griff}_r(W_{\mathbb{C}}) \otimes L_0H_1(E) \otimes \mathbb{Q}_l/\mathbb{Z}_l \to L_rH_{2r}(W_{\mathbb{C}}) \otimes L_0H_1(E) \otimes \mathbb{Q}_l/\mathbb{Z}_l$$

is not the zero map, and hence that

$$\mathrm{Griff}_r(W_{\mathbb{C}}) \otimes L_0H_1(E) \otimes \mathbb{Q}_l/\mathbb{Z}_l \to L_rH_{2r+1}(W_{\mathbb{C}} \times_{\mathbb{C}} E) \otimes \mathbb{Q}_l/\mathbb{Z}_l$$

is non-zero. □

Note that Schoen's examples give *torsion* cycles that lie in the kernel of the classical Abel–Jacobi map and are algebraically equivalent to zero. The previous result shows that these torsion cycles entail the existence of *non-torsion* classes in

$$(\ker(\Phi_r) \cap CH_r(X)_{\mathrm{alg}\sim 0}) / \ker(\Phi_r^{\mathrm{mor}}).$$

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Mark E. Walker mwalker5@math.unl.edu
Department of Mathematics, University of Nebraska–Lincoln, Lincoln, NE 68588-0130, USA

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