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NEW GLOBAL LOGARITHMIC STABILITY RESULTS ON THE CAUCHY PROBLEM FOR ELLIPTIC EQUATIONS

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Abstract

We prove the global logarithmic stability of the Cauchy problem for H^2-solutions of an anisotropic elliptic equation in a Lipschitz domain. The result is based on existing techniques used to establish stability estimates for the Cauchy problem combined with related tools used to study an inverse medium problem.

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Throughout, Ω is a Lipschitz bounded domain of $\mathbb{R}^n$, $n \geq 2$, and Γ is a nonempty open subset of $\partial\Omega$. Consider the divergence form elliptic operator L that acts by

$$Lu(x) = \mathrm{div}(A(x)\nabla u(x)),$$

where $A = (a^{ij})$ is a symmetric matrix with coefficients in $W^{1,\infty}(\Omega)$, so that there exist $\kappa > 0$ and $\lambda \geq 1$ for which

$$\lambda^{-1}|\xi|^2 \leq A(x)\xi \cdot \xi \leq \lambda|\xi|^2 \quad \text{for } x \in \Omega, \xi \in \mathbb{R}^n,$$

and

$$\sum_{k=1}^{n} \left| \sum_{i,j=1}^{n} \partial_k a^{ij}(x)\xi_i\xi_j \right| \leq \kappa|\xi|^2 \quad \text{for } x \in \Omega, \xi \in \mathbb{R}^n.$$

The Cauchy problem that we consider here can be stated as follows: Given $(F, f, g) \in L^2(\Omega) \times L^2(\Gamma) \times L^2(\Gamma)^n$, find $u \in H^2(\Omega)$ satisfying the boundary value problem

$$\begin{cases} Lu(x) = F(x) & \text{almost everywhere in } \Omega, \\ u(x) = f & \text{almost everywhere on } \Gamma, \\ \nabla u(x) = g & \text{almost everywhere on } \Gamma. \end{cases} \tag{1}$$

It is well known that this problem may not have a solution and, according to the classical uniqueness of continuation from Cauchy data, the boundary value

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problem (1) has at most one solution. Moreover, even if the solution of (1) exists, the continuous dependence of the solution on the data (F, f, g) is not in general Lipschitz. In other words, the Cauchy problem is ill-posed in Hadamard's sense. As shown by Hadamard [8], the modulus of continuity of the mapping $(F, f, g) \mapsto u$ can be of logarithmic type. Therefore, for the general Cauchy problem, a logarithmic stability estimate is the best that one can expect.

We aim to prove the following result.

THEOREM 1. *Let $0 < s < \frac{1}{2}$. Then there exist two constants $c > 0$ and $C > 0$, only depending on s, Ω, Γ, λ and κ, and δ_0 only depending on Ω, so that, for any $u \in H^2(\Omega)$, $0 < \delta < \delta_0$ and $j = 0, 1$,*

$$C\|u\|_{H^j(\Omega)} \leq \delta^{s/(j+1)}\|u\|_{H^{j+1}(\Omega)} + e^{e^{c/\delta}}(\|u\|_{L^2(\Gamma)} + \|\nabla u\|_{L^2(\Gamma)} + \|Lu\|_{L^2(\Omega)}). \tag{2}$$

As usual, the interpolation inequality (2) yields a double logarithmic stability estimate. Precisely, we have the following corollary in which

$$\Psi^c_{s,j}(\rho) = \begin{cases} (\ln\ln\rho)^{-s/(j+1)} & \text{if } \rho > c, \\ \rho & \text{if } 0 < \rho < c, \end{cases}$$

for $j = 0, 1$, extended by continuity at $\rho = 0$ by setting $\Psi^c_{s,j}(0) = 0$, where $c > e$.

COROLLARY 2. *Let $0 < s < \frac{1}{2}$. Then there exist two constants $c > e$ and $C > 0$, only depending on s, Ω, Γ, λ and κ, so that, for any $u \in H^2(\Omega)$, $u \neq 0$ and $j = 0, 1$,*

$$C\|u\|_{H^j(\Omega)} \leq \|u\|_{H^{j+1}(\Omega)}\Psi^c_{s,j}\left(\frac{\|u\|_{H^{j+1}(\Omega)}}{\|u\|_{L^2(\Gamma)} + \|\nabla u\|_{L^2(\Gamma)} + \|Lu\|_{L^2(\Omega)}}\right).$$

As we observed above, according to the classical uniqueness of continuation from Cauchy data for elliptic equations, if $u \in H^2(\Omega)$ satisfies $Lu = 0$ in Ω, $u = 0$ on Γ and $\nabla u = 0$ on Γ, then $u = 0$.

To our knowledge the optimal stability estimate for the Cauchy problem for an elliptic equation holds in two cases: (i) a Lipschitz domain and $C^{1,\alpha}$-solutions and (ii) a $C^{1,1}$ domain and H^2-solutions. This optimal stability estimate is of single logarithmic type. For case (i), we refer to [4] under an additional geometric condition on the domain. This condition was removed in [1] (see also [5]). A similar result was obtained in [3] for the Laplace operator. Case (ii) was established in [2] for the Laplace operator. However, the results in [2] can be extended to an anisotropic elliptic operator in divergence form. In the present paper we deal with the case of a Lipschitz domain and H^2-solutions. For this case we are only able to get a stability estimate of double logarithmic type (Corollary 2). We do not know whether this result can be improved to a single logarithmic type.

Let us explain briefly the main steps to obtain the global stability estimate for the Cauchy problem. The first step consists in continuing well-chosen interior data to the boundary. In the second step we continue the data from an interior subdomain to another subdomain. The continuation of the Cauchy data to some interior subdomain constitutes the third step. For the last two steps it is sufficient to assume that the domain is Lipschitz and the solutions have H^2-regularity, while in the first step it is necessary to assume that either the domain is $C^{1,1}$ or the solutions have $C^{1,\alpha}$-regularity. Apart from these two cases we do not know how to prove the continuation result in the first step. It is worth mentioning that the last two steps give rise to a stability estimate of Hölder type and for the first step the stability estimate we obtain is of logarithmic type.

Since we cannot use this classical scheme to prove Theorem 1, we modify it slightly to avoid the use of the first step. The main idea consists of refining the second step. Precisely, we show that we can continue the data, away from the boundary, from a ball with arbitrarily small radius to another ball with the same radius, with an exact dependence of the constants on the radius. This new step yields a stability estimate of double logarithmic type. It turns out that this result is optimal if one proves it using three-ball inequalities. For this reason we think that techniques based on three-ball inequalities cannot be used to improve Theorem 1.

As we already mentioned, the proof of Theorem 1 consists of an adaptation of existing results. The following proposition is proved in [4] under an additional geometric condition and for a Lipschitz domain in [1] (see also [5]).

Henceforward, C_0 is a generic constant only depending on Ω, λ and κ, while C_1 is a generic constant only depending on Ω, Γ, λ and κ.

Proposition 3. *There exist a constant $\gamma > 0$ and a ball B in $\mathbb{R}^n$ satisfying $B \cap \Omega \neq \emptyset$, $B \cap (\mathbb{R}^n \setminus \overline{\Omega}) \neq \emptyset$ and $B \cap \partial\Omega \Subset \Gamma$, only depending on Ω, Γ, λ and κ, so that, for any $u \in H^2(\Omega)$ and $\epsilon > 0$,*

$$C_1\|u\|_{H^1(B\cap\Omega)} \leq \epsilon^{\gamma}\|u\|_{H^1(\Omega)} + \epsilon^{-1}(\|u\|_{L^2(\Gamma)} + \|\nabla u\|_{L^2(\Gamma)} + \|Lu\|_{L^2(\Omega)}). \tag{3}$$

Proof of Theorem 1. Let B be as in the preceding proposition and pick $\tilde{x} \in B \cap \partial\Omega$. As $B \cap \Omega$ is Lipschitz, it contains a cone with vertex at $\tilde{x}$. That is we can find $R > 0$, $\theta \in]0, \pi/2[$ and $\xi \in \mathbb{S}^{n-1}$ so that

$$C(\tilde{x}) = \{x \in \mathbb{R}^n : 0 < |x - \tilde{x}| < R, (x - \tilde{x}) \cdot \xi > |x - \tilde{x}| \cos\theta\} \subset B \cap \Omega.$$

Let $x_\delta = \tilde{x} + \delta\xi/(3\sin\theta)$, with $\delta < (3R\sin\theta)/2$. Then $\mathrm{dist}(x_\delta, \partial(B \cap \Omega)) > 3\delta$. For $\delta > 0$, define

$$\Omega^\delta = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \delta\},$$
$$\Omega_\delta = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \delta\}$$

and set

$$\delta^* = \sup\{\delta > 0 : \Omega^\delta \neq \emptyset\}.$$

Let $0 < \delta \leq \delta^*/3$. A slight modification of the proof of [6, Theorem 2.1, Step 1] yields, for any $u \in H^2(\Omega)$, $y, y_0 \in \Omega^{3\delta}$ and $\epsilon > 0$,

$$C_0\|u\|_{L^2(B(y,\delta))} \leq \epsilon^{1/(1-\psi(\delta))}\|u\|_{L^2(\Omega)} + \epsilon^{-1/\psi(\delta)}(\|Lu\|_{L^2(\Omega)} + \|u\|_{L^2(B(y_0,\delta))}). \tag{4}$$

Here ψ is of the form $\psi(\delta) = se^{-C_0/\delta}$, with $0 < s < 1$ only depending on Ω, λ and κ.

Putting together (3) and (4) with $y_0 = x_\delta$, we find, for any $u \in H^2(\Omega)$, $y \in \Omega^{3\delta}$, $0 < \delta < \delta_0 := \min(\delta^*/3, (3R\sin\theta)/2)$, $\epsilon > 0$ and $\eta > 0$,

$$C_1\|u\|_{L^2(B(y,\delta))} \le \epsilon^{1/(1-\psi(\delta))}\|u\|_{L^2(\Omega)} + \epsilon^{-1/\psi(\delta)}[\|Lu\|_{L^2(\Omega)} + \eta^\gamma\|u\|_{H^1(\Omega)} + \eta^{-1}(\|u\|_{L^2(\Gamma)} + \|\nabla u\|_{L^2(\Gamma)} + \|Lu\|_{L^2(\Omega)})]. \quad (5)$$

In (5), take

$$\eta = \epsilon^{1/(\gamma\psi(\delta)(1-\psi(\delta)))}$$

to obtain

$$C_1\|u\|_{L^2(B(y,\delta))} \le \phi_0(\epsilon,\delta)\|u\|_{H^1(\Omega)} + \phi_1(\epsilon,\delta)(\|u\|_{L^2(\Gamma)} + \|\nabla u\|_{L^2(\Gamma)} + \|Lu\|_{L^2(\Omega)}), \quad (6)$$

where

$$\phi_0(\epsilon,\delta) = \epsilon^{1/(1-\psi(\delta))},$$
$$\phi_1(\epsilon,\delta) = \epsilon^{-1/\psi(\delta)}\max(1, \epsilon^{-1/(\gamma\psi(\delta)(1-\psi(\delta)))}).$$

On the other hand, it is straightforward to check that $\Omega^{3\delta}$ can be covered by at most k^n balls with centre in $\Omega^{3\delta}$ and radius δ, where $k = [c/\delta]$, the constant c only depending on n and the diameter of Ω. In this way, from (6),

$$C_1\|u\|_{L^2(\Omega^{3\delta})} \le \delta^{-n}\phi_0(\epsilon,\delta)\|u\|_{H^1(\Omega)} + \delta^{-n}\phi_1(\epsilon,\delta)(\|u\|_{L^2(\Gamma)} + \|\nabla u\|_{L^2(\Gamma)} + \|Lu\|_{L^2(\Omega)}). \quad (7)$$

Now, according to Hardy's inequality (see for instance [7, Theorem 1.4.4.4, page 29]), for $0 < s < \frac{1}{2}$, there exists $\varkappa$, only depending on Ω and s, so that

$$\|u\|_{L^2(\Omega_{3\delta})} \le (3\delta)^s\left\|\frac{u}{\mathrm{dist}(x,\partial\Omega)^s}\right\|_{L^2(\Omega)} \le \varkappa\delta^s\|u\|_{H^s(\Omega)}.$$

As $H^1(\Omega)$ is continuously embedded in $H^s(\Omega)$, changing $\varkappa$ if necessary,

$$\|u\|_{L^2(\Omega_{3\delta})} \le \varkappa\delta^s\|u\|_{H^1(\Omega)}. \quad (8)$$

Henceforth, $0 < s < \frac{1}{2}$ is fixed and C is a generic constant that only depends on Ω, Γ, λ, κ and s. Putting together (7) and (8),

$$C\|u\|_{L^2(\Omega)} \le (\delta^{-n}\phi_0(\epsilon,\delta) + \delta^s)\|u\|_{H^1(\Omega)} + \delta^{-n}\phi_1(\epsilon,\delta)(\|u\|_{L^2(\Gamma)} + \|\nabla u\|_{L^2(\Gamma)} + \|Lu\|_{L^2(\Omega)}). \quad (9)$$

We choose ϵ in (9) such that $\delta^{-n}\phi_0(\epsilon,\delta) = \delta^s$ or equivalently $\epsilon = \delta^{(n+s)\psi(\delta)}$. Then elementary computations yield

$$C\|u\|_{L^2(\Omega)} \le \delta^s\|u\|_{H^1(\Omega)} + e^{e^{c/\delta}}(\|u\|_{L^2(\Gamma)} + \|\nabla u\|_{L^2(\Gamma)} + \|Lu\|_{L^2(\Omega)}). \quad (10)$$

This inequality corresponds to (2) when $j = 0$.

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Next, noting that $H^1(\Omega)$ can be seen as an interpolated space between $L^2(\Omega)$ and $H^2(\Omega)$, for $\epsilon > 0$,

$$C_\Omega \|u\|_{H^1(\Omega)} \le \epsilon \|u\|_{H^2(\Omega)} + \epsilon^{-1}\|u\|_{L^2(\Omega)},$$

the constant C_Ω only depending on Ω. This inequality with $\epsilon = \delta^{s/2}$ and (10) yields

$$C\|u\|_{H^1(\Omega)} \le \delta^{s/2}\|u\|_{H^2(\Omega)} + e^{e^{c/\delta}}(\|u\|_{L^2(\Gamma)} + \|\nabla u\|_{L^2(\Gamma)} + \|Lu\|_{L^2(\Omega)}).$$

That is, we have proved (2) in the case $j = 1$. □

REMARK 4.

(i) It is worth mentioning that Theorem 1 still holds if L is replaced by $L + L_1$, where L_1 is a first-order partial differential operator with bounded coefficients. In that case the constants c and C in the statement of Theorem 1 may also depend on bounds for the coefficients of L_1.

(ii) For $0 < t < 2$, we have the following interpolation inequality, with $u \in H^2(\Omega)$ and $\epsilon > 0$:

$$C_\Omega \|u\|_{H^t(\Omega)} \le \epsilon^{t/(2-t)}\|u\|_{H^2(\Omega)} + \epsilon^{-1}\|u\|_{L^2(\Omega)}.$$

We can then proceed as in the preceding proof in order to show, for $u \in H^2(\Omega)$ and $0 < \delta < \delta_0$,

$$C\|u\|_{H^t(\Omega)} \le \delta^{st/2}\|u\|_{H^2(\Omega)} + e^{e^{c/\delta}}(\|u\|_{L^2(\Gamma)} + \|\nabla u\|_{L^2(\Gamma)} + \|Lu\|_{L^2(\Omega)}).$$

Here the constants c and C only depend on s, t, Ω, Γ, λ and κ, and δ_0 only depends on Ω.

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