This figure also gives an idea of how such a trapezium with given side-lengths a, $a + d$, $a + 3d$, $a + 2d$ can be constructed with the help of a compass and a ruler.

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107.11 The Steiner-Lehmus Theorem à la Ceva

The Steiner-Lehmus theorem (SL) states that *if the internal angle bisectors of two angles of a triangle are equal, then the corresponding sides are equal*. Despite being easy to state, it is by no means trivial to prove. One of the scores of proofs of this theorem can be found on p. 396 in [1].

In this Note, we first use the SL to prove a theorem, Theorem 1, that we call *the Steiner-Lehmus theorem inside out* (SLIO). We then show in Theorem 2 that the SL itself is a consequence of SLIO. Thus the SL and the SLIO are equivalent. In Theorem 3, we give a fairly simple proof of (a stronger form of) the SLIO using Ceva's theorem and what is often referred to as *the open mouth theorem*. The latter is illustrated in Figure 3 below. Thus we essentially have a proof of SL based on Ceva's theorem. This, hopefully, justifies the title of this Note. The Note ends with a stronger form of the SL, and with a description of the context that led to the SLIO.

Theorem 1 (The Steiner-Lehmus theorem inside out)

Let O be the circumcentre of the acute-angled triangle ABC, and let BE and CF be the cevians through O, as in Figure 1. If $AE = AF$, then $AB = AC$.

Proof

Let the line through A parallel to FC meet the extension of BE at N, and let the line through O parallel to BA meet AN at M; see Figure 1.

By elementary angle-chasing, AE is the bisector of $\angle NAO$ and OM is the bisector of $\angle AON$. By construction, $AFOM$ is a parallelogram and so $OM = AF = AE$. Hence by SL on triangle NAO we see that $NO = NA$ and so $\angle NOA = \angle NAO$. It follows that $\angle FCA = \angle EBA$ and so $\angle ACB = \angle ABC$ and $AB = AC$ as desired.

The proof above also reveals that the Steiner-Lehmus theorem follows from Theorem 1. For the convenience of the reader, we formulate this in the next theorem.

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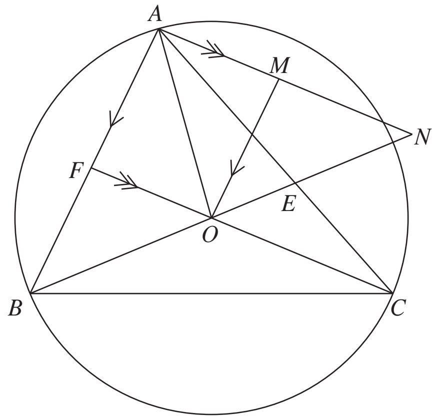

FIGURE 1

Theorem 2

Theorem 1 implies the Steiner-Lehmus theorem.

Proof

Let ABC be a triangle whose internal angle bisectors BE and CF are equal. We shall use Theorem 1 to prove that $AB = AC$.

Referring to Figure 2, let the line through B parallel to FC meet the extension of AC at K, and let the line through C parallel to BA meet BK at H and the extension of BE at G.

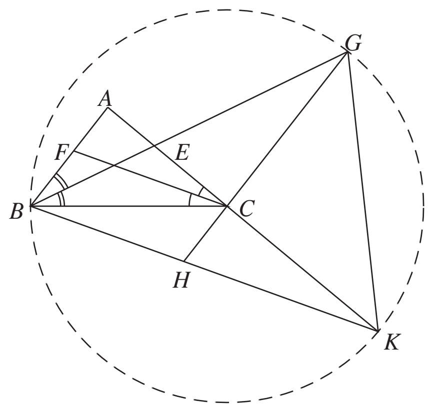

FIGURE 2

By elementary angle-chasing, $\angle GBC = \angle CGB$ and $\angle CBK = \angle CKB$ and so $CB = CG$ and $CB = CK$. Hence C is the circumcentre of BKG. By construction, $BHCF$ is a parallelogram and so $BE = CF = BH$. Hence, by Theorem 1, $BG = BK$ and so $\angle BKG = \angle BGK$. Hence $\angle CGB = \angle CKB$, $\angle ABE = \angle ACF$ and $\angle ACB = \angle ABC$ as required.

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In view of Theorem 2, a *new* proof of the Steiner-Lehmus theorem can be obtained if an appropriate, preferably simple, proof of Theorem 1 is given. This is done in Theorem 3 below, where a considerably stronger form of Theorem 1, with a proof that is essentially a *proof without words*, is established. The proof depends on Ceva's theorem and also uses what is often referred to as *the open mouth theorem*, or *the scissors' lemma*. The latter is illustrated in Figure 3 below.

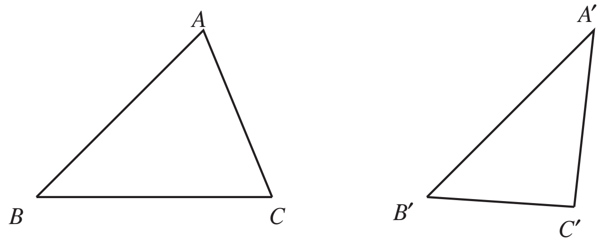

FIGURE 3

The open mouth theorem states that if $AB = A'B'$ and $AC = A'C'$. then $BC > B'C' \Leftrightarrow \angle BAC > \angle B'A'C'$.

Theorem 3 (A strong Steiner-Lehmus theorem inside out)

Let M be a point inside triangle ABC such that $MB = MC$, and let BE and CF be the cevians through M. Then $AB > AC \Leftrightarrow AF > AE$. Consequently, $AF = AE \Leftrightarrow AB = AC$.

Proof

It is clearly sufficient to prove that if $AB > AC$, then $AF > AE$. Thus we assume that $AB > AC$ and

$$AE \geqslant AF, \tag{1}$$

and we seek a contradiction.

Referring to Figure 4, we let U be the point where the extension of AM meets BC. It follows from the open mouth theorem, applied to triangles AMC and AMB, that $\angle AMC < \angle AMB$, and hence $\angle CMU > \angle UMB$. The open mouth theorem, applied now to triangles UMC and UMB, implies that

$$UC > BU. \tag{2}$$

It also follows from $AF \leqslant AE$ and $AB > AC$ that

$$FB > CE. \tag{3}$$

It finally follows from (1), (2), and (3) that

$$(EA)(FB)(UC) > (AF)(BU)(CE),$$

contradicting Ceva's theorem.

This completes the proof.

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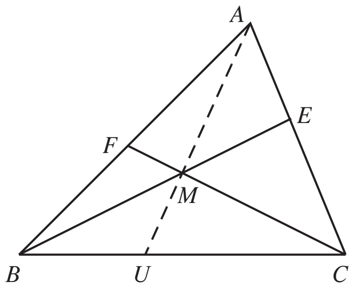

FIGURE 4

We have seen in Theorems 1 and 2 that the classical Steiner-Lehmus theorem and the Steiner-Lehmus theorem inside out are equivalent. On the other hand, Theorem 3, while considerably stronger than Theorem 1, was seen to be quite easy to prove. Does that not make the Steiner-Lehmus theorem even easier? Is Theorem 3 equivalent to some stronger form of the Steiner-Lehmus theorem? In this respect, it is worth mentioning that a stronger form of the Steiner-Lehmus theorem was indeed established in [2]. For the convenience of the reader, we state it below.

Theorem 4 (A strong Steiner-Lehmus theorem)

If cevians BE and CF of triangle ABC intersect at a point that lies on the internal bisector of A, and if $BE = CF$, then $AB = AC$.

Other stronger forms of the Steiner-Lehmus theorem have appeared in the recent literature. For a most recent one, we refer the reader to [3].

Theorems 1 and 3 pertain to the configuration consisting of a triangle ABC, an interior point P, and the cevians AD, BE and CF through P, as shown in Figure 5. The first says that if $PA = PB = PC$, and $AE = AF$, then $AB = AC$. The second says that if $PB = PC$, and $AE = AF$, then $AB = AC$.

We observe that P is the circumcentre if, and only if, $PA = PB = PC$ and that P is the Gergonne centre if, and only if, $AE = AF$, $BF = BD$ and $CE = CD$, we call a point P for which $PB = PC$ an A-semi-circumcentre, and a point P for which $AE = AF$ an A-semi-Gergonne centre.

Now we can restate Theorem 1 as saying that if the circumcentre coincides with an A-semi-Gergonne centre, then $AB = AC$, Similarly, we can restate Theorem 3 as saying that if an A-semi-circumcentre coincides with an A-semi-Gergonne centre, then $AB = AC$. Of course, it follows from this that if the circumcentre coincides with the Gergonne centre, then ABC is equilateral.

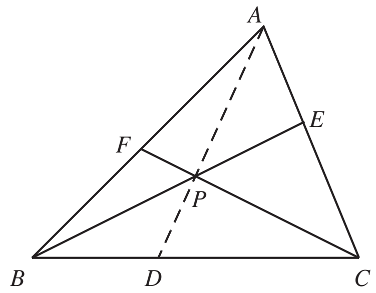

FIGURE 5

Since a triangle has many centres, the paragraph above can be a source for composing problems of the olympiad type. This line of research is pursued in [4] and also in [5].

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107.12 The Steiner-Lehmus theorem à la Euclid

The Steiner-Lehmus theorem states that if ABC is a triangle in which BE and CF are the (internal) angle bisectors of angles B and C respectively, and if $BE = CF$, then $AB = AC$; see Figure 1. Proofs of this abound in the literature, and many of these prove the stronger statement

$$AB > AC \Rightarrow BE > CF. \qquad (1)$$

This Note provides yet another proof of (1) which is short and has the advantage of being closest to Euclidean, in the sense that Euclid himself would have found a place for it somewhere in his *Elements*, namely in Book VI. The proof also meets the standards of being purely geometric that were imposed by Professor C. L. Lehmus when he first posed the problem in 1840.