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HOLDER-EXTENDIBLE EUROPEAN OPTION: CORRECTIONS AND EXTENSIONS

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Abstract

Financial contracts with options that allow the holder to extend the contract maturity by paying an additional fixed amount have found many applications in finance. Closed-form solutions for the price of these options have appeared in the literature for the case when the contract for the underlying asset follows a geometric Brownian motion with constant interest rate, volatility and nonnegative dividend yield. In this paper, option price is derived for the case of the underlying asset that follows a geometric Brownian motion with time-dependent drift and volatility, which is more important for real life applications. The option price formulae are derived for the case of a drift that includes nonnegative or negative dividend. The latter yields a solution type that is new to the literature. A negative dividend corresponds to a negative foreign interest rate for foreign exchange options, or storage costs for commodity options. It may also appear in pricing options with transaction costs or real options, where the drift is larger than the interest rate.

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1. Model

Financial contracts with options that allow the holder to extend the contract maturity by paying an additional fixed amount have found many applications in finance. The European option with extendible maturity (written on the underlying asset X_t) can be exercised by the holder on a decision time T_1 using strike K_1. The holder may also exercise the option later at some maturity $T_2 > T_1$, using strike K_2, by paying an extra premium $A > 0$ at time T_1. Denote the value of this option at time $t \le T_1$ as $Q(X_t, t; K_1, K_2, T_1, T_2)$; we want to find the fair value of this option today at time $t = T_0 = 0$. At time T_1, the payoffs for the holder-extendible call and put options are

$$Q_C(X_{T_1}, T_1; K_1, K_2, T_1, T_2) = \max(X_{T_1} - K_1, C(X_{T_1}, T_1; K_2, T_2) - A, 0), \qquad (1.1)$$

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and

$$Q_P(X_{T_1}, T_1; K_1, K_2, T_1, T_2) = \max(K_1 - X_{T_1}, P(X_{T_1}, T_1; K_2, T_2) - A, 0), \tag{1.2}$$

respectively. Here, $C(X_t, t; K, T)$ and $P(X_t, t; K, T)$ are the standard European call and put options (so-called *vanilla options*) at time t, respectively, for the underlying asset value X_t (referred to as *spot value*), strike K and maturity at time T, that is, their payoffs at maturity are $\max(X_T - K, 0)$ and $\max(K - X_T, 0)$, respectively.

Applications of these options include extendible options on foreign exchange, nondividend and continuous dividend yield stocks, real estate, bonds, and so on. For example, the standard holder-extendible option in foreign exchange (FX) allows the holder to extend the maturity of an FX vanilla option by paying an extra premium. An option on real estate often allows the option holder to extend the contract expiry date by paying an additional amount to the option writer. In general, any contract that involves rescheduling payments can be viewed as a contract with extendible option. Closed-form solutions for these options were presented by Longstaff [6], Haug [4, p. 48], Chung and Johnson [2], and Chateau and Wu [1] for the case when the underlying asset X_t follows a geometric Brownian motion with a constant drift and volatility. Of course, the Monte Carlo method, direct integration and other numerical methods can also be used to price these options. For example, Ibrahim et al. [5] have applied fast Fourier transform to the valuation of extendible options.

In this paper, we consider a geometric Brownian motion model with a time-dependent drift and volatility, which is still important for practical applications, and derive a closed-form solution for holder-extendible options in the case of a drift that can include nonnegative or negative dividend. The latter case yields a new solution type that has not been studied in the literature. We also fix several typographical errors in the formula for the holder-extendible put option presented in Longstaff [6, equation 12] and Haug [4, equation 2.15, p. 48].

Let $\mathbb{Q}$ be a risk-neutral probability measure under which the underlying asset X_t follows the stochastic risk-neutral process

$$dX_t = X_t\mu(t)\,dt + X_t\sigma(t)\,dW_t, \tag{1.3}$$

where W_t is the standard Brownian motion, $\sigma(t)$ is the instantaneous volatility, $\mu(t) = r(t) - q(t)$ is the risk-neutral drift, $r(t)$ is the risk-free domestic interest rate and $q(t)$ is a known continuous function of time (hereafter referred to as dividend). This model is often used for pricing a holder-extendible option on a foreign exchange rate, where $q(t)$ corresponds to the foreign interest rate. In the case of an option on a dividend paying stock, $q(t)$ corresponds to the continuous dividend yield. Assuming constant drift and volatility, Longstaff [6] and Chung and Johnson [2] have considered the case of zero dividend $q(t) = 0$; Haug [4] and Chateau and Wu [1] have considered the case of nonnegative dividend $q(t) \geq 0$. In this paper, we allow for a negative dividend $q(t)$ (for example, negative foreign interest rate in the case of FX options), leading to a new solution type that has not been considered in the literature. Also, the drift and volatility are allowed to be time-dependent.

For the stochastic process (1.3), the joint distribution of $\ln X_{T_1}$ and $\ln X_{T_2}$, given X_0, is a bivariate normal distribution with

$$\begin{aligned} \mathrm{E}[\ln X_{T_i} \mid \ln X_0] &= \ln X_0 + \int_0^{T_i} \Big(r(\tau) - q(\tau) - \frac{1}{2}\sigma^2(\tau) \Big)\, d\tau, \quad i = 1, 2; \\ \mathrm{Cov}[\ln X_{T_i}, \ln X_{T_j} \mid \ln X_0] &= \int_0^{\min(T_i, T_j)} \sigma^2(\tau)\, d\tau, \quad i, j = 1, 2. \end{aligned} \tag{1.4}$$

According to the standard option valuation methodology, the Black–Scholes framework generalized by Harrison and Pliska [3], a fair price of the holder-extendible option at $t = 0$ is a conditional expectation (with respect to the risk-neutral probability measure $\mathbb{Q}$)

$$Q(X_0, 0; K_1, K_2, T_1, T_2) = \exp\Big(-\int_0^{T_1} r(\tau)\, d\tau\Big) \mathrm{E}^{\mathbb{Q}}[Q(X_{T_1}, T_1; K_1, K_2, T_1, T_2) | X_0], \tag{1.5}$$

where $Q(X_{T_1}, T_1; K_1, K_2, T_1, T_2)$ is given by (1.1) and (1.2) for the holder-extendible call and put options, respectively. The above expectation can be calculated using (1.4) and integral identities (see the Appendix) in closed-form as demonstrated in the following sections. We derive option price formulas for both the holder-extendible call and the holder-extendible put options, and the formulae are derived for the cases of nonnegative and negative dividend.

2. Notation and definitions

Hereafter, the following notation and identities are used.

- Model parameters are

$$q_{ij} = \frac{1}{T_j - T_i} \int_{T_i}^{T_j} q(\tau)\, d\tau, \quad r_{ij} = \frac{1}{T_j - T_i} \int_{T_i}^{T_j} r(\tau)\, d\tau, \quad \mu_{ij} = r_{ij} - q_{ij},$$

$$\sigma_{ij}^2 = \frac{1}{T_j - T_i} \int_{T_i}^{T_j} \sigma^2(\tau)\, d\tau \quad \text{and} \quad \rho = \frac{\sigma_{01}\sqrt{T_1}}{\sigma_{02}\sqrt{T_2}}$$

  for $T_i < T_j$ and $i, j = 0, 1, 2$.
- Transformation functions are

$$g_1(y) = \frac{\ln(y/X_0) - \mu_{01} T_1 + \sigma_{01}^2 T_1/2}{\sigma_{01}\sqrt{T_1}}, \quad \widetilde{g}_1(y) = g_1(y) - \sigma_{01}\sqrt{T_1},$$

$$g_2(y) = \frac{\ln(y/X_0) - \mu_{02} T_2 + \sigma_{02}^2 T_2/2}{\sigma_{02}\sqrt{T_2}}, \quad \widetilde{g}_2(y) = g_2(y) - \sigma_{02}\sqrt{T_2},$$

  with their inverses

$$\begin{aligned} g_1^{-1}(y) &= X_0 \exp\Big(\mu_{01} T_1 - \tfrac{1}{2}\sigma_{01}^2 T_1 + \sigma_{01}\sqrt{T_1}\, y\Big), \\ g_2^{-1}(y) &= X_0 \exp\Big(\mu_{02} T_2 - \tfrac{1}{2}\sigma_{02}^2 T_2 + \sigma_{02}\sqrt{T_2}\, y\Big). \end{aligned} \tag{2.1}$$

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- Critical values of the underlying asset defining exercise regions on a decision time are denoted as $\{I_1, I_2, I_3\}$ for the holder-extendible call option, and $\{J_0, J_1, J_2\}$ for the holder-extendible put option.
- We use $N(\cdot)$ and $N_2(\cdot,\cdot;\rho)$ to denote the standard normal distribution and the standard bivariate normal distribution with correlation ρ, respectively; their densities are denoted as $n(x)$ and $n_2(x, y; \rho)$, respectively. We use $M_2(a, b, c, d; \rho)$ to denote the probability of the standard bivariate normal density with correlation ρ for the region $[a, b] \times [c, d]$, and $M(a, b)$ to denote the probability of the standard normal density in the interval $[a, b]$ that can be expressed through $N(\cdot)$ and $N_2(\cdot,\cdot;\rho)$ as given in Section 5.
- The standard European call and put prices at time T_i with maturity $T_j > T_i$ are

$$\begin{aligned}
C(x, T_i; K, T_j) &= xe^{(\mu_{ij}-r_{ij})(T_j-T_i)}N(d_1) - Ke^{-r_{ij}(T_j-T_i)}N(d_2),\\
P(x, T_i; K, T_j) &= Ke^{-r_{ij}(T_j-T_i)}N(-d_2) - xe^{(\mu_{ij}-r_{ij})(T_j-T_i)}N(-d_1),\\
d_1 &= \frac{\ln(x/K) + (\mu_{ij} + \sigma_{ij}^2/2)(T_j - T_i)}{\sigma_{ij}\sqrt{T_j - T_i}}, \quad d_2 = d_1 - \sigma_{ij}\sqrt{T_j - T_i}.
\end{aligned}$$

- To compare the calculus with Longstaff [6], one has to set

$$\sigma_{01} = \sigma_{02} = \sigma_{12} = \sigma, \quad \mu_{01} = \mu_{02} = \mu_{12} = r_{01} = r_{02} = r_{12} = r. \tag{2.2}$$

Some notations are chosen for the purpose of easier comparison with existing results in the literature.

3. Holder-extendible call

The decision at $t = T_1$ to extend or exercise the call option is determined by comparing two risky payoffs

$$C(X_{T_1}, T_1; K_2, T_2) - A \quad \text{and} \quad \max(X_{T_1} - K_1, 0),$$

and choosing the largest payoff. If the first payoff is larger then the option is extended, otherwise it is exercised when $X_{T_1} > K_1$ or expires worthless when $X_{T_1} \le K_1$; for an illustrative example, see Figure 1. Note that the standard European call option $C(x, T_1; K_2, T_2)$ is calculated at time T_1 for maturity at T_2.

Denote the region of $X_{T_1} = x$ where the option is extended as

$$\Omega_C = \{x \ge 0 \mid C(x, T_1; K_2, T_2) - A > \max(x - K_1, 0)\},$$

and the region where it is exercised as

$$\overline{\Omega}_C = \{x > K_1 \mid x - K_1 \ge C(x, T_1; K_2, T_2) - A\}.$$

For all other values of X_{T_1}, the option expires worthless. Then, using transformation of X_{T_1} and X_{T_2} to the random variables, $Z_1 = g_1(X_{T_1})$ and $Z_2 = g_1(X_{T_2})$, from the standard normal distribution, today's price (1.5) of the holder-extendible call option

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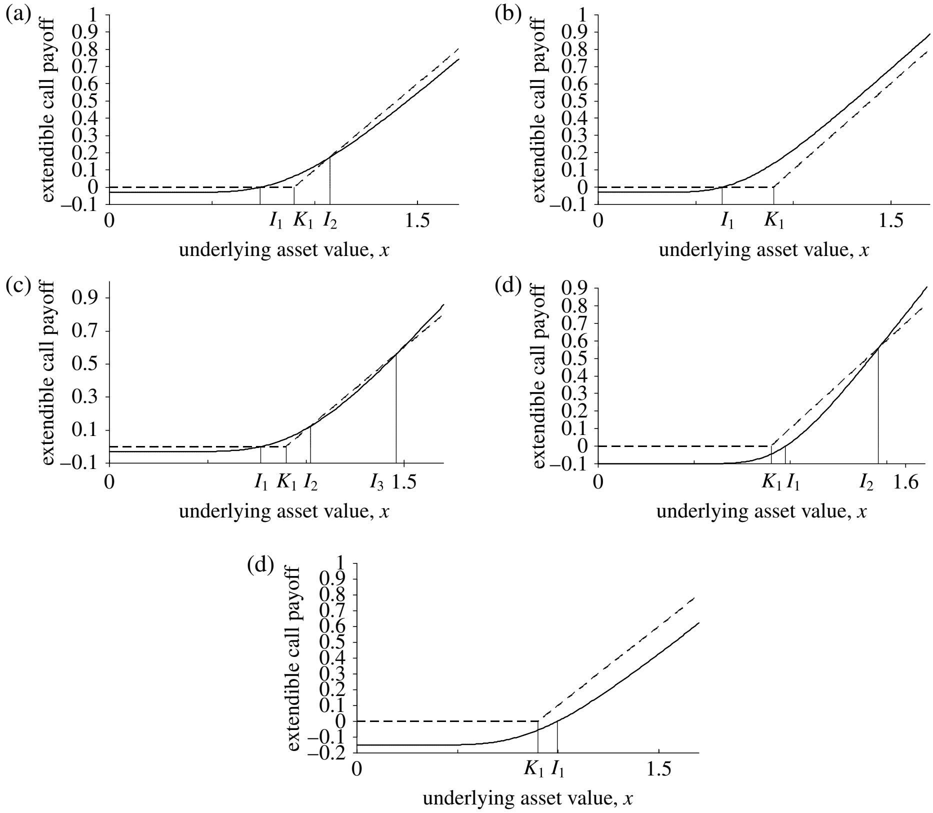

FIGURE 1. Some possible holder-extendible call option payoffs on a decision time T_1. The payoff is determined by choosing the largest value between the solid line $C(x, T_1; K_2, T_2) - A$ and the dashed line $\max(x - K_1, 0)$. The case of positive dividend $q_{12} > 0$ may lead to cases in (a) and (e), $q_{12} = 0$ may lead to cases in (a), (b) and (e); and $q_{12} < 0$ may lead to cases in (b–d).

$$\begin{aligned}
Q_C(X_0, 0; K_1, K_2, T_1, T_2) \\
&= e^{-r_{01}T_1} \int_{-\infty}^{\infty} \max(C(x_1, T_1; K_2, T_2) - A, x_1 - K_1, 0) n(z_1)\, dz_1 \\
&= e^{-r_{02}T_2} \int_{x_1 \in \Omega_C} dz_1 \int_{g_2(K_2)}^{\infty} (x_2 - K_2) n_2(z_1, z_2; \rho)\, dz_2 \\
&\quad - e^{-r_{01}T_1} A \int_{x_1 \in \Omega_C} n(z_1)\, dz_1 + e^{-r_{01}T_1} \int_{x_1 \in \overline{\Omega}_C} n(z_1)(x_1 - K_1)\, dz_1. \qquad (3.1)
\end{aligned}$$

Here $x_1 = g_1^{-1}(z_1)$ and $x_2 = g_2^{-1}(z_2)$ are functions of z_1 and z_2, respectively, as in (2.1).

The regions Ω_C and $\overline{\Omega}_C$ are determined using solutions (critical asset values) of nonlinear equations

$$f_1^C(x) = C(x, T_1; K_2, T_2) - A = 0, \qquad x \geq 0, \qquad (3.2)$$

and

$$f_2^C(x) = C(x, T_1; K_2, T_2) - x + K_1 - A = 0, \quad x > K_1. \tag{3.3}$$

These can be solved numerically using, for instance, the Newton–Raphson algorithm combined with the standard bisection algorithm.

The equation (3.2) has one solution denoted by $x = I_1$, which is bounded as

$$Ae^{q_{12}(T_2-T_1)} \le I_1 \le Ae^{q_{12}(T_2-T_1)} + K_2 e^{-\mu_{12}(T_2-T_1)}.$$

The second equation $f_2^C(x) = 0$ may have two, one or no solution depending on the option characteristics (strikes, maturities, model parameters) that will determine today's option price. If they exist, the solutions will be denoted as I_2, I_3. Figure 1 illustrates some of the possible cases. Below, we consider two distinct cases of nonnegative and negative dividend, that is, the cases $q_{12} \ge 0$ and $q_{12} < 0$, respectively. This is so, because if $q_{12} \ge 0$ then $f_2^C(x) = 0$ may have one or no solution, and if $q_{12} < 0$ then $f_2^C(x) = 0$ may have two solutions. Note that the extendible call formula in Longstaff [6, equation 7] corresponds to the case of zero dividend, that is, $q_{12} = 0$.

All conditions listed in Sections 3.1 and 3.2 on option characteristics to determine the solution type can be proved using the facts that the European call option price $C(x, T_1; K_2, T_2)$ is a continuous and increasing function of x, and its first derivative is

$$\begin{aligned}\Delta_C(x) &= \frac{\partial C(x, T_1; K_2, T_2)}{\partial x} \\ &= e^{-q_{12}(T_2-T_1)} N\Big(\frac{\ln(x/K_2) + (\mu_{12} + \sigma_{12}^2/2)(T_2 - T_1)}{\sigma_{12}\sqrt{T_2 - T_1}}\Big).\end{aligned} \tag{3.4}$$

It is important to note that $0 \le \Delta_C(x) \le 1$ when $q_{12} \ge 0$, however, if $q_{12} < 0$ then $\Delta_C(x) > 1$ is possible.

3.1. Nonnegative dividend

Consider the case of a nonnegative dividend, $q_{12} \ge 0$.

- If $I_1 \ge K_1$, then the call option is never extended (that is, $f_2^C(x) = 0$ has no solutions for $x > K_1$), and thus,

  $$Q_C(X_0, 0; K_1, K_2, T_1, T_2) = C(X_0, 0; K_1, T_1),$$

  which is a standard European call option. This is the case illustrated by Figure 1(e).
- If $I_1 < K_1$, then the nonlinear equation $f_2^C(x) = 0$ (for $x > K_1$) has either one solution denoted as I_2, or none as illustrated by Figure 1(a) and 1(b), respectively. In the case of one solution I_2, the call option is extended when $I_1 < X_{T_1} < I_2$, exercised when $X_{T_1} \ge I_2$, and it expires worthless when $X_{T_1} \le I_1$. If there is no solution, then the call option is extended when $I_1 < X_{T_1}$, and it expires worthless when $I_1 \ge X_{T_1}$. In particular,

  (I) if $q_{12} > 0$, then there is a finite I_2,
  (II) if $q_{12} = 0$, then I_2 is finite when $K_1 - A - K_2 e^{-r_{12}(T_2-T_1)} < 0$; and $f_2^C(x) = 0$ has no finite solution when $K_1 - A - K_2 e^{-r_{12}(T_2-T_1)} \ge 0$.

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Then, today's price of the holder-extendible call option can be calculated by integrating (3.1) with $\Omega_C = [I_1, I_2]$ and $\overline{\Omega}_C = [I_2, \infty)$ to obtain

$$\begin{aligned} Q_C(X_0, 0; K_1, K_2, T_1, T_2) = C(X_0, 0; K_1, T_1) & \\ + X_0 e^{(\mu_{02} - r_{02})T_2} M_2(-\widetilde{g}_1(I_2), -\widetilde{g}_1(I_1), -\infty, -\widetilde{g}_2(K_2); \rho) & \\ - K_2 e^{-r_{02}T_2} M_2(-g_1(I_2), -g_1(I_1), -\infty, -g_2(K_2); \rho) & \\ - A e^{-r_{01}T_1} M(-g_1(I_2), -g_1(I_1)) & \\ - X_0 e^{-q_{01}T_1} M(-\widetilde{g}_1(I_2), -\widetilde{g}_1(K_1)) & \\ + K_1 e^{-r_{01}T_1} M(-g_1(I_2), -g_1(K_1)). & \end{aligned} \tag{3.5}$$

Note that the case when $f_2^C(x) = 0$ has no solution can be treated by setting $I_2 = \infty$. This expression (3.5) reduces to the original Longstaff [6, equation 7] formula for the holder-extendible call option after setting parameters as in (2.2), and using the Longstaff [6] notation

$$\gamma_1 = -\widetilde{g}_1(I_2), \quad \gamma_2 = -\widetilde{g}_1(I_1), \quad \gamma_3 = -\widetilde{g}_2(K_2), \quad \gamma_4 = -\widetilde{g}_1(K_1).$$

Example 3.1. Consider the holder-extendible call option with initial maturity of $T_1 = 1$ year that can be extended to $T_2 = 2$ years. The model parameters are: spot value $X_0 = 0.9$, strike on decision $K_1 = 0.9$, strike at final maturity $K_2 = 0.95$, interest rate $r = 0.02$, dividend $q = 0$, volatility $\sigma = 0.3$ and extra premium $A = 0.03$. The payoffs and critical values are shown in Figure 1(a). Solving nonlinear equations (3.2) and (3.3) via the bisection algorithm gives critical values $I_1 \approx 0.734$ and $I_2 \approx 1.074$. Finally, using formula (3.5), it is found that today's price of the holder-extendible call option is $Q_C(X_0, 0; K_1, K_2, T_1, T_2) \approx 0.129$.

3.2. Negative dividend The case of a negative dividend, $q_{12} < 0$, is slightly complicated. This is because the first derivative of the European call option $\Delta_C(x)$ (see (3.4)) can become greater than one.

- If $I_1 > K_1$ then nonlinear equation (3.3) has one finite solution I_2, and the call option is extended when $X_{T_1} > I_2$, as shown in Figure 1(d). The price is calculated by integrating (3.1) with $\Omega_C = [I_2, \infty)$ and $\overline{\Omega} = [K_1, I_2]$

$$\begin{aligned} Q_C(X_0, 0; K_1, K_2, T_1, T_2) = C(X_0, 0; K_1, T_1) & \\ + X_0 e^{-q_{02}T_2} N_2(-\widetilde{g}_1(I_2), -\widetilde{g}_2(K_2); \rho) & \\ - K_2 e^{-r_{02}T_2} N_2(-g_1(I_2), -g_2(K_2); \rho) & \\ - A e^{-r_{01}T_1} N(-g_1(I_2)) - X_0 e^{-q_{01}T_1} N(-\widetilde{g}_1(I_2)) & \\ + K_1 e^{-r_{01}T_1} N(-g_1(I_2)). & \end{aligned}$$

- If $I_1 \le K_1$ then nonlinear equation (3.3) has either two solutions I_2 and I_3 (with $I_3 \ge I_2 \ge I_1$, as illustrated in Figure 1(c), and the call option is extended if $I_1 < X_{T_1} < I_2$ or $X_{T_1} > I_3$) or none, as illustrated in Figure 1(b). For the latter, the call option is extended if $X_{T_1} > I_1$. Specifically, $f_2^C(x)$ has a minimum at $x = x_c^*$ where $df_2^C(x)/dx = 0$. Using (3.4), it is found that

$$x_c^* = K_2 \exp\Big(\sigma_{12}\sqrt{T_2 - T_1} F_N^{-1}(e^{q_{12}(T_2-T_1)}) - (\mu_{12} + \tfrac{1}{2}\sigma_{12}^2)(T_2 - T_1)\Big),$$
$$f_2^C(x_c^*) = K_1 - A - K_2 e^{-r_{12}(T_2-T_1)} N(F_N^{-1}(e^{q_{12}(T_2-T_1)})),$$

where $F_N^{-1}(\cdot)$ is the inverse of the standard normal distribution function. Thus, if $f_2^C(x_c^*) < 0$ then there are two finite solutions, otherwise there is no solution. In the case of no solution, the price is given by (3.5) with I_2 set to ∞. In the case of two solutions, I_2 and I_3 can be found, for example, via the bisection algorithm for $[K_1, x_c^*)$ and $[x_c^*, \infty)$, respectively, and integration in (3.1) with $\Omega_C = \{(I_1, I_2) \cup (I_3, \infty)\}$ and $\overline{\Omega}_C = [I_2; I_3]$ gives

$$\begin{aligned}
Q_C(X_0, 0; K_1, K_2, T_1, T_2) &= C(X_0, 0; K_1, T_1)\\
&+ X_0 e^{-q_{02}T_2}[M_2(-\widetilde{g}_1(I_2), -\widetilde{g}_1(I_1), -\infty, -\widetilde{g}_2(K_2); \rho) + N_2(-\widetilde{g}_1(I_3), -\widetilde{g}_2(K_2); \rho)]\\
&- K_2 e^{-r_{02}T_2}[M_2(-g_1(I_2), -g_1(I_1), -\infty, -g_2(K_2); \rho) + N_2(-g_1(I_3), -g_2(K_2); \rho)]\\
&- A e^{-r_{01}T_1}[M(g_1(I_1), g_1(I_2)) + N(-g_1(I_3))]\\
&+ X_0 e^{-q_{01}T_1}[M(\widetilde{g}_1(I_2), \widetilde{g}_1(I_3)) - N(-\widetilde{g}_1(K_1))]\\
&- K_1 e^{-r_{01}T_1}[M(g_1(I_2), g_1(I_3)) - N(-g_1(K_1))].
\end{aligned} \tag{3.6}$$

EXAMPLE 3.2. Consider the holder-extendible call option with initial maturity $T_1 = 1$ year, which can be extended to $T_2 = 2$ years. The model parameters are: spot value $X_0 = 0.9$, strike on decision $K_1 = 0.9$, strike at final maturity $K_2 = 1.4$, interest rate $r = 0.02$, dividend $q = -0.28$, volatility $\sigma = 0.3$ and extra premium $A = 0.03$. The payoffs and critical values are shown in Figure 1(c). Solving the nonlinear equation (3.2) via the bisection algorithm gives critical value $I_1 \approx 0.771$. Using conditions listed in this section, it is easy to find that the nonlinear equation (3.3) has two solutions $I_2 \approx 1.024$ and $I_3 \approx 1.459$, obtained via the bisection algorithm for $[K_1, x_c^*)$ and $[x_c^*, \infty)$, respectively. Finally, using formula (3.6), one may find that today's price of the holder-extendible call is $Q_C(X_0, 0; K_1, K_2, T_1, T_2) \approx 0.357$.

4. Holder-extendible put

The decision at T_1 to extend or exercise the put option is determined by comparing two risky payoffs

$$P(X_{T_1}, T_1; K_2, T_2) - A \quad \text{and} \quad \max(K_1 - X_{T_1}, 0)$$

and choosing the largest payoff. If the first payoff is larger then the option is extended, otherwise it is exercised when $X_{T_1} < K_1$, or expires worthless when $X_{T_1} \geq K_1$; for an illustrative example, see Figure 2. Note that the standard European put option $P(x, T_1; K_2, T_2)$ is calculated at time T_1 for maturity at T_2.

Denote the region of X_{T_1} values where the put option is extended as

$$\Omega_P = \{x \geq 0 : P(x, T_1; K_2, T_2) - A > \max(K_1 - x, 0)\},$$

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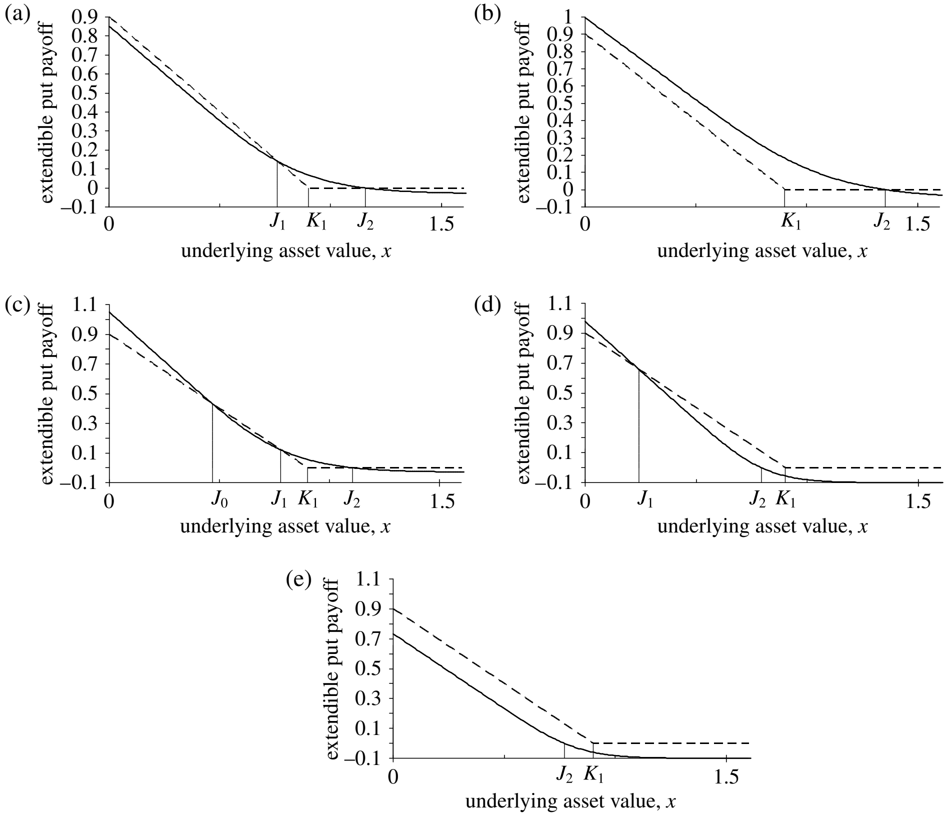

FIGURE 2. Possible holder-extendible put payoffs on a decision time T_1. The payoff is determined by choosing the largest value between solid line $P(x, T_1; K_2, T_2) - A$ and dashed line $\max(K_1 - x, 0)$. The case of positive dividend $q_{12} \geq 0$ may lead to cases shown in (a), (b) and (e), and $q_{12} < 0$ may lead to cases in (a–e).

and the region where it is exercised as

$$\overline{\Omega}_P = \{0 \leq x < K_1 : K_1 - x \geq P(x, T_1; K_2, T_2) - A\}.$$

For all other values of X_{T_1}, the option expires worthless. Then the holder-extendible put option price can be written as

$$\begin{aligned} &Q_P(X_0, 0; K_1, K_2, T_1, T_2) \\ &\quad = e^{-r_{01}T_1} \int_{-\infty}^{\infty} \max(P(x_1, T_1; K_2, T_2) - A, K_1 - x_1, 0) n(z_1)\, dz_1 \\ &\quad = e^{-r_{02}T_2} \int_{x_1 \in \Omega_P} dz_1 \int_{-\infty}^{g_2(K_2)} (K_2 - x_2) n_2(z_1, z_2; \rho)\, dz_2 \\ &\qquad - e^{-r_{01}T_1} A \int_{x_1 \in \Omega_P} n(z_1)\, dz_1 + e^{-r_{01}T_1} \int_{x_1 \in \overline{\Omega}_P} n(z_1)(K_1 - x_1)\, dz_1, \quad (4.1) \end{aligned}$$

where $x_1 = g_1^{-1}(z_1)$ and $x_2 = g_2^{-1}(z_2)$ are functions of z_1 and z_2, as given by (2.1).

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The regions Ω_P and $\overline{\Omega}_P$ can be determined using critical asset values which are solutions of nonlinear equations

$$f_1^P(x) = P(x, T_1; K_2, T_2) - K_1 + x - A = 0, \quad 0 \le x < K_1, \tag{4.2}$$

and

$$f_2^P(x) = P(x, T_1; K_2, T_2) - A = 0, \quad x \ge 0. \tag{4.3}$$

As in the case of the holder-extendible call option, these can be solved numerically using the Newton–Raphson algorithm combined with the standard bisection algorithm.

If $A > P(0, T_1; K_2, T_2) = K_2 e^{-r_{12}(T_2 - T_1)}$, then $f_2^P(x) < 0$ for all $x \ge 0$, and thus the put option is never extended, that is, $Q_P(X_0, 0; K_1, K_2, T_1, T_2) = P(X_0, 0; K_1, T_1)$. Otherwise, $f_2^P(x) = 0$ has one solution denoted as $x = J_2$, and this case is considered hereafter.

The first equation $f_1^P(x) = 0$ may have no solution, one solution (denoted as J_1) or two solutions (denoted as J_0 and J_1) depending on the option characteristics (strikes, maturities, model parameters) that will determine today's option price. Figure 2 illustrates all such possibilities. Below, we consider the cases of nonnegative and negative dividend, that is, the cases $q_{12} \ge 0$ and $q_{12} < 0$, respectively. Similar to the holder-extendible call option, $f_1^P(x) = 0$ may have one or no solution if $q_{12} \ge 0$, and two solutions if $q_{12} < 0$. Note that the extendible put formula in Longstaff [6, equation 12] corresponds to the case of zero dividend $q_{12} = 0$.

All conditions listed in Sections 4.1 and 4.2 on option characteristics to determine solution type can easily be proved using the facts that the European put price $P(x, T_1; K_2, T_2)$ is continuous and a decreasing function of x, and its first derivative

$$\begin{aligned}\Delta_P(x) &= \frac{\partial P(x, T_1; K_2, T_2)}{\partial x} \\ &= e^{-q_{12}(T_2 - T_1)} \left[N\left(\frac{\ln(x/K_2) + (\mu_{12} + \sigma_{12}^2/2)(T_2 - T_1)}{\sigma_{12}\sqrt{T_2 - T_1}} \right) - 1 \right]\end{aligned} \tag{4.4}$$

is negative. It is important to note that $-1 \le \Delta_P(x) \le 0$ when $q_{12} \ge 0$, however, if $q_{12} < 0$ then $\Delta_P(x) < -1$ is possible.

4.1. Nonnegative dividend

Here we consider the case of a nonnegative dividend $q_{12} \ge 0$.

- If $J_2 \le K_1$ then the put option is never extended, and thus,

  $$Q_P(X_0, 0; K_1, K_2, T_1, T_2) = P(X_0, 0; K_1, T_1),$$

  which is a standard European put option, as shown in Figure 2(e).
- If $J_2 > K_1$ the nonlinear equation (4.2) may have one solution J_1 (that is, the option is extended if $J_1 < X_{T_1} < J_2$) or none as shown in Figure 2(a) and 2(b), respectively. The latter case corresponds to the put option, which is extended for $X_{T_1} < J_2$. In particular, if $K_1 < K_2 e^{-r_{12}(T_2 - T_1)} - A$ then there are no solutions,

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otherwise, there is one finite solution J_1. Then, today's price can be calculated by integrating (4.1) with $\Omega_P = [J_1, J_2]$ and $\overline{\Omega}_P = [0, J_1]$ (the case when $f_1^P(x)$ has no solution can be treated by setting $J_1 = 0$) to obtain

$$\begin{aligned} Q_P(X_0, 0; K_1, K_2, T_1, T_2) = P(X_0, 0; K_1, T_1) \\ - X_0 e^{(\mu_{02} - r_{02})T_2} M_2(\widetilde{g}_1(J_1), \widetilde{g}_1(J_2), -\infty, \widetilde{g}_2(K_2); \rho) \\ + K_2 e^{-r_{02}T_2} M_2(g_1(J_1), g_1(J_2), -\infty, g_2(K_2); \rho) \\ - A e^{-r_{01}T_1} M(-g_1(J_2), -g_1(J_1)) \\ + X_0 e^{(\mu_{01} - r_{01})T_1} M(-\widetilde{g}_1(K_1), -\widetilde{g}_1(J_1)) \\ - K_1 e^{-r_{01}T_1} M(-g_1(K_1), -g_1(J_1)). \end{aligned} \tag{4.5}$$

This formula appeared in the literature with typographical errors. To make a comparison easier, rewrite the formula using Longstaff's [6] notation

$$\gamma_1 = -\widetilde{g}_1(J_2), \quad \gamma_2 = -\widetilde{g}_1(J_1), \quad \gamma_3 = \widetilde{g}_2(K_2), \quad \gamma_4 = -\widetilde{g}_1(K_1).$$

Then the holder-extendible put option can be written as

$$\begin{aligned} Q_P(X_0, 0; K_1, K_2, T_1, T_2) = P(X_0, 0; K_1, T_1) \\ - \underline{X_0 e^{(\mu_{02} - r_{02})T_2} M_2(-\gamma_2, -\gamma_1, -\infty, -\gamma_3; \rho)} \\ + \underline{K_2 e^{-r_{02}T_2} M_2(\sigma_{01}\sqrt{T_1} - \gamma_2, \sigma_{02}\sqrt{T_2} - \gamma_1, -\infty, \sigma_{02}\sqrt{T_2} - \gamma_3; \rho)} \\ - A e^{-r_{01}T_1} M(\gamma_1 - \sigma_{01}\sqrt{T_1}, \gamma_2 - \sigma_{01}\sqrt{T_1}) \\ + X_0 e^{(\mu_{01} - r_{01})T_1} M(\gamma_4, \gamma_2) \\ - K_1 e^{-r_{01}T_1} M(\gamma_4 - \sigma_{01}\sqrt{T_1}, \gamma_2 - \sigma_{01}\sqrt{T_1}). \end{aligned}$$

After setting parameters as in (2.2), the difference between this formula (see underlined terms) and Longstaff's formula [6, equation 12] is clear. For the latter, γ_3, $\gamma_3 - \sigma\sqrt{T_2}$ and ρ should be changed to $-\gamma_3$, $-\gamma_3 + \sigma\sqrt{T_2}$ and $-\rho$, respectively, and the factor in the third term, $\exp(-r(T_2 - T_1))$, should be replaced with $\exp(-rT_2)$. Also, note that the formula for the holder-extendible put option in Haug [4, equation 2.15, p. 48] has a typographical error where ρ should be changed to $-\rho$. When comparing the formulas the following symmetry property is useful:

$$M_2(a, b, c, d, \rho) = M_2(-b, -a, c, d, -\rho).$$

Example 4.1. Consider the holder-extendible put option with initial maturity $T_1 = 1$ year that can be extended to $T_2 = 2$ years. The model parameters are: spot value $X_0 = 0.9$, strike on decision $K_1 = 0.9$, strike at final maturity $K_2 = 0.9$, interest rate $r = 0.02$, dividend $q = 0$, volatility $\sigma = 0.3$, extra premium $A = 0.03$. The payoffs and critical values are shown in Figure 2(a). Solving nonlinear equations (4.2) and (4.3) via the bisection algorithm gives critical values $J_1 \approx 0.758$ and $J_2 \approx 1.157$. Finally, using formula (4.5), it is found that today's price of the holder-extendible put is $Q_P(X_0, 0; K_1, K_2, T_1, T_2) \approx 0.113$.

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4.2. Negative dividend The case of a negative dividend, $q_{12} < 0$, is slightly more complicated due to the fact that the first derivative of the European put option Δ_P may become less than -1 (see (4.4)).

- If $J_2 < K_1$ then nonlinear equation (4.2) has either one finite solution J_1, and the put option is extended if $X_{T_1} < J_1$, or none, as shown in Figure 2(d) and Figure 2(e), respectively. Specifically, if $K_1 < K_2 e^{-r_{12}(T_2-T_1)} - A$, then there is one solution, otherwise there is no solution. If there is no solution then the put option is never extended, that is, $Q_P(X_0, 0; K_1, K_2, T_1, T_2) = P(X_0, 0; K_1, T_1)$. Otherwise, the price is calculated by integrating (4.1) with $\Omega_P = [0, J_1]$ and $\overline{\Omega}_P = [J_1, K_1]$ to obtain

$$\begin{aligned} Q_P(X_0, 0; K_1, K_2, T_1, T_2) = P(X_0, 0; K_1, T_1) &- X_0 e^{-q_{02}T_2} N_2(\widetilde{g}_1(J_1), \widetilde{g}_2(K_2); \rho) \\ &+ K_2 e^{-r_{02}T_2} N_2(g_1(J_1), g_2(K_2); \rho) - A e^{-r_{01}T_1} N(g_1(J_1)) \\ &+ X_0 e^{-q_{01}T_1} N(\widetilde{g}_1(J_1)) - K_1 e^{-r_{01}T_1} N(g_1(J_1)). \end{aligned}$$

- If $J_2 \geq K_1$ then nonlinear equation (4.2) has either two solutions J_0 and J_1 (the put option is extended if $0 < X_{T_1} < J_0$ or $J_1 < X_{T_1} < J_2$), one solution (the put option is extended if $J_1 < X_{T_1} < J_2$) or none (the put option is extended if $X_{T_1} > J_1$). These three cases are shown in Figures 2(c), 2(a) and 2(b), respectively. Specifically, $f_1^P(x)$ has a minimum at $x = x_p^*$ where $df_1^P(x)/dx = 0$. Using (4.4), it is easy to find that

$$\begin{aligned} x_p^* &= K_2 \exp\Big(\sigma_{12}\sqrt{T_2 - T_1} F_N^{-1}(d) - (\mu_{12} + \tfrac{1}{2}\sigma_{12}^2)(T_2 - T_1)\Big), \\ f_1^P(x_p^*) &= K_2 e^{-r_{12}(T_2-T_1)} N\Big(\sigma_{12}\sqrt{T_2 - T_1} - d\Big) - A - K_1, \end{aligned}$$

where $F_N^{-1}(\cdot)$ is the inverse of the standard normal distribution function and $d = F_N^{-1}(1 - e^{q_{12}(T_2-T_1)})$. Thus, if $f_1^P(x_p^*) > 0$ then there is no solution, and the price can be calculated using (4.5) with J_1 set to zero. If $f_1^P(x_p^*) \leq 0$ and $K_1 > K_2 e^{-r_{12}(T_2-T_1)}$ then there is one finite solution J_1, and the price can be calculated using (4.5). If $f_1^P(x_p^*) \leq 0$ and $K_1 \leq K_2 e^{-r_{12}(T_2-T_1)}$ then there are two finite solutions $J_0 \leq J_1$. For the last case, J_0 and J_1 can be found via the bisection algorithm for $[0, x_p^*]$ and $[x_p^*, K_1]$, respectively, and the integration in (4.1) with $\Omega = \{[0, J_0) \cup (J_1, J_2)\}$ and $\overline{\Omega}_P = [J_0, J_1]$ gives

$$\begin{aligned} Q_P(X_0, 0; K_1, K_2, T_1, T_2) &= P(X_0, 0; K_1, T_1) \\ &- X_0 e^{-q_{02}T_2}[N_2(\widetilde{g}_1(J_0), \widetilde{g}_2(K_2); \rho) + M_2(\widetilde{g}_1(J_1), \widetilde{g}_1(J_2), -\infty, \widetilde{g}_2(K_2); \rho)] \\ &+ K_2 e^{-r_{02}T_2}[N_2(g_1(J_0), g_2(K_2); \rho) + M_2(g_1(J_1), g_1(J_2), -\infty, g_2(K_2); \rho)] \\ &- A e^{-r_{01}T_1}[N(g_1(J_0)) + M(g_1(J_1), g_1(J_2))] \\ &- X_0 e^{-q_{01}T_1}[M(\widetilde{g}_1(J_0), \widetilde{g}_1(J_1)) - N(\widetilde{g}_1(K_1))] \\ &+ K_1 e^{-r_{01}T_1}[M(g_1(J_0), g_1(J_1)) - N(g_1(K_1))]. \end{aligned} \tag{4.6}$$

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EXAMPLE 4.2. Consider the holder-extendible put with initial maturity $T_1 = 1$ year that can be extended to $T_2 = 2$ years. The model parameters are: spot value $X_0 = 0.9$, strike on decision $K_1 = 0.9$, strike at final maturity $K_2 = 1.1$, interest rate $r = 0.02$, dividend $q = -0.28$, volatility $\sigma = 0.3$ and extra premium $A = 0.03$. The payoffs and critical values are shown in Figure 2(c). Solving the nonlinear equation (4.3) via the bisection algorithm gives critical value $J_2 \approx 1.107$. Using conditions listed in this section, it is easy to find that the nonlinear equation (4.2) has two solutions $J_0 \approx 0.468$ and $J_1 \approx 0.779$ that we find via the bisection algorithm applied to regions $[0, x_p^*]$ and $[x_p^*, K_1]$, respectively. Finally, using formula (4.6), we find that today's price of the holder-extendible put is $Q_P(X_0, 0; K_1, K_2, T_1, T_2) \approx 0.034$.

5. Conclusion

We have derived closed-form formulas for the holder-extendible call and put options in the presence of a dividend yield that can be zero, positive or negative. A negative dividend can correspond to a negative foreign interest rate for FX options or storage costs for a commodity option. It may also appear in pricing options with transaction costs or real options, where the drift is larger than the interest rate. Previously, the zero dividend case was studied by Longstaff [6] and Chung and Johnson [2] and the nonnegative dividend case was treated by Haug [4] and Chateau and Wu [1]. It is important to note that a negative dividend may lead to solutions involving three critical asset values defining decision regions, while the nonnegative dividend case leads to solutions involving only two critical values. Finally, all formulas are derived for the case of geometric Brownian motion with a time-dependent drift and volatility which is important for practical applications.

Appendix. Integral formulas and identities

All integrals involved in calculation of today's option price (3.1) and (4.1) can be found using closed-form integrals

$$\int_{-\infty}^{a}\int_{-\infty}^{b} n_2(x,y;\rho)e^{\beta x}\,dx\,dy = \exp\Big(\frac{\beta^2}{2}\Big)N_2(a-\beta, b-\beta\rho;\rho),$$
$$\int_{a}^{\infty}\int_{b}^{\infty} n_2(x,y;\rho)e^{\beta x}\,dx\,dy = \exp\Big(\frac{\beta^2}{2}\Big)N_2(\beta-a, \beta\rho-b;\rho),$$
$$\int_{a}^{\infty} n(x)e^{\beta x}\,dx = \exp\Big(\frac{\beta^2}{2}\Big)N(\beta-a),$$
$$\int_{-\infty}^{a} n(x)e^{\beta x}\,dx = \exp\Big(\frac{\beta^2}{2}\Big)N(a-\beta).$$

Also, the following relationships for the probability functions are used throughout the

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paper to simplify the formulas:

$$M_2(a, b, c, d; \rho) = N_2(b, d; \rho) - N_2(a, d; \rho) - N_2(b, c; \rho) + N_2(a, c; \rho),$$
$$M_2(a, b, -\infty, d; \rho) = N_2(b, d; \rho) - N_2(a, d; \rho),$$
$$M(a, b) = N(b) - N(a).$$

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