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REPRESENTATIONS OF THE $(p^2 - 1)$-DIMENSIONAL LIE ALGEBRAS OF R. E. BLOCK

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ABSTRACT. For all algebras G, such that $G/\operatorname{rad} G \cong H\big(2;\mathbf{1};\Phi(\tau)\big)^{(1)}$ is an algebra mentioned in the title, the modules of dimension $\leq p^2$ are determined. The module homomorphisms from the tensor product of these modules into a third module of the same type are described. We also give the central extensions of the algebras $H\big(2;\mathbf{1};\Phi(\tau)\big)^{(1)}$.

Introduction. In 1958 R. E. Block [Bl-58] defined a class of simple Lie algebras over any field of positive characteristic p, named $L(G,\delta,f)$. Here G is a $GF(p)$-vector space, δ some appropriate element of G and f a biadditive form on $G\times G$. R. D. Schafer [Sch-60] gave a realization of these algebras in terms of a Lie multiplication on a truncated polynomial ring. The class of these algebras form a subclass of the hamiltonian Lie algebras, which in turn are of Cartan type [W-76]. The restricted, or more generally, the graded members of this class are pretty well known, while only little information in the general case has yet been published. Unfortunately some of these algebras occur naturally (and then play an obstructive role) in the present approach to the classification of the simple modular Lie algebras. In fact, a finite dimensional simple Lie algebra over an algebraically closed field of characteristic $p > 7$ is of classical or Cartan type, provided it has no subalgebra, whose semisimple quotient is an algebra of type $L(G,\delta,f)$ of dimension $p^2 - 1$. It is the interest in the classification problem of the simple Lie algebras which mainly motivated this note.

There is up to isomorphism only one algebra of dimension $p^2 - 1$ of type $L(G,\delta,f)$ [Oeh-65]. It is denoted here by $H\big(2;\mathbf{1};\Phi(\tau)\big)^{(1)}$.

We investigate Lie algebras G, for which $G/\operatorname{rad} G \cong H\big(2;\mathbf{1};\Phi(\tau)\big)^{(1)}$. The list of all modules of dimension $\leq p^2$ for such algebras is given (Theorems 4.9, 5.1, 6.5). The determination of the modules has been applied in another paper on the classification to solve the classical case [St-2]. As a further consequence, we determine the central extensions of $H\big(2;\mathbf{1};\Phi(\tau)\big)^{(1)}$ (Theorems 6.2, 6.3). In addition, the module homomorphisms from the tensor product of two of these modules into a third one of the same type are determined to some extent (Theorems 7.4, 7.5). These results apply to a situation when a simple Lie algebra has such a subalgebra G and subspaces which are G-modules of dimension p^2. The multiplication of such subspaces is governed by the above mentioned results on the module homomorphisms. This information will be used in some forthcoming paper to construct subalgebras in some simple Lie algebras in order to apply the

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Recognition Theorem, thus proving a further class of simple algebras to be of Cartan type.

In addition to results which are essential for a solution of the classification problem this note gives an example for the application of some of the methods in representation theory: every module M for a Lie algebra G is also a module for any p-envelope G_p; if M is irreducible it admits a character χ; if G is filtered, the dimension of the module can be estimated in terms of χ and the length of the filtration; M is a homomorphic image of an induced module $u(G_p,\chi)\otimes_{u(K,\chi|K)} N$, where $u(G_p,\chi)$ is the universal enveloping algebra of G_p reduced by the character χ, K is a restricted subalgebra and N is a K-module. This note establishes a first step of investigating the representations of nongraded Cartan type Lie algebras.

1. **The algebra** $\operatorname{Der} H\big(2;\mathbf{1};\Phi(\tau)\big)^{(1)}$. *In this section we assume that the underlying field F has characteristic $p > 3$.* The following are well-known facts, published in various papers. We gather this material without proofs, but give some references. The main reference is [Sch-60].

Let $A(2;\mathbf{1}) := F[x,y]$, $x^p = y^p = 0$, denote the p^2-dimensional truncated polynomial ring in two generators and define a distinguished element $\Lambda := 1 - x^{p-1}y^{p-1}$. $A(2;\mathbf{1})$ carries a Lie algebra structure by putting

$$\{f,g\} := [(\partial_1 f)(\partial_2 g) - (\partial_2 f)(\partial_1 g)]\Lambda \text{ with } \partial_1 := \partial/\partial x,\ \partial_2 := \partial/\partial y.$$

The algebra $\big(A(2;\mathbf{1}),\{\,\}\big)$ is isomorphic to $H\big(2;\mathbf{1};\Phi(\tau)\big)$, as it is defined in [BW-88] (cf. [St-1] Theorems VII.1 and VII.2). We have that $\big(A(2;\mathbf{1}),\{\,\}\big)^{(1)} \cap F1 = 0$ and $\big(A(2;\mathbf{1}),\{\,\}\big)^{(1)}$ is a simple Lie algebra of dimension p^2-1. For every $f \in \big(A(2;\mathbf{1}),\{\,\}\big)$ the mapping $\{f,?\}$ is a derivation of the truncated polynomial ring and so there is a homomorphism $D\colon \big(A(2;\mathbf{1}),\{\,\}\big) \longrightarrow W(2;\mathbf{1})$ $D(f) := \{f,?\}$ from $\big(A(2;\mathbf{1}),\{\,\}\big)$ into the restricted generalized Witt algebra $W(2;\mathbf{1})$. $D(f)$ is given by

$$D(f) = \Lambda\partial_1(f)\partial_2 - \Lambda\partial_2(f)\partial_1.$$

Obviously, $\ker D = F_{\mathbf{1}}$. Therefore D is injective on $\big(A(2;\mathbf{1}),\{\,\}\big)^{(1)}$.

For the rest of this note we introduce the following

ABBREVIATION. $H := \big(A(2;\mathbf{1}),\{\,\}\big)^{(1)}$.

$D(H)$ and hence H is not a restricted algebra, since we have

$$D(x)(y) = \{x,y\} = \Lambda, D(x)^p(y) = D(x)^{p-1}(\Lambda) = -(p-1)!\,x^{p-1},$$

showing that $D(x)^p = x^{p-1}\partial_2 \notin D(H)$. Similarly, we have

$$D(y)^p(x) = -D(y)^{p-1}(\Lambda) = -(-1)^p(p-1)!\,y^{p-1},$$

hence $D(y)^p = -y^{p-1}\partial_1$.

Every finite dimensional Lie algebra G can be embedded into a finite dimensional restricted Lie algebra $(G_p,[p])$, with some embedding $\iota\colon G \longrightarrow G_p$, such that G_p itself is the restricted subalgebra generated by $\iota(G)$ [SF-88]. We call $(G_p,[p],\iota)$ (or simpler G_p) a p-envelope of G.

PROPOSITION 1.1. *1)* $\mathrm{Der}\, H \cong D(H) \oplus Fx^{p-1}\partial_2 \oplus Fy^{p-1}\partial_1$.
2) $\mathrm{Der}\, H$ *is a p-envelope of* $D(H)$.

PROOF. The right-hand-side vector space is embedded into the derivation algebra via the ad-representation. Its dimension is $p^2 + 1$, which is also the dimension of $\mathrm{Der}\, H$ [Bl-58]. Since every p-envelope of $D(H)$ in $W(2;\mathbf{1})$ has to contain $D(x)^p$ and $D(y)^p$, and $\mathrm{Der}\, H$ as a full derivation algebra is closed under pth powers, the result follows. ■

In order to simplify the notation I prefer to argue in $H \subset \big(A(2;\mathbf{1}), \{\ \}\big)$, rather than in $W(2;\mathbf{1})$. Thus we introduce the notion of two elements Γ, Θ (corresponding to $x^{p-1}\partial_2$, $-y^{p-1}\partial_1$) and of the Lie algebra

$$L := H \oplus F\Gamma \oplus F\Theta$$

with multiplication and p-mapping $[p]$

$$\begin{aligned}
\{\Gamma, x^a y^b\} &:= b x^{p-1+a} y^{b-1}\\
\{\Theta, x^a y^b\} &:= -a x^{a-1} y^{p-1+b}\\
\{\Gamma, \Theta\} &:= -\Lambda = -1 + x^{p-1}y^{p-1}\\
(x^a y^b)^{[p]} &= \begin{cases} 0 & \text{if } (a,b) \notin \{(0,1),(1,0),(1,1)\}\\ xy & \text{if } a = b = 1\\ \Gamma & \text{if } a = 1,\ b = 0\\ \Theta & \text{if } a = 0,\ b = 1\end{cases}\\
\Gamma^{[p]} &= 0,\\
\Theta^{[p]} &= 0.
\end{aligned}$$

It is a straightforward calculation that $(L, [p])$ is a restricted algebra isomorphic to $\mathrm{Der}\, H$, the isomorphism being given by

$$\psi : L \longrightarrow W(2;\mathbf{1})$$
$$\psi(f + \alpha\Gamma + \beta\Theta) = D(f) + \alpha x^{p-1}\partial_2 - \beta y^{p-1}\partial_1.$$

$W(2;\mathbf{1})$ is canonically filtered (even graded) by

$$W(2;\mathbf{1})_{(i)} := \sum_{r+s>i} F x^r y^s \partial_1 + F x^r y^s \partial_2.$$

With this notation $W(2;\mathbf{1}) = W(2;\mathbf{1})_{(-1)}$, $W(2;\mathbf{1})_{(2p-3)} \neq 0$, $W(2;\mathbf{1})_{(2p-2)} = 0$.

We remind the reader, that $\{x^a y^b, x^c y^d\} = (ad - bc)x^{a+c-1}y^{b+d-1}$, whenever $a + b + c + d > 2$. This simplified version of the product in a special situation will frequently be used. We also remark (which is easily proved by induction), that

$$\begin{aligned}
(\mathrm{ad}\, x)^i (\mathrm{ad}\, y)^j (\Lambda) &= (-1)^{i+1} i!\, j!\, x^{p-1-j} y^{p-1-i} \quad 0 \le i, j \le p-1,\ 1 \le i + j \le 2p-3\\
&= -\Lambda \quad i = j = p-1
\end{aligned}$$

PROPOSITION 1.2. *1) H and L carry filtrations $\big(H_{(i)}\big)_{-1\le i\le 2p-4}$, $\big(L_{(i)}\big)_{-1\le i\le 2p-4}$, respectively defined by*

$$\begin{aligned}
H_{(i)} &:= H \cap \psi^{-1}\big(W(2;\mathbf{1})_{(i)}\big) = \mathrm{span}\{x^a y^b \mid i + 2 \le a + b \le 2p-3\} + F\Lambda\\
L_{(i)} &:= L \cap \psi^{-1}\big(W(2;\mathbf{1})_{(i)}\big) = H_{(i)} + F\Gamma + F\Theta \quad \textit{if } i \le p-2,\\
&\phantom{:= L \cap \psi^{-1}\big(W(2;\mathbf{1})_{(i)}\big)} = H_{(i)} \quad \textit{if } i > p-2.
\end{aligned}$$

2) *a)* $\dim L/L_{(0)} = 2$
b) $L_{(0)}/L_{(1)} \cong \mathrm{sl}(2)$
c) $L^{(1)} = H = H^{(1)}$, $\left(H_{(0)}\right)^{(1)} = \mathrm{span}\{x^a y^b \mid 2 \leq a+b \leq 2p-3\}$
d) $\left(L_{(0)}\right)^{(1)} = H_{(0)}$, $\left(L_{(0)}\right)^{(2)} = \left(L_{(0)}\right)^{(n)}$ *for all* $n \geq 3$
e) $L_{(0)} = \left(L_{(0)}\right)^{(2)} \oplus (F\Gamma \oplus F\Theta \oplus F\Lambda)$ *and* $(F\Gamma \oplus F\Theta \oplus F\Lambda)$ *is a Heisenberg algebra*
f) $\{H, L_{(i)}\} = \{H, H_{(i)}\} = \mathrm{span}\{x^a y^b \mid i+1 \leq a+b \leq 2p-3\}$ *for* $0 \leq i \leq 2p-4$. *Thus* $H_{(i-1)} = \{H, H_{(i)}\} + F\Lambda$.

3) $H_{(0)}$ and $L_{(0)}$ are restricted subalgebras of L.

PROOF. 1) follows from the fact, that ψ is a homomorphism.

2)a),b): $L = L_{(0)} \oplus Fx \oplus Fy, L_{(0)} = L_{(1)} \oplus Fx^2 \oplus Fxy \oplus Fy^2$. $Fx^2 \oplus Fxy \oplus Fy^2$ is isomorphic to sl(2).

c): The simplicity of H gives $H^{(1)} \subset L^{(1)} \subset H \subset H^{(1)}$. To prove that $\left(H_{(0)}\right)^{(1)} \supset \mathrm{span}\{x^a y^b \mid 2 \leq a+b \leq 2p-3\}$ we consider any element $x^a y^b \in H$ with $0 < a+b < 2p-2$. If $a < p-1$, and $b \neq 0$ then $x^a y^b = [2(a+1)]^{-1}\{x^{a+1}y^{b-1}, y^2\}$, while for $b = 0, a \neq 0$, $x^a = a^{-1}\{x^a, xy\}$. Similarly, we treat the case $b \neq p-1$. To prove the reverse inclusion we observe that for $a+b \geq 2$, $c+d \geq 2$ a product $\{x^a y^b, x^c y^d\} = (ad-bc)x^{a+c-1}y^{b+d-1}$ vanishes, whenever $(a+c-1) = (b+d-1) = p-1$.

d): According to c) we have $H_{(0)} = \left(H_{(0)}\right)^{(1)} + F\Lambda$. An easy computation yields $\{H_{(0)}, \Lambda\} = 0$, hence $\left(H_{(0)}\right)^{(2)} = \left(H_{(0)}\right)^{(1)}$. As $\{\Gamma, \Theta\} = -\Lambda$, this proves $L^{(1)}_{(0)} = H_{(0)}$ and $\left(L_{(0)}\right)^{(n)} = (H_{(0)})^{(n-1)} = (H_{(0)})^{(1)} = \left(L_{(0)}\right)^{(2)}$ for $n > 2$.

e): is obvious.

f): For $i \leq p-2$ we have $\{H, H_{(i)}\} \subset \{H, L_{(i)}\} = \{H_{(0)}, H_{(i)}\} + \sum_{a+b \geq i+2} F\{x, x^a y^b\} + F\{x, \Lambda\} + \sum_{a+b \geq i+2} F\{y, x^a y^b\} + F\{y, \Lambda\} + \{H_{(0)}, \Gamma\} + \{H_{(0)}, \Theta\} + F\{x, \Gamma\} + F\{x, \Theta\} + F\{y, \Gamma\} + F\{y, \Theta\} \subset \mathrm{span}\{x^r y^s \mid i+1 \leq r+s \leq 2p-3\}$. To prove the reverse inclusion we consider an element $x^a y^b$ with $a < p-1$ ($b < p-1$ is treated similarly), $a+b > 0$. Then $x^a y^b = -(a+1)^{-1}\{y, x^{a+1}y^b\} \in \{H, H_{(a+b-1)}\}$. For $i > p-2$ one proceeds similarly.

3) The mapping $[p]$ maps a basis of $L_{(0)}$ and $H_{(0)}$ into $L_{(0)}$ and $H_{(0)}$, respectively, and therefore leaves each of these algebras invariant. ∎

We need some information about subalgebras of suitable size. An analogue is well-known for the restricted hamiltonian algebra of dimension $p^2 - 2$.

PROPOSITION 1.3. *1) If* $g \in H$, $g \notin H_{(0)}$ *then*

$$\{g, H_{(1)}\} + H_{(1)} = H_{(0)}.$$

2) If K *denotes a subalgebra of* H *which satisfies* $H_{(1)} \subset K + H_{(2)}$, *then*

$$H_{(1)} \subset K + F\Lambda.$$

3) $H_{(0)}$ *is generated as an algebra by* $\{x^2, y^2, x^3, \Lambda\}$.

4) $L_{(0)}$ is generated as an algebra by either of $\{x^2, y^2, x^3, \Gamma, \Theta\}$ or $\{x^2, y^2, y^3, \Gamma, \Theta\}$.

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PROOF. 1) Write $g = \alpha x + \beta y + h$, $h \in H_{(0)}$. The result follows from the ensuing computations mod $H_{(1)}$

$$\{g, x^3\} \equiv -3\beta x^2, \quad \{g, x^2 y\} \equiv \alpha x^2 - 2\beta xy,$$
$$\{g, y^3\} \equiv 3\alpha y^2, \quad \{g, xy^2\} \equiv 2\alpha xy - \beta y^2.$$

2) Assume inductively, that $H_{(1)} \subset K + H_{(r)}$ for some $r \geq 1$ and let $x^a y^b$, $a + b = r + 3$, be an element of $H_{(r+1)}$. The induction hypothesis yields that there are elements (we treat the cases $a = 0$ or $b = 0$ simultaneously)

$$ax^{a-1}y^b,\ bx^a y^{b-1},\ x^2 y,\ xy^2 \in H_{(1)} \cap K + H(r).$$

Thus $K + H_{(r+1)}$ contains

$$\{ax^{a-1}y^b, x^2 y\} = a\big((a-1) - 2b\big)x^a y^b,$$
$$\{bx^a y^{b-1}, xy^2\} = b\big(2a - (b-1)\big)x^a y^b.$$

If $a = 0$ then $b = r+3 \neq 1$ and clearly $0 < b < p$. The second equation yields the result. If $a \neq 0$, $b \neq 0$, one of the factors is nonzero, provided $p \neq 3$ and $(a,b) \neq (p-1, p-1)$. Hence we obtain by induction $H_{(1)} \subset K + H_{(2p-4)} = K + F\Lambda$.

3), 4) The algebra G generated by $\{x^2, y^2, x^3\}$ contains

$$\{x^2, y^2\} = 4xy,\ \{x^3, y^2\} = 6x^2 y,\ \{x^2 y, y^2\} = 4xy^2, \{xy^2, y^2\} = 2y^3.$$

Thus $H_{(1)} \subset H_{(0)} \subset G + H_{(2)}$ and part (2) yields $H_{(1)} \subset G + F\Lambda$. This proves 3). As $\{\Gamma, \Theta\} = -\Lambda$, the first case of 4) follows as well. The other case is treated similarly. ∎

PROPOSITION 1.4.

1) If K is a subalgebra of H with $\dim H/K \leq 2$, *then $K = H$ or $K = H_{(0)}$.*

2) If K is a restricted subalgebra of L and $\dim L/K \leq 2$, *then $K = L$ or $K = L_{(0)}$.*

3) If K is an ideal of L, then $H \subset K$ or $K = 0$. In particular, L has no nontrivial p-ideal.

PROOF. 1) If $K \subset H_{(0)}$, the assumption $\dim H/K \leq 2 = \dim H/H_{(0)}$ yields $K = H_{(0)}$. Thus assume that $K \not\subset H_{(0)}$.

If $H_{(0)} \subset K + H_{(1)}$, we observe that $H_{(0)}/H_{(1)} \cong H_{(0)} \cap K/H_{(1)} \cap K$. Since $H_{(1)}/H_{(2)}$ and $H/H_{(0)}$ are irreducible $H_{(0)}/H_{(1)}$-modules, having nonzero submodules $H_{(1)} \cap K/H_{(2)} \cap K$ and $K/H_{(0)} \cap K$, respectively, we obtain $H_{(1)} \subset K + H_{(2)}$, $H \subset K + H_{(0)}$. The first result yields in accordance with Proposition 1.3(2) that $H_{(1)} \subset K + F\Lambda$. The second result and the above assumption then gives $H \subset K + F\Lambda$. As $\{H_{(0)}, \Lambda\} = 0$, we obtain $H_{(0)}^{(1)} \subset K$, $xy \in K$, and $x, y \in K$. Consequently, $\Lambda = \{x, y\} \in K$ and $K = H$.

If $H_{(1)} \subset K + H_{(2)}$, then (as we assume $K \not\subset H_{(0)}$) Proposition 1.3(1) applies and yields $H_{(0)} \subset K + H_{(1)}$. This is the former case.

Thus assume that $H_{(1)} \not\subset K+H_{(2)}, H_{(0)} \not\subset K+H_{(1)}$, i.e. $\dim H_{(1)}/\big(H_{(1)}\cap K+H_{(2)}\big) \neq 0$, $\dim H_{(0)}/\left(H_{(0)} \cap K + H_{(1)}\right) \neq 0$. Observe that

$$\begin{aligned}\sum_{-1\leq i\leq 1} \dim H_{(i)}/\big(H_{(i)} \cap K + H_{(i+1)}\big) \\ &= \sum_{-1\leq i\leq 1} \dim\big(H)_{(i)} + K\big)/\big(H_{(i+1)} + K\big) = \dim H/\big(H_{(2)} + K\big) \\ &= \dim H/K - \dim\big(H_{(2)} + K\big)/K = 2 - \dim H_{(2)}/H_{(2)} \cap K.\end{aligned}$$

The present assumption yields $\dim H/\left(K + H_{(0)}\right) = 0$, $H_{(2)} \subset K$. Then

$$H_{(1)} \subset \{H, H_{(2)}\} + F\Lambda \subset \{K,K\} + \{H_{(0)}, H_{(2)}\} + F\Lambda \subset K + H_{(2)},$$

a contradiction.

2) Let $\pi: L \longrightarrow L/K$ denote the canonical linear mapping. Then $\dim \pi(H) \leq \dim L/K \leq 2$ and therefore $H \cap K = \ker \pi|H$ is a subalgebra of H of codimension ≤ 2. By the preceding result the only possibilities are $H\cap K = H$ or $H\cap K = H_{(0)}$. In the first case, $H \subset K$ and as K is a restricted algebra, it contains the p-envelope L of H. In the second case, $H\cap K = H_{(0)}$ has codimension 4 in L. Let $\alpha x + \beta y + \gamma\Gamma + \delta\Theta + g$, $g \in H_{(0)}$ be an element in K. Then, as H is an ideal of L,

$$\{xy, \alpha x + \beta y + \gamma\Gamma + \delta\Theta + g\} = -\alpha x + \beta y + \{xy, g\} \in \{H\cap K, K\} \subset H\cap K = H_{(0)},$$

proving $\alpha = \beta = 0$. Hence $K \subset L_{(0)}$, and as $\dim L/K = \dim L/L_{(0)}$, this yields the result.

3) If $H\cap K = 0$, then $\{H,K\} \subset H\cap K = 0$ and K centralizes H. This is possible in L only if $K = 0$. If $H \cap K \neq 0$, then the simplicity of H yields $H \subset K$.

Under the assumption of K being a restricted nonzero ideal, it contains the p-envelope L of H. Therefore L has no nontrivial restricted ideals. ∎

2. **Methods in representation theory.** *In this section we assume the ground field to be algebraically closed.* The generalization to arbitrary fields can be done by an almost obvious procedure.

Let G be an arbitrary Lie algebra and G_p a p-envelope of G. Given any G-module M this representation can be extended to a representation of G_p ([SF-88], Theorem V.1.1). This extension, however, is in general not unique.

PROPOSITION 2.1. *Let G be a Lie algebra, $\rho: G \longrightarrow \mathrm{gl}(M)$ a representation, G_p a p-envelope of G and $\rho': G_p \longrightarrow \mathrm{gl}(M)$ an extension of ρ.*

1) *For any linear form $\lambda \in (G_p)^*$ such that $\lambda(G) = 0$, $\rho' + \lambda\ \mathrm{id}_M$ is also an extension of ρ.*
2) *Let $\chi: G_p \longrightarrow \mathrm{gl}(M)$ be another extension of ρ. If M is G-irreducible, then there is $\lambda \in (G_p)^*$, $\lambda(G) = 0$, with $\rho' - \chi = \lambda\ \mathrm{id}_M$.*

PROOF. Note that $(G_p)^{(1)} \subset G$.

1) As $\lambda\left(G_p^{(1)}\right) = \lambda(G) = 0$, $\rho' + \lambda\ \mathrm{id}_M$ is a representation and an extension of ρ.

2) As for all $g \in G_p$, $h \in G$,

$$\begin{aligned}[\rho'(g) - \chi(g), \rho(h)] &= [\rho'(g), \rho(h)] - [\chi(g), \rho(h)] = [\rho'(g), \rho'(h)] - [\chi(g), \chi(h)] \\ &= \rho'([g,h]) - \chi([g,h]) = \rho([g,h]) - \rho([g,h]) = 0,\end{aligned}$$

the irreducibility of M shows that $\rho' - \chi$ is a linear mapping from G_p into $F\,\mathrm{id}_M$. ■

Let $(G, [p])$ be a restricted Lie algebra and $\rho\colon G \longrightarrow \mathrm{gl}(M)$ be an irreducible representation. According to ([SF-88], Theorem V.2.5) there is a linear form $\mu \in G^*$ with

$$\rho(g)^p - \rho(g^{[p]}) = \mu(g)^p \,\mathrm{id}_M.$$

It is useful in this context to introduce the notion of a μ-reduced universal enveloping algebra $u(G, \mu)$. According to ([SF-88], Chapter V.3) ρ extends uniquely to an associative representation $u(G, \mu) \longrightarrow \mathrm{End}\, M$. If, in addition, $G_{(0)}$ denotes a restricted subalgebra of G and M_0 is a $G_{(0)}$-submodule of M, then M is a homomorphic image of the *induced module* $u(G, \mu) \otimes_{u(G_{(0)}, \mu|_{G(0)})} M_0$ ([SF-88], Theorem V.6.3).

In many cases it is much more convenient to determine the modules for restricted algebras rather than for arbitrary ones. We will here proceed in this way. Then there are two steps to make: first we determine some of the modules for a p-envelope (Proposition 2.1 explains that there is some degree of freedom) and then interpret this information on the G_p-module M as giving information on the G-module structure of M.

PROPOSITION 2.2. *Let* $H = H\big(2; \mathbf{1}; \Phi(\tau)\big)^{(1)}$ *and* $L = \mathrm{Der}\, H$ *denote the algebras described in* § 1. $\rho\colon H \longrightarrow \mathrm{gl}(M)$ *denotes an irreducible representation.*

1) *There is an extension of* ρ *to a representation* $\rho'\colon L \longrightarrow \mathrm{gl}(M)$ *with character* μ, *such that* $\mu(x) = \mu(y) = 0$.

2) *The unique eigenvalue of* $\rho'(x^a y^b)$ *is*

$$\begin{array}{ll} \mu(x^a y^b) & \text{if } 2 \leq a + b \leq 2p - 3,\ (a, b) \neq (1, 1), \\ \mu(\Gamma)^{1/p} & \text{if } a = 1,\ b = 0, \\ \mu(\Theta)^{1/p} & \text{if } a = 0,\ b = 1. \end{array}$$

The unique eigenvalue of $\rho'(\Lambda)$ *is given by* $\mu(\Lambda)$.

PROOF. 1) Let ρ' be any extension of ρ and μ' the associated character. Define the linear form λ on L by $\lambda(H) = 0$, $\lambda(\Gamma) = \mu'(x)^p$, $\lambda(\Theta) = \mu'(y)^p$. According to Proposition 2.1 $\chi := \rho' + \lambda\,\mathrm{id}_M$ is also an extension of ρ. The character μ of χ is given by

$$\mu(g)^p \,\mathrm{id}_M = \chi(g)^p - \chi(g^{[p]}) = \{\mu'(g)^p + \lambda(g)^p - \lambda(g^{[p]})\}\,\mathrm{id}_M.$$

Observing that $x^{[p]} = \Gamma$, $y^{[p]} = \Theta$, $\lambda(x) = \lambda(y) = 0$, this gives the result.

2) For $2p - 3 \geq a + b \geq 2$, $(a, b) \neq (1, 1)$ we have $(x^a y^b)^{[p]} = 0$. Thus $\rho'(x^a y^b)^p = \mu(x^a y^b)^p \,\mathrm{id}_M$ and therefore $\{\rho'(x^a y^b) - \mu(x^a y^b)\,\mathrm{id}_M\}^p = 0$. For $(a, b) = (1, 0)$ (and similarly for $(a, b) = (0, 1)$) we obtain

$$\{\rho'(x) - \mu(\Gamma)^{1/p}\,\mathrm{id}_M\}^{p^2} = \{\rho'(x)^p - \mu(\Gamma)\,\mathrm{id}_M\}^p = \rho'(\Gamma)^p - \mu(\Gamma)^p\,\mathrm{id}_M = 0.$$

Observing that $\Lambda^{[p]} = 0$ we get the final assertion. ■

Using information on the eigenvalues of a representation one gets a lower bound for the dimension of the representation by the following result.

THEOREM 2.3 [ST-77]. *Let h, k be subalgebras of a Lie algebra G over an algebraically closed field K of characteristic $p > 0$, $G = h + \sum_{1 \le i \le n} Ke_i$. Let M_G and M_h be finite dimensional G-, respectively h-modules, M_h h-irreducible, and $M_h \subset M_G$. Let T denote the corresponding representation. Assume*

(1) k is an ideal of h

(2) there exist $f_1, \ldots, f_m \in k$ $(m \le n)$ so that $T([e_i, f_j])$ is nilpotent if $i \ne j$ and invertible if $i = j$,

(3) $[e_i, f_j]$, $\big[[e_i, f_j], e_l\big] \in h$ for all i, j, l,

(4) $k + \sum_{i,j} K[e_i, f_j]$ generates a Lie subalgebra R such that $T(g)$ is nilpotent for all $g \in [h, k] + [R, R]$.

Then $\dim M_G \ge p^m \dim M_h$. ■

This theorem is in particular useful if h defines a suitable filtration on G.

THEOREM 2.4. *Assume that $G = G_{(-1)} \supset \cdots \supset G_{(s)} = 0$ $(s > 1)$ is a filtered Lie algebra, $\rho\colon G \longrightarrow \mathrm{gl}(M)$ is a representation, and $\lambda \in G^*$, such that $\rho(g) - \lambda(g)\,\mathrm{id}_M$ is nilpotent for all $g \in G_{(1)}$.*

1) Let $M_{(0)}$ denote an irreducible $G_{(0)}$-submodule. If

(i) there is $l \ge 2$, such that $[G_{(0)}, G_{(l)}] + [G_{(l-1)}, G_{(l-1)}] \subset \ker\lambda$,

(ii) there are $f_1, \ldots, f_m \in G_{(l)}$ and $e_1, \ldots, e_m \in G$ with $\lambda([e_i, f_j]) = 0$ for all $1 \le i < j \le m$, and $\lambda([e_i, f_i]) \ne 0$, $(1 \le i \le m)$, then

$$\dim M \ge p^m \dim M_{(0)}.$$

2) Let $M_{(1)}$ denote an irreducible $G_{(1)}$-submodule. If $G = G_{(0)}$ and

(i) there is $l \ge 1$, such that $[G_{(1)}, G_{(l)}] \subset \ker\lambda$,

(ii) there are $f_1, \ldots, f_m \in G_{(l)}$ and $e_1, \ldots, e_m \in G$ with $\lambda([e_i, f_j]) = 0$ for all $1 \le i < j \le m$, and $\lambda([e_i, f_i]) \ne 0$, $(1 \le i \le m)$ then

$$\dim M \ge p^m \dim M_{(1)}.$$

PROOF. 1) In Theorem 2.3 put $h := G_{(0)}$, $k := G_{(l)}$. We check the assumptions. (1) Clearly, k is an ideal of h. (2) $[e_i, f_j] \in G_{(1)}$ is nilpotent or invertible, depending on whether the eigenvalue $\lambda([e_i, f_j])$ vanishes or not. Since the matrix $\lambda([e_i, f_j])_{1 \le i,j \le m}$ is a triangular matrix, a suitable substitution of the f_j turns it into a diagonal matrix. (3) follows from the property of a filtration and the fact, that $l \ge 2$. (4) $[h, k] + [R, R] \subset [G_{(0)}, G_{(l)}] + [G_{(l-1)}, G_{(l-1)}] \subset \ker\lambda$.

2) In Theorem 2.3 put $h := G_{(1)}$, $k := G_{(l)}$ and argue as in 1). ■

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3. **Commutation rules.** For future reference we derive some rules for how elements in an associative algebra commute. Let A be an associative algebra and suppose that $z, x_1, \ldots, x_n$ are elements of A. We use the multi-index notation $x^t := x_1^{t_1} \cdots x_n^{t_n}$, $s < t$ if and only if $s_i < t_i$ for all i, $\binom{t}{s} = \prod \binom{t_i}{s_i}$. We set for $t = (t_1, \ldots, t_n)$

$$\{z, x; t\} := [\ldots [z, \underbrace{x_1] \ldots, x_1]}_{t_1 \text{ times}}, \underbrace{x_2], \ldots, x_2]}_{t_2 \text{ times}}, \underbrace{\ldots x_n], x_n]}_{t_n \text{ times}}.$$

if $t > 0$ and $\{z, x; 0\} = z$. We quote

LEMMA 3.1 ([SF-88],(V.7.1)). *Let $z, x_1, \ldots, x_n$ be elements of an associative algebra A. Then*

$$zx^s = \sum_{0 \le t \le s} \binom{t}{s} x^{s-t} \{z, x; t\}. \qquad \blacksquare$$

Let H and L be the algebras defined in §1 and $\rho: L \longrightarrow \mathrm{gl}\, M$ a representation. Lemma 3.1 will be employed for $A = \mathrm{End}\, M$.

LEMMA 3.2. *Assume that $0 \le a, b, i, j \le p - 1$, $0 < a + b$, and put*

$$g := (-1)^{i+j} (\mathrm{ad}\, y)^j (\mathrm{ad}\, x)^i (x^a y^b) = \{x^a y^b, (x, y); (i, j)\}.$$

We obtain for g

a) $b > i \ge 0$:

$$g = \begin{cases} 0 & a < j \\ (-1)^i [a!/(a-j)!][b!/(b-i)!] x^{a-j} y^{b-i} & a \ge j \end{cases}$$

b) $b = i$:

$$g = \begin{cases} (-1)^{b+1+j-a} a!\, b! (j-a)!\, x^{p-1+a-j} y^{p-1} & a < j \\ (-1)^b a! b! \Lambda & a = j \\ (-1)^b [a!/(a-j)!] b! x^{a-j} & a > j \end{cases}$$

c) $0 \le b < i$:

$$g = \begin{cases} (-1)^{b+1+j} b!\, j! (i-b)!\, x^{p-1-j} y^{p-1+b-i} & a = 0 \\ 0 & a > 0. \end{cases}$$

PROOF. a) $i = 0$: Since $b > 0$ we have $\{x^a y^b, y\} = a x^{a-1} y^b$ and in general $g = a! \{\ldots \{y^b, y\}, \ldots, y\} = 0$ for $a < j$, $g = a \ldots (a + 1 - j) x^{a-j} y^b$ for $a \ge j$. We now proceed by induction on i. For $i > 0$ one obtains in case a) $b > 1$ and then

$$g = -\{\ldots \{x, x^a y^b\}, x\} \ldots, x\}, y\} \ldots, y\} = -b \{\ldots \{x^a y^{b-1}, x\} \ldots, x\}, y\} \ldots, y\}.$$

The induction hypotheses yields the result.

b) If $a > 0$, induction on b yields directly

$$g = (-1)^b b! \{\dots\{x^a, \underbrace{y\}, \dots, y\}}_{j \text{ times}},$$

while for $a = 0$ (observe that in this case $b = i \neq 0$)

$$g = (-1)^b b! \{\dots\{\Lambda, \underbrace{y\}, \dots, y\}}_{j \text{ times}}.$$

For $a > j$ only the first case occurs, which yields the result immediately. Assume that $a \leq j$. We obtain in both the cases $a > 0$ and $a = 0$ (as $\Lambda = \{x, y\}$)

$$g = (-1)^b b!\, a! \{\dots\{x, \underbrace{y\}, \dots, y\}}_{(j-a+1) \text{ times}}\big).$$

This is the result for $a = j$. In case $a < j$

$$\begin{aligned} g &= (-1)^b a!\, b! \{\dots\{\Lambda, \underbrace{y\} \dots, y\}}_{(j-a) \text{ times}} \\ &= (-1)^b a!\, b! \big[(p-1)\dots\big(p-1-(j-a)+1\big)\big](-x^{p-1-(j-a)}y^{p-1}) \\ &= (-1)^{b+1+(j-a)} a!\, b!(j-a)! x^{p-1+a-j} y^{p-1}. \end{aligned}$$

c) Consider the case $a \neq 0$. Then $(\operatorname{ad} x)^i(x^a y^b) = b!(\operatorname{ad} x)^{i-b}(x^a) = 0$. So assume that $a = 0$. Then $b \neq 0$ and $p - 1 - (i - b) \neq 0$.

$$\begin{aligned} g &= (-1)^{b-1} b! \{\dots\{y, \underbrace{x\}, \dots x}_{(i-b+1) \text{ times}} \}, \underbrace{y\}, \dots, y\}}_{j \text{ times}} \\ &= (-1)^b b! \{\dots\{\Lambda, \underbrace{x\}, \dots, x\}}_{(i-b) \text{ times}}, \underbrace{y\}, \dots, y\}}_{j \text{ times}} \\ &= (-1)^i b! \big[(p-1)\dots\big(p-1-(i-b)+1\big)\big] \\ &\qquad \big[(p-1)\dots(p-1-j+1)\big](-x^{p-1-j}y^{p-1-(i-b)}). \end{aligned}$$

■

PROPOSITION 3.3. *Let $\rho: L \longrightarrow \mathrm{gl}(M)$ be a representation and assume that there is $u \in M$ with $\rho(x^i y^j)u = 0$ for all i, j with $3 \leq i + j \leq 2p - 3$.*

Put for $0 \leq r \leq b \leq p - 1, 0 \leq s \leq a \leq p - 1, 3 \leq a + b \leq 2p - 3$

$$g := \rho(x^a y^b)\rho(x)^r \rho(y)^s u.$$

We obtain for g

a) $a \geq s + 3$:	$g = 0$
b) $a + b \geq r + s + 3$:	$g = 0$
c) $a = s,\ b = r + 2$:	$g = (-1)^b a!, b!(1/2)\rho(y^2)u$

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d) $a = s + 2$, $b = r$: $\qquad g = (-1)^b a!\, b!(1/2)\rho(x^2)u.$

PROOF. 1) According to Lemma 3.1 we have

$$g = \sum_{\substack{0 \le i \le r \\ 0 \le j \le s}} \binom{r}{i}\binom{s}{j} \rho(x)^{r-i}\rho(y)^{s-j}\rho(\{\ldots\{x^a y^b, x\}, \ldots, x\}, y\} \ldots, y\})u.$$

The assumption on u yields that all summands with $i = j = 0$ or $(a-j)+(b-i) \ge 3$ vanish. Thus only the following pairs (i,j) yield a contribution to the sum:

$$(i,j) \in \{(b,a), (b,a-1), (b,a-2), (b-1,a), (b-1,a-1), (b-2,a)\} =: \mathcal{S}.$$

The condition $i \le r, j \le s$ yields in the respective cases

a) $j + 3 \le a : \mathcal{S} = \emptyset$,

b) $i + j + 3 \le a + b : \mathcal{S} = \emptyset$,

c) $i \le b - 2 : \mathcal{S} = \{(b-2,a)\} = \{(r,s)\}$,

d) $j \le a - 2 : \mathcal{S} = \{(b,a-2)\} = \{(r,s)\}$.

The computation of the respective summands is done by Lemma 3.2 (with the particular side condition, that $b = i+2$, $a = j$ in case c) and $b = i$, $a = j+2$ in case d)). ∎

PROPOSITION 3.4. *Let $\rho: L \longrightarrow \mathrm{gl}(M)$ be a representation and assume that there is $u \in M$ with $\rho(x^i y^j)u = 0$ for all i,j with $2 \le i+j \le 2p-3$, $\rho(\Lambda)u = 0$.*

1)

$$\rho(\Lambda)\rho(x)^{p-1}\rho(y)^{p-2}u = -\rho(x)u$$
$$\rho(\Lambda)\rho(x)^{p-2}\rho(y)^{p-1}u = \rho(y)u$$
$$\rho(\Lambda)\rho(x)^{p-1}\rho(y)^{p-1}u = 0.$$

2)

$$\rho(xy^2)\rho(x)^r\rho(y)^s u = r(-2s + r - 1)\rho(x)^{r-1}\rho(y)^s u$$
$$\rho(x^2y)\rho(x)^r\rho(y)^s u = s(-2r + s - 1)\rho(x)^r\rho(y)^{s-1}u.$$

3) Assume in addition, that $\left(\rho(x)^i\rho(y)ju\right)_{0 \le i,j \le p-1}$ is a basis of M. If $g \in M$ satisfies $\rho(xy^2)g = 0 = \rho(x^2y)g$, then $g \in Fu + F\rho(x)u + F\rho(y)u + F\rho(x)^{p-1}\rho(y)^{p-1}u$.

PROOF. 1) a) Applying Lemma 3.1 we obtain

$$\rho(\Lambda)\rho(x)^{p-1}\rho(y)^{p-2}u =$$
$$\sum \binom{p-1}{i}\binom{p-2}{j}\rho(x)^{p-1-i}\rho(y)^{p-2-j}\rho(\{\ldots\{1 - x^{p-1}y^{p-1}, x\} \ldots, x\}, y, \ldots y\})u.$$

The only nonvanishing summand occurs for the tuple $(i,j) = (p-1, p-2)$, which then gives the result. Similar reasoning works for the second case of 1). In the third case we obtain

$$g := \rho(\Lambda)\rho(x)^{p-1}\rho(y)^{p-1}u$$
$$= \sum \binom{p-1}{i}\binom{p-1}{j}\rho(x)^{p-1-i}\rho(y)^{p-1-j}\rho(\{\ldots\{1 - x^{p-1}y^{p-1}, x\} \ldots, x\}, y, \ldots y\})u.$$

The only nonvanishing summands occur for the tuples $(i,j) \in \{(p-1,p-2), (p-2,p-1)\}$. We obtain

$$\begin{aligned} g &= -\rho(y)[-(p-1)!(p-1)!]\rho(x)u + (p-1)\rho(x)[-(-1)^{p-2}(p-1)!(p-1)!]\rho(y)u \\ &= -\rho(\Lambda)u = 0. \end{aligned}$$

2)
$$\begin{aligned} g &:= \rho(xy^2)\rho(x)^r\rho(y)^s u \\ &= \sum_{\substack{0\le i\le r\\ 0\le j\le s}} \binom{r}{i}\binom{s}{j}\rho(x)^{r-i}\rho(y)^{s-j}\rho(\{\ldots\{xy^2,x\}\ldots,x\},y,\ldots y\})u. \end{aligned}$$

The only nonvanishing summands occur for the tuples $(i,j) \in \{(1,1), (2,0)\}$. Put $a := 1$, $b := 2$ in Lemma 3.2 to obtain

$$g = [-2rs + r(r-1)]\rho(x)^{r-1}\rho(y)^s u.$$

We proceed similarly with $\rho(x^2y)\rho(x)^r\rho(y)^s u$.

3) Write $g = \sum \alpha_{rs}\rho(x)^r\rho(y)^s u$. The assumptions yield that

$$\alpha_{rs}r(-2s+r-1) = 0, \quad \alpha_{rs}s(-2r+s-1) = 0 \text{ for all } r, s.$$

Assume that $\alpha_{rs} \neq 0$ for some $(r,s) \neq (0,0)$. If $s = 0$, the first equation yields $r = 1$. Similarly, the assumption $s \neq 0$, $r = 0$ yields $s = 1$. Assume that $r \neq 0$, $s \neq 0$. Then $-r+2s = -1$, $2r-s = -1$. This system of linear equation has a unique solution, which then is $r = p-1$, $s = p-1$. ∎

PROPOSITION 3.5. *Let $\rho: L \longrightarrow \mathrm{gl}(M)$ be a representation and assume that there is $u \in M$ with $\rho(x^iy^j)u = 0$ for all i,j with $3 \le i+j \le 2p-3$, $\rho(y^2)u = 0$, $\rho(xy)u \subset Fu$, $\rho(\Lambda)u = 0$. Then for $1 \le r \le p-1$*

1) $\rho(y)^r\rho(x)u = \rho(x)\rho(y)^r u$

2) $\rho(x)^r\rho(y)u = \rho(y)\rho(x)^r u$

3) $\rho(y)\rho(x)^{p-1}\rho(y)^{p-1}u - \rho(x)^{p-1}\rho(y)^p u \subset F\rho(y)u$.

PROOF.

1)
$$\begin{aligned} \rho(x)\rho(y)^r u - \rho(y)^r\rho(x)u &= \sum_{1\le i\le r}\binom{r}{i}\rho(y)^{r-i}\rho(\{\ldots\{x,y\}\ldots,y\})u \\ &= \sum_{1\le i\le r}\binom{r}{i}\rho(y)^{r-i}\rho(\{\ldots\{\Lambda,\underbrace{y\}\ldots,y}_{i-1\text{ times}}\})u \\ &= 0. \end{aligned}$$

2) is similar to 1).

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3)
$$\begin{aligned}
&\rho(y)\rho(x)^{p-1}\rho(y)^{p-1}u-\rho(x)^{p-1}\rho(y)^{p}u\\
&\quad=\sum_{1\le i\le r}(-1)^{i}\rho(x)^{p-1-i}\rho(\{\ldots\{y,x\}\ \ldots,x\})\rho(y)^{p-1}u\\
&\quad\subset\sum_{1\le i,j\le p-1}F\rho(x)^{p-1-i}\rho(y)^{p-1-j}\rho(x^{p-1-j}y^{p-i})u\\
&\quad\subset F\rho(y)u.
\end{aligned}$$

PROPOSITION 3.6. *Let $\rho\colon L\longrightarrow \mathrm{gl}(M)$ denote a representation,* $1\le r\le p-1$.

1) $[\rho(\Theta),\rho(x)^r]=\sum_{1\le i\le r}-\binom{r}{i}(i-1)!\,\rho(x)^{r-i}\rho(y^{p-i})$
2) $[\rho(\Gamma),\rho(y)^r]=\sum_{1\le i\le r}(-1)^{i-1}\binom{r}{i}(i-1)!\,\rho(y)^{r-i}\rho(x^{p-i})$
3) $[\rho(y^2),\rho(x)^r]=-2r\rho(x)^{r-1}\rho(y)+r(r-1)\rho(x)^{r-2}\rho(\Lambda)$
$-2\sum_{3\le i\le r}\binom{r}{i}(i-2)!\,\rho(x)^{r-i}\rho(x^{p-1}y^{p+1-i})$
4) $[\rho(x^2),\rho(y)^r]=2r\rho(y)^{r-1}\rho(x)+r(r-1)\rho(y)^{r-2}\rho(\Lambda)$
$-2\sum_{3\le i\le r}\binom{r}{i}(-1)^{i}(i-2)!\,\rho(y)^{r-i}\rho(x^{p+1-i}y^{p-1})$
5) $[\rho(y^3),\rho(x)^r]=-3r\rho(x)^{r-1}\rho(y^2)+3r(r-1)\rho(x)^{r-2}\rho(y)$
$-r(r-1)(r-2)\rho(x)^{r-3}\rho(\Lambda)+6\sum_{4\le i\le r}\binom{r}{i}(i-3)!\,\rho(x)^{r-i}\rho(x^{p-1}y^{p+2-i})$
6) $[\rho(x^3),\rho(y)^r]=3r\rho(y)^{r-1}\rho(x^2)+3r(r-1)\rho(y)^{r-2}\rho(x)+r(r-1)(r-2)\rho(y)^{r-3}\rho(\Lambda)$
$+6\sum_{4\le i\le r}\binom{r}{i}(i-3)!\,\rho(y)^{r-i}\rho(x^{p+2-i}y^{p-1})$

PROOF. 1) We apply Lemma 3.2 putting $a=0$, $b=p-1$, $i=l-1$, $j=0$

$$\begin{aligned}
[\rho(\Theta),\rho(x)^r]&=\sum_{1\le l\le r}\binom{r}{l}\rho(x)^{r-l}\rho(\{\ldots\{\Theta,\underbrace{x\}\ \ldots,x\}}_{l\text{ times}})\\
&=\sum_{1\le l\le r}\binom{r}{l}\rho(x)^{r-l}\rho(\{\ldots\{-y^{p-1},\underbrace{x\}\ \ldots,x\}}_{l-1\text{ times}})\\
&=\sum_{1\le l\le r}-\binom{r}{l}(-1)^{l-1}[(p-1)!/(p-l)!]\rho(x)^{r-l}\rho(y^{p-l})\\
&=\sum_{1\le l\le r}-\binom{r}{l}(l-1)!\,\rho(x)^{r-l}\rho(y^{p-l}).
\end{aligned}$$

2) is done by similar computations putting $a=p-1$, $b=0$, $i=0$, $j=l-1$ in Lemma 3.2.

3)
$$\begin{aligned}
[\rho(y^2),\rho(x)^r]&=\sum_{1\le l\le r}\binom{r}{l}\rho(x)^{r-l}\rho\big(\{\ldots\{y^2,\underbrace{x\},\ldots,x\}}_{l\text{ times}})\big)\\
&=-2r\rho(x)^{r-1}\rho(y)+r(r-1)\rho(x)^{r-2}\rho(\Lambda)\\
&\quad+2\sum_{3\le l\le r}\binom{r}{l}\rho(x)^{r-l}\rho\big(\{\ldots\{1-x^{p-1}y^{p-1},\underbrace{x\},\ldots,x\}}_{l-2\text{ times}})\big)
\end{aligned}$$

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Apply Lemma 3.2 with $a = b = p - 1$, $i = l - 2$, $j = 0$ to obtain

$$= -2r\rho(x)^{r-1}\rho(y) + r(r-1)\rho(x)^{r-2}\rho(\Lambda)$$
$$+ 2\sum_{3\leq l\leq r}\binom{r}{l}(-1)^{l-2}[(p-1)!/(p+1-l)!]\rho(x)^{r-l}\rho(-x^{p-1}y^{p+1-l}).$$

This is the result.

4)

$$[\rho(x^2), \rho(y)^r] = \sum_{1\leq l\leq r}\binom{r}{l}\rho(y)^{r-l}\rho\big(\{\ldots\{x^2, \underbrace{y\}, \ldots, y\}}_{l\text{ times}}\big)$$
$$= 2r\rho(y)^{r-1}\rho(x) + r(r-1)\rho(y)^{r-2}\rho(\Lambda)$$
$$+ 2\sum_{3\leq l\leq r}\binom{r}{l}\rho(y)^{r-l}\rho\big(\{\ldots\{1 - x^{p-1}y^{p-1}, \underbrace{y\}, \ldots, y\}}_{l-2\text{ times}}\big)$$

Apply Lemma 3.2 with $a = b = p - 1$, $i = 0$, $j = l - 2$ to obtain

$$= 2r\rho(y)^{r-1}\rho(x) + r(r-1)\rho(y)^{r-2}\rho(\Lambda)$$
$$+ 2\sum_{3\leq l\leq r}\binom{r}{l}[(p-1)!/(p+1-l)!]\rho(y)^{r-l}\rho(-x^{p+1-l}y^{p-1}).$$

5), 6) are done analogously. ∎

4. **Irreducible H-modules of dimension $\leq p^2$.** *In this section we assume that the ground field F is algebraically closed and has characteristic $p > 5$.* Let $\rho: H \longrightarrow \mathrm{gl}(M)$ be an irreducible representation of dimension $\leq p^2$ (in particular $\rho(H)M \neq 0$). We extend ρ to a representation of $L = \mathrm{Der}\ H$ with a character μ according to Proposition 2.2 such that $\mu(x) = \mu(y) = 0$ and denote this representation again by ρ. The first step in the determination of ρ is the successive construction of a one-dimensional submodule for $L_{(0)}$.

LEMMA 4.1. *$\mu(x^a y^b) = 0$ for $3 \leq a + b \leq 2p - 3$.*

PROOF. Assume that the lemma is false. Then $\{\, i{+}j \mid 3 \leq i{+}j \leq 2p{-}3, \mu(x^i y^j) \neq 0\}$ is nonempty and hence has a maximum k. Put $b := \max\{j \mid j \leq k, \mu(x^{k-j}y^j) \neq 0\}$, $a := k - b$. This definition implies that, whenever $3 \leq i + j \leq 2p - 3$,

$$\mu(x^i y^j) = 0 \text{ if } (i + j > a + b) \text{ or } (i + j = a + b \text{ and } j > b),$$

and

$$\mu(x^a y^b) \neq 0.$$

Proposition 2.2. (2) yields that $\rho(x^a y^b)$ is invertible, but $\rho(x^i y^j)$ is nilpotent for $(i{+}j > a{+}b)$ or $(i{+}j = a{+}b$ and $j > b)$. Put in Theorem 2.4(1) $G = H$ with the filtration $\big(H_{(n)}\big)$

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defined in § 1, put $l = k-1 \geq 2$, and observe that $H_{(0)}^{(1)} = \mathrm{span}\{x^iy^j \mid 2 \leq i+j \leq 2p-3\}$ (Proposition 1.2.(2)).

a) Consider the case, that $a, b < p-1$. Put $e_1 := y, e_2 := x, f_1 := x^{a+1}y^b, f_2 := x^ay^{b+1}$. If M_0 denotes an irreducible $H_{(0)}$-submodule, then Theorem 2.4.(1) yields the estimate

$$\dim M \geq p^2 \dim M_0.$$

As $\dim M \leq p^2$, we obtain $\dim M_0 = 1$. In particular, $(H_{(0)})^{(1)}$ annihilates M_0, which implies that $\mu(H_{(1)} \cap (H_{(0)})^{(1)}) = 0$. This, however, contradicts our assumption, since $x^ay^b \in H_{(1)} \cap H_{(0)}^{(1)}$.

b) Consider the case $a = p-1$ (the case $b = p-1$ is similar). Then $b \leq 2p-3-a = p-2$. Let M_0 denote an irreducible $H_{(0)}$-submodule and put in Theorem 2.4.(1) $e := x$, $f := x^{p-1}y^{b+1}$, to obtain the upper bound

$$\dim M_0 \leq p.$$

We now apply Theorem 2.4.(2) with $G := H_{(0)}$, $e_1 := x^2$, $e_2 := xy$, $f_1 := x^{p-2}y^{b+1}$, $f_2 := x^{p-1}y^b$ and obtain the contradiction $\dim M_0 \geq p^2$. ∎

LEMMA 4.2. *Let M_0 be an irreducible $H_{(0)}$-submodule of M. Then*

1) $\rho(x^ay^b)m = 0$ for all $m \in M_0$, $3 \leq a+b \leq 2p-3$

2) $\rho(\Lambda)m = \mu(\Lambda)m$ for all $m \in M_0$

3) M_0 is an irreducible module for $Fx^2 + Fxy + Fy^2 \cong \mathrm{sl}(2)$.

PROOF. $J := \mathrm{span}\{x^ay^b \mid 3 \leq a+b \leq 2p-3\}$ is an ideal of $H_{(0)}$ with $H_{(1)} \cap (H_{(0)})^{(1)} = J$. Since $J \subset \ker\mu$ by (4.1), J consists of nilpotent transformations and hence acts nilpotently on M_0. The irreducibility of M_0 yields that $\rho(J)M_0 = 0$. As Λ centralizes $H_{(0)}$, $\rho(\Lambda)|M_0$ is contained in $F\,\mathrm{id}_{M_0}$, the scalar given by the unique eigenvalue $\mu(\Lambda)$. Then M_0 is an irreducible module for $Fx^2 + Fxy + Fy^2$. ∎

LEMMA 4.3. *1) Every irreducible $(H_{(0)}+F\Gamma)$-submodule M_0 of M is $H_{(0)}$-irreducible and satisfies $\rho(\Gamma)m = \mu(\Gamma)m$ for all $m \in M_0$.*

2) If $\mu(\Lambda) = 0$, then every irreducible $L_{(0)}$-submodule of M is $H_{(0)}$-irreducible and satisfies $\rho(\Gamma)m = \mu(\Gamma)m$, $\rho(\Theta)m = \mu(\Theta)m$ for all $m \in M_0$.

PROOF. 1) Put $G := H_{(0)}+F\Gamma$, $I := H_{(1)}+F\Gamma$. G is a subalgebra of L and I is an ideal of G. Note that $\{G, I\} \subset \mathrm{span}\{x^ay^b \mid 3 \leq a+b \leq 2p-3\} \subset I \cap \ker\mu$. Thus $\{G, I\}$ is an ideal of G consisting of nilpotent transformations. Let M_0 be an irreducible G-submodule of M. Then the ideal $\{G, I\}$ annihilates M_0, which in turn means, that $\rho(I)|M_0 \subset F\,\mathrm{id}_{M_0}$. In particular, the choice of the extension of ρ to L yields $\rho(\Gamma)|M_0 = \mu(\Gamma)\,\mathrm{id}_{M_0}$ and therefore M_0 is already irreducible as an $H_{(0)}$-module.

2) Put $G := L_{(0)}, I := L_{(1)}$. In the present case $\{L_{(0)}, L_{(1)}\} \subset H_{(1)} \subset \ker\mu$. Now proceed as in 1). ∎

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LEMMA 4.4. *1) Every irreducible $H_{(0)}$-submodule has dimension ≤ 2.*

2) $\mu(x^2) = 0$, $\mu(y^2) = 0$.

3) If every irreducible $(H_{(0)} + F\Gamma)$-submodule has dimension 2, then each such module is of the form $M_0 = Fu \oplus F\rho(x^2)u$, with

$$\rho(y^2)u = 0,\ \rho(x^2)^2u = 0,\ \rho(xy)u = u,\ 2\rho(x)u = \rho(y)\rho(x^2)u.$$

PROOF. 1), 2) Let M_0 denote an irreducible $H_{(0)}$-module. According to (4.2) it is an irreducible module for $Fx^2 + Fxy + Fy^2$. Take any vector $u \in M_0$ which is an eigenvector with respect to $\rho(xy)$. The family

$$\left(\rho(x)^i\rho(y)^ju\right)_{0\leq i,j\leq p-3} \cup \left(\rho(x)^i\rho(y)^j\rho(x^2)u\right)_{0\leq i,j\leq p-3}$$

has more than p^2 elements and is therefore linearly dependent. There is a relation with not all coefficients vanishing

$$\sum_{0\leq i,j\leq p-3} \alpha_{ij}\rho(x)^i\rho(y)^ju + \sum_{0\leq i,j\leq p-3} \beta_{ij}\rho(x)^i\rho(y)^j\rho(x^2)u = 0.$$

Put

$$k := \max\{\, i+j \mid 0 \leq i,j \leq p-3, \alpha_{ij} \neq 0 \text{ or } \beta_{ij} \neq 0\}$$
$$s := \max\{ j \mid \alpha_{k-j,j} \neq 0 \text{ or } \beta_{k-j,j} \neq 0\}, \quad r := k - s.$$

If $k = 0$ then u, $\rho(x^2)u$ are linearly dependent. Since they correspond to different eigenvalues with respect to $\rho(xy)$, this is only possible if $\rho(x^2)u = 0$.

If $k > 0$ then Proposition 3.3 yields

$$\begin{aligned} 0 &= \rho(x^{s+2}y^r)\Big(\sum_{0\leq i,j\leq p-3} \alpha_{ij}\rho(x)^i\rho(y)^ju + \sum_{0\leq i,j\leq p-3} \beta_{ij}\rho(x)^i\rho(y)^j\rho(x^2)u\Big) \\ &= (-1)^r(s+2)!\, r!(1/2)\alpha_{r,s}\rho(x^2)u + (-1)^r(s+2)!\, r!(1/2)\beta_{r,s}\rho(x^2)^2u. \end{aligned}$$

As $\rho(x^2)u$ and $\rho(x^2)^2u$ correspond to different eigenvalues with respect to $\rho(xy)$, we obtain $\alpha_{rs}\rho(x^2)u = \beta_{rs}\rho(x^2)^2u = 0$. By definition, one of the coefficients is nonzero, which is possible only if $\rho(x^2)^2u = 0$.

Thus M_0 is one-dimensional, or it is the two-dimensional irreducible sl(2)-module. In the latter case there is a basis $\big(u, \rho(x^2)u\big)$ such that

$$\rho(y^2)u = 0,\ \rho(x^2)^2u = 0,\ \rho(xy)u = u.$$

This proves 1). Since $\rho(x^2)$, $\rho(y^2)$ act nilpotently on M_0, the unique eigenvalue is $\mu(x^2) = 0$ and $\mu(y^2) = 0$, respectively. This proves 2).

3) Now assume that M_0 denotes an irreducible $(H_{(0)} + F\Gamma)$-module. According to Lemma 4.3 it is irreducible as a $H_{(0)}$-module and $\rho(\Gamma)m = \mu(\Gamma)m$ for all $m \in M_0$. Thus we can apply the results of 1) and obtain that either M_0 is one-dimensional (which is impossible

under the present assumption) or is two-dimensional with a basis of the above form. It remains to prove that $v := \rho(y)\rho(x^2)u - 2\rho(x)u$ vanishes.

Taking into account Lemma 4.2 we obtain

$$\begin{aligned}
\rho(x^3)v &= \rho(\{x^3, y\})\rho(x^2)u = 3\rho(x^2)^2u = 0,\\
\rho(x^2)v &= \rho(\{x^2, y\})\rho(x^2)u + \rho(y)\rho(x^2)^2u - 2\rho(x^2)\rho(x)u = 0,\\
\rho(y^2)v &= \rho(y)\rho(\{y^2, x^2\})u - 2\rho(\{y^2, x\})u = -4\rho(y)\rho(xy)u + 4\rho(y)u = 0.
\end{aligned}$$

Similarly, one computes $\rho(\Gamma)v = \mu(\Gamma)v$ and $\rho(\Lambda)v = \mu(\Lambda)v$. Proposition 1.3.(3) implies that Fv is an $(H_{(0)} + F\Gamma)$-submodule. Our present assumption yields $v = 0$. ∎

LEMMA 4.5. $\mu(\Lambda) = 0$.

PROOF. Let M_0 be a $(H_{(0)}+F\Gamma)$-submodule and choose $u \in M_0$, such that $\rho(y^2)u = 0$. Consider the family

$$\big(\rho(\Theta)^i\rho(y)^ju\big)_{0\le i,j\le p-3} \cup \big(\rho(\Theta)^i\rho(y)^j\rho(x)u\big)_{0\le i,j\le p-3}.$$

This family contains more than p^2 elements and therefore is linearly dependent. Thus there is a relation with not all coefficients vanishing

$$\sum_{0\le i,j\le p-3} \alpha_{ij}\rho(\Theta)^i\rho(y)^ju + \sum_{0\le i,j\le p-3} \beta_{ij}\rho(\Theta)^i\rho(y)^j\rho(x)u = 0.$$

If not all of the β_{ij} vanish, the application of $\rho(y^2)$ (in combination with $\rho(y^2)u = 0$) yields a nontrivial relation

$$-2\sum_{0\le i,j\le p-3} \beta_{ij}\rho(\Theta)^i\rho(y)^{j+1}u = 0.$$

Considering eigenvectors with respect to $\rho(xy)$ we see in either case, that there is $r \le p-2$ and a relation

$$\sum_{0\le i\le k} \gamma_i\rho(\Theta)^i\rho(y)^ru = 0, \quad k \le p-3,\ \gamma_k \ne 0.$$

Multiplying by $\rho(y)^{p-r}$ (if $r \ne 0$) we end up with a relation

$$\sum_{0\le i\le k} \delta_i\rho(\Theta)^iu = 0, \quad k \le p-2,\ \delta_k \ne 0.$$

Choose k minimal and assume that $\mu(\Lambda) \ne 0$. Then

$$\begin{aligned}
0 &= \big(\rho(\Gamma) - \mu(\Gamma)\,\mathrm{id}\big)\Big(\sum_{0\le i\le k} \delta_i\rho(\Theta)^iu\Big)\\
&= \sum_{0\le i\le k} \delta_i[\rho(\Gamma), \rho(\Theta)^i]u = \sum_{0\le i\le k} i\delta_i\rho(\Theta)^{i-1}\rho(-\Lambda)u\\
&= -\mu(\Lambda)\sum_{0\le i\le k} i\delta_i\rho(\Theta)^{i-1}u.
\end{aligned}$$

The minimality of k yields $k = 0$, and $u = 0$, a contradiction. ∎

We are now ready to prove the first main result, namely that $L_{(0)}$ has a one-dimensional submodule.

THEOREM 4.6. *M contains a one-dimensional $L_{(0)}$-submodule.*

The proof is done in several steps, deriving relations. Assume that the theorem is not true.

STEP 1. There exists an irreducible two-dimensional $L_{(0)}$-module $Fu \oplus F\rho(x^2)u$ with $\rho(y^2)u = 0$, $\rho(xy)u = u$, $\rho(x^2)^2u = 0$, $2\rho(x)u = \rho(y)\rho(x^2)u$.

PROOF. Every irreducible $L_{(0)}$-module is $H_{(0)}$-irreducible and $(H_{(0)}+F\Gamma)$-irreducible (Lemma 4.3) and hence has dimension at most two. Apply Lemma 4.4. ∎

STEP 2. $\rho(x)^{p-1}\rho(x^2)u = 0$, $\mu(\Gamma) = 0$, $\rho(x)^pu = 0$, $\rho(x)^{p-1}\rho(y)^{p-3}\rho(x)u = 0$.

PROOF.

$$\begin{aligned}
\rho(x^2)\rho(x)^{p-1}\rho(x^2)u &= \rho(x)^{p-1}\rho(x^2)\rho(x^2)u = 0,\\
\rho(x^3)\rho(x)^{p-1}\rho(x^2)u &= \rho(x)^{p-1}\rho(x^2)\rho(x^3)u = 0,\\
\rho(\Gamma)\rho(x)^{p-1}\rho(x^2)u &= \rho(x)^{p-1}\rho(x^2)\rho(\Gamma)u = \mu(\Gamma)\rho(x)^{p-1}\rho(x^2)u.
\end{aligned}$$

Application of (3.6) yields

$$\begin{aligned}
\rho(y^2)\rho(x)^{p-1}\rho(x^2)u &= \rho(x)^{p-1}\rho(x^2)\rho(y^2)u + [\rho(y^2), \rho(x)^{p-1}]\rho(x^2)u\\
&\quad + \rho(x)^{p-1}[\rho(y^2), \rho(x^2)]u\\
&= 2\rho(x)^{p-2}\rho(y)\rho(x^2)u + 2\rho(x)^{p-3}\rho(\Lambda)\rho(x^2)u\\
&\quad - 2\sum_{3\le i\le p-1}(-1)^i(i-2)!\,\rho(x)^{p-1-i}\rho(x^{p-1}y^{p+1-i})\rho(x^2)u\\
&\quad - 4\rho(x)^{p-1}\rho(xy)u\\
&= 2\rho(x)^{p-2}\rho(y)\rho(x^2)u - 4\rho(x)^{p-1}u = 0,\\
\rho(\Theta)\rho(x)^{p-1}\rho(x^2)u &= \rho(x)^{p-1}\rho(x^2)\rho(\Theta)u + [\rho(\Theta), \rho(x)^{p-1}]\rho(x^2)u\\
&\quad + \rho(x)^{p-1}[\rho(\Theta), \rho(x^2)]u\\
&= \mu(\Theta)\rho(x)^{p-1}\rho(x^2)u\\
&\quad - \sum_{1\le i\le p-1}(-1)^i(i-1)!\,\rho(x)^{p-1-i}\rho(y^{p-i})\rho(x^2)u\\
&\quad - 2\rho(x)^{p-1}\rho(xy^{p-1})u\\
&= \mu(\Theta)\rho(x)^{p-1}\rho(x^2)u - (p-2)!\rho(y)\rho(x^2)u\\
&\quad + (p-3)!\,\rho(x)\rho(y^2)\rho(x^2)u\\
&= \mu(\Theta)\rho(x)^{p-1}\rho(x^2)u + (p-3)!\Big(2\rho(y)\rho(x^2)u\\
&\quad + \rho(x)\big(-4\rho(xy)\big)\Big)u\\
&= \mu(\Theta)\rho(x)^{p-1}\rho(x^2)u.
\end{aligned}$$

According to (1.3) $F\rho(x)^{p-1}\rho(x^2)u$ is a $L_{(0)}$-module and hence has to vanish. Then

$$\mu(\Gamma)\rho(x^2)u = \rho(\Gamma)\rho(x^2)u = \rho(x)^p\rho(x^2)u = 0$$

and $\mu(\Gamma) = 0$. Consequently, $\rho(x)^p u = \rho(\Gamma)u = \mu(\Gamma)u = 0$. Finally we apply (3.5) to obtain

$$\rho(x)^{p-1}\rho(y)^{p-3}\rho(x)u = \rho(x)^p\rho(y)^{p-3}u = \rho(\Gamma)\rho(y)^{p-3}u = [\rho(\Gamma), \rho(y)^{p-3}]u = 0. \qquad \blacksquare$$

STEP 3. $\rho(y)^{p-1}u = 0$, $\mu(\Theta) = 0$.

PROOF. Similar to Step 2 we compute

$$\begin{aligned}
&\rho(y^2)\rho(y)^{p-1}u = 0, \rho(y^3)\rho(y)^{p-1}u = 0, \rho(\Theta)\rho(y)^{p-1}u = \mu(\Theta)\rho(y)^{p-1}u,\\
&\rho(x^2)\rho(y)^{p-1}u = \rho(y)^{p-1}\rho(x^2)u - 2\rho(y)^{p-2}\rho(x)u + 2\rho(y)^{p-3}\rho(\Lambda)u\\
&\qquad - 2\sum_{3\le i\le p-1}(i-2)!\,\rho(y)^{p-1-i}\rho(x^{p+1-i}y^{p-1})u\\
&\qquad = \rho(y)^{p-2}[\rho(y)\rho(x^2)u - 2\rho(x)u] = 0,\\
&\rho(\Gamma)\rho(y)^{p-1}u = \rho(y)^{p-1}\rho(\Gamma)u - \sum_{1\le i\le p-1}(i-1)!\,\rho(y)^{p-1-i}\rho(x^{p-i})u\\
&\qquad = -(p-3)!\,\rho(y)\rho(x^2)u - (p-2)!\,\rho(x)u = 0.
\end{aligned}$$

Hence $F\rho(y)^{p-1}u$ is a $L_{(0)}$-submodule. So we obtain $\rho(y)^{p-1}u = 0$ and

$$\mu(\Theta)u = \rho(\Theta)u = \rho(y)^p u = 0. \qquad \blacksquare$$

STEP 4. $\rho(x)^{p-1}\rho(y)^{p-2}u = 0$.

PROOF.

$$\begin{aligned}
\rho(y^2)\rho(x)^{p-1}\rho(y)^{p-2}u &= [\rho(y^2), \rho(x)^{p-1}]\rho(y)^{p-2}u\\
&= 2\rho(x)^{p-2}\rho(y)^{p-1}u + 2\rho(x)^{p-3}\rho(\Lambda)\rho(y)^{p-2}u\\
&\qquad - 2\sum_{3\le i\le p-1}(-1)^i(i-2)!\,\rho(x)^{p-1-i}\rho(x^{p-1}y^{p+1-i})u\\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\rho(y^3)\rho(x)^{p-1}\rho(y)^{p-2}u &= [\rho(y^3), \rho(x)^{p-1}]\rho(y)^{p-2}u\\
&= 3\rho(x)^{p-2}\rho(y^2)\rho(y)^{p-2}u\\
&\qquad + 6\rho(x)^{p-3}\rho(y)^{p-1}u + 6\rho(x)^{p-4}\rho(\Lambda)\rho(y)^{p-2}u\\
&\qquad + 6\sum_{4\le i\le p-1}(-1)^i(i-3)!\,\rho(x)^{p-1-i}\rho(x^{p-1}y^{p+2-i})\rho(y)^{p-2}u\\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\rho(\Theta)\rho(x)^{p-1}\rho(y)^{p-2}u &= [\rho(\Theta), \rho(x)^{p-1}]\rho(y)^{p-2}u\\
&= -(p-2)!\,\rho(y)^{p-1}u\\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\rho(x^2)\rho(x)^{p-1}\rho(y)^{p-2}u &= \rho(x)^{p-1}\rho(y)^{p-2}\rho(x^2)u + \rho(x)^{p-1}[\rho(x^2), \rho(y)^{p-2}]u\\
&= \rho(x)^{p-1}\rho(y)^{p-2}\rho(x^2)u - 4\rho(x)^{p-1}\rho(y)^{p-3}\rho(x)u\\
&= -2\rho(x)^{p-1}\rho(y)^{p-3}\rho(x)u = 0,
\end{aligned}$$

$$\rho(\Gamma)\rho(x)^{p-1}\rho(y)^{p-2}u = \rho(x)^{p-1}[\rho(\Gamma),\rho(y)^{p-2}]u + \mu(\Gamma)\rho(x)^{p-1}\rho(y)^{p-2}u$$
$$= \rho(x)^{p-1}\sum_{1\leq i\leq p-2}(-1)^{i-1}\binom{p-2}{i}(i-1)!\,\rho(y)^{p-2-i}\rho(x^{p-i})u$$
$$+\mu(\Gamma)\rho(x)^{p-1}\rho(y)^{p-2}u = (p-3)!\,\rho(x)^{p-1}\rho(x^2)u = 0. \quad \blacksquare$$

PROOF OF THEOREM 4.6. Under the assumption that M has no one-dimensional $L_{(0)}$-submodule, we derive under Step 4

$$0 = \rho(x)^p\rho(y)^{p-2}u = \rho(\Gamma)\rho(y)^{p-2}u$$
$$= \sum_{1\leq i\leq p-2}(-1)^{i-1}\binom{p-2}{i}(i-1)!\,\rho(y)^{p-2-i}\rho(x^{p-i})u = (p-3)!\,\rho(x^2)u.$$

Thus $\rho(x^2)u = 0$, contradicting Step 1. $\blacksquare$

PROPOSITION 4.7. *Let M be an irreducible H-module of dimension $\leq p^2$ and Fu a one-dimensional $L_{(0)}$-submodule.*

1) $\dim M \in \{1, p^2-1, p^2\}$

2) If $\dim M = p^2$, *then* $\{\rho(x)^i\rho(y)^ju \mid 0\leq i,j\leq p-1\}$ *is a basis of* M.

3) If $\dim M = p^2-1$, *then*

a) $\{\rho(x)^i\rho(y)^ju \mid 0\leq i,j\leq p-1, i+j\leq 2p-3\}$ *is a basis of* M

b) $\rho(x)^{p-1}\rho(y)^{p-1}u = -u$

c) $\mu(\Gamma) = \mu(\Theta) = 0$.

PROOF. 1) Assume that the family $(\rho(x)^i\rho(y)^ju)_{0\leq i,j\leq p-1, i+j\leq 2p-3}$ is linearly dependent. By multiplication (if necessary) we obtain a relation

$$\sum \alpha_{ij}\rho(x)^i\rho(y)^ju = 0, \quad \alpha_{p-1,p-2}\neq 0 \text{ or } \alpha_{p-2,p-1}\neq 0.$$

Then Proposition 3.4 yields

$$0 = \rho(\Lambda)\sum\alpha_{ij}\rho(x)^i\rho(y)^ju = -\alpha_{p-1,p-2}\rho(x)u + \alpha_{p-2,p-1}\rho(y)u.$$

Application of $\rho(x^2)$ and $\rho(y^2)$ yields $\rho(x)u = \rho(y)u = 0$. Then Fu is a trivial H-module. Thus if $\dim M \notin \{1,p^2\}$, then this family is a basis of M. This proves 1).

2) follows from the fact that $\sum_{0\leq i,j\leq p-1} F\rho(x)^i\rho(y)^ju$ is an H-submodule.

3) The proof of 1) shows that in the present situation the family $\left(\rho(x)^i\rho(y)^ju\right)_{\substack{0\leq i,j\leq p-1\\ i+j\leq 2p-3}}$ is a basis of M. Thus

$$\rho(x)^{p-1}\rho(y)^{p-1}u = \sum_{0\leq i+j\leq 2p-3}\alpha_{ij}\rho(x)^i\rho(y)^ju.$$

Considering eigenvectors with respect to $\rho(xy)$, we may assume that $\alpha_{ij} = 0$ for $i\neq j$. Thus

$$\rho(x)^{p-1}\rho(y)^{p-1}u = \sum_{0\leq i\leq p-2}\alpha_{ii}\rho(x)^i\rho(y)^iu.$$

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Then

$$\sum_{0\leq i\leq p-2} \alpha_{ii}\rho(x)^{i+1}\rho(y)^i u = \rho(x)^p\rho(y)^{p-1}u = \rho(\Gamma)\rho(y)^{p-1}u$$
$$= [\rho(\Gamma), \rho(y)^{p-1}]u + \rho(y)^{p-1}\rho(\Gamma)u$$
$$= -\sum_{1\leq l\leq p-1} (l-1)!\,\rho(y)^{p-1-l}\rho(x^{p-l})u + \mu(\Gamma)\rho(y)^{p-1}u$$
$$= -(p-2)!\,\rho(x)u + \mu(\Gamma)\rho(y)^{p-1}u.$$

This shows that $\alpha_{ii} = 0$ for $i \neq 0$, $\alpha_{00} = -(p-2)! = -1$, $\mu(\Gamma) = 0$. (3.5(3)) yields

$$\mu(\Theta)\rho(x)^{p-1}u = \rho(x)^{p-1}\rho(y)^p u \in \rho(y)\rho(x)^{p-1}\rho(y)^{p-1}u + F\rho(y)u = F\rho(y)u.$$

This is only possible if $\mu(\Theta) = 0$. ■

COROLLARY 4.8. *Let Fu be the trivial $L_{(0)}$-module. The L-module induced by Fu with character $\mu = 0$ has exactly two proper submodules, namely the trivial H-module $M_1 := F\big(\rho(x)^{p-1}\rho(y)^{p-1} \otimes u + 1 \otimes u\big)$ and $M_2 := \sum_{0<i+j} F\rho(x)^i\rho(y)^j \otimes u$, which is isomorphic to H as an L-module. Moreover, it decomposes*

$$u(L,0) \otimes_{u(L_{(0)},0)} Fu = M_1 \oplus M_2.$$

PROOF. Put $M := u(L,0) \otimes_{u(L_{(0)},0)} Fu$, and let $U \neq 0$ be a submodule. If there is a proper submodule with $\dim U \notin \{1, p^2-1\}$ then M/U and U have only trivial factors in an H-composition series. As $H^{(n)} = H$ for all n, H would annihilate M, which is impossible. The same reasoning shows that every one-dimensional submodule would be maximal. Assume that U is a one-dimensional submodule, and $\psi: M \longrightarrow M/U$ the module homomorphism onto the irreducible module M/U of dimension p^2-1. The former proposition shows that $\psi\left(\rho(x)^{p-1}\rho(y)^{p-1}\otimes u\right) = \psi(-1\otimes u)$. Thus $M_1 = \ker\psi = U$.

H, considered as an L-module, has the trivial $L_{(0)}$-submodule $F\Lambda$ and it is a restricted module. Thus there is a module homomorphism $\psi: M \longrightarrow H$, with $\psi(1\otimes u) = \Lambda$. Therefore M has a one-dimensional submodule.

Since the character of M is $\mu = 0$, M_2 is a submodule of M of dimension p^2-1, necessarily irreducible. Let U be any submodule of dimension p^2-1. Then $\dim M/U = 1$, hence $\rho(H)M \subset U$. In particular, $M_2 \subset U$.

Thus M_1, M_2 are the unique submodules. Clearly $M_1 + M_2 = M$, and a dimension argument yields that $M_1 \cap M_2 = 0$. ■

The truncated polynomial ring $A(2;\mathbf{1})$ is a realization of this split induced module.

We are now able to determine the irreducible modules completely.

THEOREM 4.9. *Let M be an irreducible H-module of dimension $\leq p^2$ and Fu a one-dimensional $L_{(0)}$-submodule. Then M is one of the following*

1) $M \cong H$, the isomorphism is given by $\psi(u) = \Lambda$,

$$\psi\left(\rho(x)^i\rho(y)^j u\right) = (-1)^{i+1} i!\,j!\,x^{p-1-j}y^{p-1-i} \quad (1 \leq i+j \leq 2p-3)$$

2) $M \cong u(L,\mu) \otimes_{u(L_{(0)},\mu|L_{(0)})} Fu$, with some linear form $\mu \neq 0$.

PROOF. M has a character μ. Then M is the homomorphic image of the induced module $u(L,\mu) \otimes_{u(L_{(0)},\mu|L_{(0)})} Fu$. If $\dim M = p^2 - 1$, Proposition 4.7 shows that $\mu = 0$. Then (4.8) yields that $M \cong H$. The isomorphism maps u onto Λ, which implies the asserted equation.

Assume that $\dim M = p^2$. Then M is isomorphic to $u(L,\mu) \otimes_{u(L_{(0)},\mu|L_{(0)})} Fu$. The irreducibility of M in combination with (4.8) proves that $\mu \neq 0$. ∎

REMARK. Let M be an H-module of dimension $< p^2$. If M is not irreducible, then any factor of a composition series has dimension $< p^2 - 1$. Proposition 4.7 proves that every such factor has dimension 1. As $H^{(1)} = H$, M then is a trivial module. Thus every H-module of dimension $< p^2$ is either trivial or isomorphic to H.

In particular, the dual space H^* is isomorphic to H as an H-module. This result is also a natural consequence of Theorem 6.1 below.

5. **Reducible modules of dimension p^2.** Having determined the irreducible modules of dimension $\leq p^2$, we now derive a complete list of the nonirreducible ones. *We presuppose the same assumptions as in* § 4.

THEOREM 5.1. *Let M be a reducible H-module of dimension $\leq p^2$. Then M is one of the following*

1) *M is trivial, $\rho(H)M = 0$*
2) *$M \cong H \oplus Fu_0$ is the direct sum of an irreducible (p^2-1)-dimensional module and a one-dimensional trivial one*
3) *$M \cong H \oplus F(\alpha\Gamma + \beta\Theta)$ is indecomposible, nonirreducible*
4) *The dual M^* of M is of type 3).*

PROOF. Let $M = M_1 \supset \cdots \supset M_{t+1} = 0$ be a composition series. According to our assumption $t > 1$ holds.

a) If every factor of this composition series is trivial, then (as $H^{(n)} = H$ for all $n \geq 1$) $M$is trivial. If not all factors are trivial, then $t = 2$ and either M_1/M_2 is (p^2-1)-dimensional irreducible, M_2 one-dimensional or M_1/M_2 is one-dimensional and M_2 is (p^2-1)-dimensional irreducible. If the module splits, we are in case 2).

b) Consider the nonsplit case with M_2 (p^2-1)-dimensional. According to Theorem 4.9 M_2 is module-isomorphic to H. Choose $u_0 \notin M_2$ an eigenvector with respect to $\rho(xy)$, necessarily of eigenvalue 0. Since M/M_2 is trivial, $\rho(y)u_0 \in M_2$ and

$$\rho(y)u_0 = \sum_{0 \leq a \leq p-2} \alpha_a x^a y^{a+1} + \alpha x^{p-1}.$$

Put $u_1 := u_0 + \sum_{0 \leq a \leq p-3} \alpha_a (a+1)^{-1} x^{a+1} y^{a+1} + \alpha_{p-2}\Lambda$. Then $\rho(y)u_1 = \alpha x^{p-1}$. Write $\rho(x)u_1 := \sum_{0 \leq a \leq p-2} \beta_a x^{a+1} y^a + \beta y^{p-1}$. $\rho(\Gamma) - \rho(x)^p$ and $\rho(\Theta) - \rho(y)^p$ commute with

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$\rho(x)$, $\rho(y)$. They vanish on $\rho(x)u_1$, $\rho(y)u_1$. Thus $\rho(x)$, $\rho(y)$ vanish on $\big(\rho(\Gamma)-\rho(x)^p\big)u_1$, $\big(\rho(\Gamma)-\rho(x)^p\big)u_1 \in H$. This is only true if $\rho(\Gamma)u_1 = \rho(x)^p u_1$, $\rho(\Theta)u_1 = \rho(y)^p u_1$. Then

$$\rho(\Theta)u_1 = \rho(y)^p u_1 = \rho(y)^{p-1}(\alpha x^{p-1}) = \alpha(\operatorname{ad} y)^{p-1}(x^{p-1}) = -\alpha\Lambda$$

$$\rho(\Gamma)u_1 = \rho(x)^p u_1 = (\operatorname{ad} x)^{p-1}\Big(\sum_{0\le a\le p-2} \beta_a x^{a+1}y^a + \beta y^{p-1}\Big) = -\beta\Lambda.$$

Therefore

$$-\rho(\Lambda)u_1 = \rho(\Gamma)\rho(\Theta)u_1 - \rho(\Theta)\rho(\Gamma)u_1 = -\alpha(\operatorname{ad} x)^p(\Lambda) - \beta(\operatorname{ad} y)^p(\Lambda) = 0,$$

$$\begin{aligned} 0 &= \rho(\Lambda)u_1 = \rho(x)\rho(y)u_1 - \rho(y)\rho(x)u_1 \\ &= 0 - \{y, \sum_{0\le a\le p-2} \beta_a x^{a+1}y^a + \beta y^{p-1}\} = \sum_{0\le a\le p-2} \beta_a(a+1)x^a y^a, \end{aligned}$$

hence $\beta_a = 0$ for all a and then $\rho(x)u_1 = \beta y^{p-1}$. The actions of x and y on M determine the module structure completely, as x and y generate H as an algebra. Thus given α, β the module is uniquely determined. Then it has to be the module $H \oplus F(-\alpha\Gamma + \beta\Theta)$.

c) Consider the case that M_2 is one-dimensional. The dual module has a submodule of codimension one and is therefore of the type we considered under b). ∎

As a consequence of Theorems 4.9 and 5.1 the character of a module carries the following important information:

COROLLARY 5.2. *Let M be a nontrivial H-module of dimension $\le p^2$.*

1) M can be turned into an L-module with a character $\mu \in L^*$, $\mu(H) = 0$.
2) The following are equivalent:
   a) $\mu \ne 0$
   b) x or y act nonnilpotently on M
   c) $\dim M = p^2$, M irreducible.
3) If $\mu \ne 0$, then $M \cong u(L,\mu) \otimes_{u(L_{(0)},\mu|L_{(0)})} Fu$.

PROOF. 1) The assertion is clear if M is irreducible. If M is reducible (and nontrivial) it is one of the modules described in Theorem 5.1(2)–(4). These are restricted L-modules and hence carry the character $\mu = 0$.

2) a)⇒b): As $\mu(H) = 0$, $\mu \ne 0$, we have $\mu(\Gamma) \ne 0$ or $\mu(\Theta) \ne 0$. Observe that $\rho(x)^{p^2} = \mu(\Gamma)^p \operatorname{id}$, $\rho(y)^{p^2} = \mu(\Theta)^p \operatorname{id}$.

b)⇒c): M is none of the modules described in Theorem 4.9.(1) or Theorem 5.1. Thus it is irreducible of type (4.9.(2)).

c)⇒a): As above, M has to be of type (4.9.(2)). Hence $\mu \ne 0$.

3) M has to be of type (4.9(2)). ∎

COROLLARY 5.3. *1) The irreducible H-modules of dimension $\le p^2$ are parametrized by $(L/H)^*$ via the character on L.*

2) Two modules $H \oplus F(\alpha\Gamma + \beta\Theta)$, $H \oplus F(\gamma\Gamma + \delta\Theta)$ are isomorphic if and only if $\alpha\delta = \beta\gamma$.

3) *The indecomposable, reducible, nontrivial H-modules of dimension p^2 are parametrized by the union of two projective lines.*

PROOF. 1) obvious.

2) Since the dimension of M is p^2, we have $(\alpha, \beta) \neq (0,0)$. If $\alpha\delta = \beta\gamma$, and say $\gamma \neq 0$, then $H \oplus F(\alpha\Gamma + \beta\Theta) \cong H \oplus F\gamma\big(\alpha\Gamma + (\alpha\delta/\gamma)\Theta\big)$.

Assume that $\psi : H \oplus F(\alpha\Gamma + \beta\Theta) \longrightarrow H \oplus F(\gamma\Gamma + \delta\Theta)$ is an isomorphism. As there is exactly one submodule of dimension $p^2 - 1$, namely $\rho(H)M \cong H$, ψ induces a module automorphism of H. $F\Lambda$ is uniquely determined as the one-dimensional $L_{(0)}$-submodule of H. Hence $\psi\,|H$ is just the multiplication with a nonzero scalar τ, given by $\psi(\Lambda) = \tau\Lambda$. Therefore $\psi(\{x, \alpha\Gamma + \beta\Theta\}) = \tau\{x, \alpha\Gamma + \beta\Theta\} = \beta\tau y^{p-1}$, $\psi(\{y, \alpha\Gamma + \beta\Theta\}) = \tau\{y, \alpha\Gamma + \beta\Theta\} = -\alpha\tau x^{p-1}$, proving that $\psi(\alpha\Gamma + \beta\Theta) = \tau(\alpha\Gamma + \beta\Theta)$. On the other hand, there is $\sigma \neq 0$ such that $\psi(\alpha\Gamma + \beta\Theta) - \sigma(\gamma\Gamma + \delta\Theta) \in H$. Thus $\tau\alpha = \sigma\gamma$, $\tau\beta = \sigma\delta$.

3) As is proved under 2), the modules of type 3) of (5.1) are parametrized by a projective line. So two projective lines are needed to parametrize the modules of types 3) and 4). ∎

The following corollary is needed for the solution of the classical case of the generalized Kostrikin-Shafarevic Conjecture ([St-2], Theorem 4.5). The quoted theorem gives a characterization of the simple classical Lie algebras over algebraically closed fields of characteristic $p > 7$ in terms of the one-sections with respect to an optimal torus in some p-envelope of those algebras.

COROLLARY 5.4. *Let M be a nontrivial H-module of dimension $\leq p^2$. Put $M_1 := \{m \in M | \rho(\Lambda)m = 0\}$.*

1) *M_1 is a $L_{(0)}$-submodule,* $\dim M/M_1 = 2$.

2) *The eigenvalues of xy on M/M_1 are ± 1, both with multiplicity 1.*

PROOF. We use the classification of the modules in question.

a) The result is clear for: $M \cong H$, as in that case $M_1 = H_{(0)}$, and for completely reducible modules $H \oplus F$. Thus only the irreducible and the indecomposible reducible modules of dimension p^2 remain to be considered.

b) If $M \cong H \oplus F(\alpha\Gamma + \beta\Theta)$, then $M_1 = H_{(0)} \oplus F(\alpha\Gamma + \beta\Theta)$. This proves the result in this case.

c) Let M be the dual of $U := H \oplus F(\alpha\Gamma + \beta\Theta)$. Note that $\{\Lambda, U\} = Fx^{p-1}y^{p-2} + Fx^{p-2}y^{p-1}$ is two-dimensional. As $f \in M_1$ if and only if $\Lambda.f = 0$, i.e. if and only if $f(\{\Lambda, U\}) = 0$, M_1 has codimension 2 in M, and M/M_1 is represented by $\{\Lambda, U\}^*$. Since xy has eigenvalues ± 1 on $\{\Lambda, U\}$, it has eigenvalues $-(\pm 1)$ on the dual space.

d) Let M be irreducible. Then there is a one-dimensional $L_{(0)}$-submodule Fu, and $\{\rho(x)^i\rho(y)^j u \mid 0 \leq i,j \leq p-1\}$ is a basis (Theorem 4.9). It is direct consequence of the commutation rules of § 3, that $\{\rho(x)^i\rho(y)^j u \mid 0 \leq i+j \leq 2p-4\} \subset M_1$. (3.4) proves that $\rho(\Lambda)\rho(x)^{p-1}\rho(y)^{p-1}u = 0$.

Clearly, $M \neq M_1$ as $\ker\rho = 0$. Then $1 \leq \dim M/M_1 \leq 2$ and the only possible eigenvalues of xy on M/M_1 are ± 1. Since M/M_1 is a module for $Fx^2 + Fxy + Fy^2$ ($\cong$ sl(2)) both eigenvalues have to occur, with multiplicity one. ∎

6. **Extensions.** *We presuppose the same assumptions as in* §4. The fact, that the H-modules H and H^* are isomorphic (which is established in Theorem 4.9), can be considered also a consequence of the following theorem. The result of this theorem has far reaching consequences in cohomology theory.

THEOREM 6.1 [BL-58, THEOREM 7]. *H carries a nondegenerate invariant bilinear form.*

One can define a nondegenerate invariant bilinear form on H in the following way. The multiplication $\{\,,\}$ on the truncated polynomial ring $A(2;\mathbf{1})$ turns this into a Lie algebra. The derived algebra is H which is an ideal of codimension 1. Define a linear form

$$\lambda : A(2;\mathbf{1}) \longrightarrow A(2;\mathbf{1})/H \cong F$$

and a bilinear form, also denoted by λ

$$\lambda : A(2;\mathbf{1}) \times A(2;\mathbf{1}) \longrightarrow A(2;\mathbf{1})/H \cong F$$
$$\lambda(x^a y^b, x^r y^s) := x^{a+r} y^{b+s} + H.$$

As $(ub - va)x^{a+u+r-1}y^{b+v+s-1}\Lambda + (us - vr)x^{a+u+r-1}y^{b+v+s-1}\Lambda = \{x^u y^v, x^{a+r}y^{b+s}\} \in H$, we obtain the invariance of λ

$$\lambda(\{x^u y^v, x^a y^b\}, x^r y^s) + \lambda(x^a y^b, \{x^u y^v, x^r y^s\}) = 0.$$

Consequently, as an H-module, H^* is canonically isomorphic to H. Let us look at this isomorphism in detail. It is given by $\psi : g \mapsto \lambda(g, ?)$ and H acts via $\big(h \cdot \psi(g)\big)(f) = -\psi(g)(\{h,f\}) = -\lambda(g, \{h,f\}) = \lambda(\{h,g\},f) = \psi(\{h,g\})(f)$. H has a basis $\{x^a y^b \mid 0 < a + b \leq 2p - 3\} \cup \{\Lambda\}$. Thus $x^{p-1}y^{p-1} = 1 - \Lambda \equiv 1 \bmod (H)$, and therefore the dual basis with respect to λ is given by

$$(x^a y^b)^* = x^{p-a-1}y^{p-b-1} \quad 0 < a + b \leq 2p - 3, \quad \Lambda^* = -\Lambda.$$

THEOREM 6.2. $\mathrm{Ext}^1_{U(H)}(H, F) \cong \mathrm{Ext}^1_{U(H)}(F, H) \cong L/H$. *These* Ext-*groups are two-dimensional.*

PROOF. The above remark proves the existence of isomorphisms of the Ext-groups $\mathrm{Ext}^1_{U(H)}(H, F) \cong \mathrm{Ext}^1_{U(H)}(H^*, F^*) \cong \mathrm{Ext}^1_{U(H)}(F, H)$. Since the first one of these describes the extensions of F by H, Theorem 5.1.(3) shows, that this is given by L/H. L/H is two-dimensional. ∎

We give bases of these three spaces and by this exhibit the isomorphisms. A basis of L/H is represented by the residue classes of Γ and Θ. The elements of $\mathrm{Ext}^1_{U(H)}(F, H) = H^1(H, H)$ are represented by the extensions $H_{\alpha\Gamma+\beta\Theta} := H \oplus F(\alpha\Gamma + \beta\Theta)$ (as they were described in Theorem 5.1). Γ and Θ determine outer derivations, i.e. cocycles in $C^1(H, H)$

$$f_\Gamma, f_\Theta : H \longrightarrow H, f_\Gamma(g) := \{g, \Gamma\}, f_\Theta(g) := \{g, \Theta\}.$$

The associated elements span $\mathrm{Ext}^1_{U(H)}(F,H)$. The corresponding basis of $\mathrm{Ext}^1_{U(H)}(H,F) \cong \mathrm{Ext}^1_{U(H)}(F,H^*) \cong H^1(H,H^*)$ is obtained by dualizing this. The isomorphism $H \longrightarrow H^*$ maps $f_\Gamma(g) \mapsto \lambda(f_\Gamma(g),?) \mapsto \lambda(\{g,\Gamma\},?)$.

We obtain outer derivations

$$f^\Gamma, f^\Theta \colon H \longrightarrow H^*, \big(f^\Gamma(g)\big)(h) := \lambda(\{g,\Gamma\},h), (f^\Theta(g)(h) := \lambda(\{g,\Theta\},h).$$

The extensions of H by a one-dimensional module are given by the dual spaces $(H_{\alpha\Gamma+\beta\Theta})^*$. These can then be described by

$$(H_{\alpha\Gamma+\beta\Theta})^* = H \oplus F^{\alpha\Gamma+\beta\Theta},$$
$$g\cdot(h+\delta 1) := \{g,h\} + f^{\alpha\Gamma+\beta\Theta}(g)(h) = \{g,h\} + \lambda(\{g,\alpha\Gamma+\beta\Theta\},h)1.$$

The determination of these extensions of the module H by a one-dimensional module also gives insight into the central extensions of the algebra H. We have to observe, that under this aspect one is no longer interested in the equivalence classes of extensions but in the algebra (!) isomorphism classes.

THEOREM 6.3. *Up to algebra isomorphisms there are exactly two central extensions of H by a one-dimensional center, namely*

1) the split extension $H \oplus F$

2) the nonsplit extension $(H_\Gamma)^* = H \oplus Fw$, given by

$$\{x^a y^b + \delta w, x^r y^s + \delta' w\}^\Gamma := (as-br)x^{a+r-1}y^{b+s-1}\Lambda + f(x^a y^b, x^r y^s)w,$$
$$f(x^a y^b, x^r y^s) := s\delta_{a,0}\delta_{r,0}\delta_{b+s,p}.$$

PROOF. Let G be a nonsplit central extension, the multiplication denoted by $[\ ,\ ]$. Then G is an indecomposable H-module of dimension p^2 with a one-dimensional submodule. Thus $G \cong (H_{\alpha\Gamma+\beta\Theta})^*$ with suitable $\alpha,\beta \in F$ (not both vanishing). In terms of the multiplication this and the above determination of these modules means that $G = H \oplus Fw$,

$$[g+\delta w, h+\delta' w] = \{g,h\} + \lambda(\{g,\alpha\Gamma+\beta\Theta\},h)w.$$

Consider the linear automorphisms φ and ψ_δ (for all $\delta \in F$) of the vector space $A(2;\mathbf{1})$ given by

$$\varphi(x^a y^b) := (-1)^b x^b y^a, \quad \psi_\beta(x^a y^b) := (x+\delta y)^a y^b.$$

We have $\varphi(\Lambda) = \Lambda$, $\psi_\delta(\Lambda) = \Lambda$, and the chain rule easily proves that φ, ψ_δ are automorphisms of H. In addition, these mappings are automorphisms of the commutative truncated polynomial ring $A(2;\mathbf{1})$ and therefore are orthogonal transformations with respect to the invariant bilinear form λ, defined in Theorem 6.1. These mappings can be extended canonically to L (as L is a p-envelope of H) by

$$\varphi(\Gamma) := \varphi(x)^{[p]} = \Theta,\ \varphi(\Theta) := \varphi(y)^{[p]} = -\Gamma,$$
$$\psi_\delta(\Gamma) := (x+\delta y)^{[p]}, \quad \psi_\delta(\Theta) := y^{[p]} = \Theta.$$

We are not interested in the exact determination of $\psi_\delta(\Gamma)$, but just mention that $\psi_\delta(\Gamma) = x^{[p]} + \delta^p y^{[p]} + u_\delta = \Gamma + \delta^p\Theta + u_\delta$, with some $u_\delta \in H$. Thus by choosing a suitable automorphism ρ of L we obtain $\rho(\alpha\Gamma + \beta\Theta) = \gamma(\Gamma + u)$ for some $u \in H, \gamma \neq 0$. Define $\rho' \colon G \longrightarrow G$ by $\rho'(g + \delta w) := \rho(g) + \gamma^{-1}\delta w$. Then

$$\begin{aligned}\rho'([g+\delta w, h+\delta' w]) &= \rho(\{g,h\}) + \lambda(\{g, \alpha\Gamma + \beta\Theta\}, h)\gamma^{-1}w \\ &= \rho(\{g,h\}) + \lambda\big(\{\rho(g), \rho(\alpha\Gamma + \beta\Theta)\}, \rho(h)\big)\gamma^{-1}w \\ &= \{\rho(g), \rho(h)\} + \lambda\big(\{\rho(g), \gamma(\Gamma + u)\}, \rho(h)\big)\gamma^{-1}w \\ &= \{\rho(g), \rho(h)\} + \lambda\big(\{\rho(g), \Gamma + u\}, \rho(h)\big)w.\end{aligned}$$

Since we are only interested in isomorphisms classes, we may assume that the product $[\ ,\ ]$ makes G an H-module determined by a cocycle $f^{\Gamma+u}$. Since this differs from f^Γ by a coboundary (determined by $\varphi \colon H \longrightarrow F$, $\varphi(g) := \lambda(gu)$), we may choose a different vector space decomposition $G = V \oplus Fw$, and a vector space isomorphism $\sigma \colon G \longrightarrow G$, such that $\sigma(w) = w$, $\sigma(V) = H$ and

$$\sigma([v_1 + \delta w, v_2 + \delta' w]) = \{\sigma(v_1), \sigma(v_2)\} + \lambda\big(\{\sigma(v_1), \Gamma\}, \sigma(v_2)\big)w.$$

Thus up to isomorphisms, the multiplication of G is given by

$$[g + \delta w, h + \delta' w] = \{g,h\} + \lambda(\{g, \Gamma\}, h)w.$$

To complete the proof, we have to compute $\lambda(\{g,\Gamma\}, h)$. Put $g = x^a y^b$, $h = x^r y^s$. Then $\{x^a y^b, \Gamma\} = -bx^{p-1+a}y^{b-1}$ and interpreting $\lambda(\{g,\Gamma\}, h)$ as an element of $A(2;\mathbf{1})/H \cong F$ we obtain

$$\lambda(\{x^a y^b, \Gamma\}, x^r y^s) = -bx^{p-1+a+r}y^{b-1+s} + H = (s\delta_{r,0}\delta_{a,0}\delta_{b+s,p})1 + H. \quad \blacksquare$$

We now turn to a problem which had been left open in [St-89/3] and in this context allows a natural partial solution.

PROPOSITION 6.4. *1) Let G be a central extension of H, i.e. $G/C(G) \cong H$, and M any G-module. If $G^{(1)} \cap C(G)$ acts nonnilpotently on M, then*

$$\dim M \geq p^{(p-1)/2}.$$

2) Let G be a Lie algebra with $G/\operatorname{rad} G \cong H$, and M an irreducible G-module. Assume that $[G, \operatorname{rad} G]$ does not act nilpotently on M and $\dim M \leq p^{(1/2)(p^2-2)}$. Then G has an ideal J with the properties

a) $J^{(1)}$ acts nilpotently on M

b) $[G, J]$ does not act nilpotently on M.

PROOF. 1) Decompose $M = \oplus M_i$ into weight spaces with respect to $C(G)$. Every M_i is a G-submodule. Let $U_i \subset M_i$ be an irreducible G-submodule with representation $\rho_i \colon G \longrightarrow \mathrm{gl}(U_i)$. Since U_i is irreducible, every $x \in C(G)$ acts on U_i as a scalar multiple of id_{U_i}. Our assumption implies, that there is $k \in \mathbb{N}$ and $w \in G^{(1)} \cap C(G)$, such that $\rho_k(w) = \mathrm{id}_{U_k}$. In particular, $G/\ker\rho_k$ is not abelian. Thus substituting (if necessary) M by U_k and G by $G/\ker\rho_k$ we may assume that M is faithful and irreducible. Consequently, $C(G) = Fw$ is one-dimensional and $\rho(w) = \mathrm{id}$.

The structure of G is determined in Theorem 6.3. As G is not split, we have (up to isomorphism) $[y^b, y^s] = s\delta_{b+s,p}w$. Put $K := \sum_{1\leq i\leq p-1} Fy^i + Fw$ (the p-dimensional Heisenberg algebra) and $K_{(1)} := \sum_{(p+1)/2\leq i\leq (p-1)} Fy^i + Fw$. $K_{(1)}$ is an ideal of K of codimension $(p-1)/2$. $\rho([y^i, y^j])$ is invertible if and only if $i+j = p$. Thus Theorem 2.4.(2) applies and yields that $\dim M \geq p^{(p-1)/2}$.

2) We proceed by induction on $\dim G$. If $\rho(G)$ is solvable the result is well-known. Thus we may assume that M is a faithful module. Due to our assumptions $\mathrm{rad}\, G \neq C(G)$. Choose an ideal J minimal under the conditions

a) $C(G) \subset J$,
b) $C(G) \neq J$,
c) $J^{(1)} \subset C(G)$.

If $C(G) = 0$ then we are done. Otherwise $C(G)$ is one-dimensional and is spanned by some element c with $\rho(c) = \mathrm{id}_M$. In this case there is a skew-symmetric bilinear form $\mu\colon J \times J \longrightarrow F$ given by $[x, y] =: \mu(x, y)c$. It has the properties

$$\mu(x,y) = -\mu(y,x),\ \ \mu([g,x],y) + \mu(x,[g,y]) = 0 \text{ for all } x,y \in J, g \in G.$$

$I := \{g \in J \mid \mu(g,y) = 0\,\forall y \in J\}$ is an abelian ideal of G. If $I \neq C(G)$, then $J = I$ is abelian and, as J is not central, $[G, J] \neq 0$ and therefore does not act nilpotently on M. In this case we are done.

Thus we assume $I = C(G)$ and prove that this assumption leads to a contradiction. J is a Heisenberg algebra. Choose a maximal totally isotropic subspace $V \subset J$. V is an abelian ideal of J of codimension $(1/2)\dim J/C(G)$ in J. Note that for $x, y \in J$, $\rho([x,y]) = \mu(x,y)\,\mathrm{id}$. As in 1), Theorem 2.4 yields the estimate

$$p^{(1/2)\dim J/C(G)} \leq \dim M.$$

As $\dim M \leq p^{(1/2)(p^2-2)}$ this shows that $\dim J/C(G) \leq p^2 - 2 < p^{(p-1)/2}$. $J/C(G)$ is a G-module, and the minimality of J implies that it is irreducible. Let $\kappa\colon G \longrightarrow \mathrm{gl}\big(J/C(G)\big)$ denote the representation. If $\kappa(G)$ is solvable then ([SF-88], Lemma V.8.1) proves that J is abelian, $J = I = C(G)$, a contradiction. If $[G, \mathrm{rad}\, G]$ acts nilpotently on $J/C(G)$, then the irreducibility implies $\kappa([G, \mathrm{rad}\, G]) = 0$, i.e. $\kappa(\mathrm{rad}\, G) = C\big(\kappa(G)\big)$. The first part of this proposition yields that $\kappa(G)^{(1)} \cap C\big(\kappa(G)\big)$ acts nilpotently, hence is 0. Then $\kappa(G) = \kappa(G)^{(1)} \oplus C\big(\kappa(G)\big)$ splits and $\kappa(G)^{(1)} \cong H$ acts faithfully on $J/C(G)$. This contradicts the result on the dimension in combination with Theorem 4.9. By induction hypothesis applied to the algebra G/J and the irreducible module $J/C(G)$ there is an ideal K of G such that $[K^{(1)}, J] \subset C(G)$ and $[G, K]$ acts nonnilpotently on $J/C(G)$. Since $\kappa(K)$ is abelian and $J/C(G)$ is irreducible, there is an eigenvalue function $\lambda \in K^*$, such that $\kappa(g)-\lambda(g)\,\mathrm{id}$ acts nilpotently on $J/C(G)$ for all $g \in K$. On the other hand, the above constructed bilinear form μ yields that

$$\lambda(g)^{p^r}\mu(x,y) = \mu\big((\operatorname{ad} g)^{p^r}(x),y\big) = (-1)^{p^r}\mu\big(x,(\operatorname{ad} g)^{p^r}(y)\big) = -\lambda(g)^{p^r}\mu(x,y)$$
$$\text{for all } x,y \in J, g \in K, \text{ and suitable } r.$$

Thus $\lambda(K) = 0$. Then $\lambda([G,K]) = 0$ and $[G,K]$ acts nilpotently on $J/C(G)$, a contradiction. ∎

THEOREM 6.5. *Let G be a Lie algebra with $G/\operatorname{rad} G \cong H$. Assume that G has a faithful irreducible module M of dimension $\dim M \leq p^2$. G_p denotes a p-envelope of G, such that M is a faithful G_p-module. Then the following two mutually exclusive possibilities occur.*

1) $\operatorname{rad} G = C(G)$: *$G \cong H \oplus C(G)$ is a split central extension. M is one of the modules of Theorem 4.9.*
2) $[G,\operatorname{rad} G]$ *acts not nilpotently on M* : $\operatorname{rad} G \neq C(G)$, $(\operatorname{rad} G_p)^{(1)} = 0$. *There is a restricted subalgebra K of G_p of codimension 2 and containing* $\operatorname{rad} G_p$, *a character μ, and a one-dimensional K-submodule Fu, s.t. $M \cong u(G_p,\mu)\otimes_{u(K,\mu|K)} Fu$.*

PROOF. a) Consider the case that $[G,\operatorname{rad} G]$ *acts nilpotently on M*: as M is faithful irreducible and $[G,\operatorname{rad} G]$ is an ideal, this vanishes and hence $\operatorname{rad} G = C(G)$. Proposition 6.4 shows that $G^{(1)} \cap C(G) = 0$, i.e. G is a split central extension, $G \cong H \oplus C(G)$. The irreducibility of M yields $\dim C(G) = 1$.

b) Consider the case that $[G,\operatorname{rad} G]$ *does not act nilpotently on M*: Proposition 6.4 applies to $G + \operatorname{rad} G_p$ and yields the existence of an abelian ideal $J \not\subset C(G_p)$. Clearly J is an ideal of G_p. Let λ denote the eigenvalue function on J, i.e. $\rho(g) - \lambda(g)\,\mathrm{id}$ is nilpotent for all $g \in J$. Put $K := \{g \in G_p \mid \lambda([g,J]) = 0\}$. K is a restricted subalgebra containing J. Let M_1 be an irreducible K-submodule of M. Theorem 2.4 implies that $\dim M \geq p^{\dim G_p/K} \dim M_1$. Consequently, $\dim G_p/K \leq 2$ and $\dim M_1 \leq p^{2-\dim G_p/K}$.

b_1) If $K + \operatorname{rad} G_p \neq G_p$, then $(K \cap G + \operatorname{rad} G_p)/\operatorname{rad} G_p$ is a subalgebra of H of codimension at most 2 (and different from H). According to Proposition 1.4 this subalgebra is $H_{(0)}$ and has codimension 2. As $\dim G_p/K \leq 2$, this in turn means that $K + \operatorname{rad} G_p = K$. Then $\dim M_1 = 1$ and $(\operatorname{rad} G_p)^{(1)}$ annihilates M_1. As $\{m \in M \mid \rho(g)m = 0$ for all $g \in (\operatorname{rad} G_p)^{(1)}\}$ is a submodule, it is all of M. The faithfulness of M yields that $\operatorname{rad} G_p$ is abelian.

b_2) If $K + \operatorname{rad} G_p = G_p$, we put $K' := \cap K^{(n)}$. Observe that $K'/K' \cap \operatorname{rad} G_p \cong \cap_{n>0}(G_p/\operatorname{rad} G_p)^{(n)} \cong H$ is simple. In combination with the fact that $K'^{(1)} = K'$, this implies that $K' \cap \operatorname{rad} G_p$ is the unique maximal ideal of K', in particular $K' \cap \operatorname{rad} G_p = \operatorname{rad} K'$. K' acts on G_p/K, which is at most two-dimensional. Inductively, K' acts on every composition factor trivially. As $K'^{(1)} = K'$, we obtain that $[K',G_p] \subset K$, hence even $[K',G_p] \subset K'$ and K' is an ideal of G_p. Again by induction we obtain, that K' annihilates any composition factor of M_1, hence K' annihilates M_1. As K' is an ideal of G_p, this implies as above, that K' annihilates M, hence $K' = 0$ and K is solvable. But then G is solvable as well, a contradiction. ∎

We finally give an example that the situation becomes much more complicated if the module M under consideration is not irreducible.

Let λ be the invariant form defined in the beginning of this section. $U := \ker\lambda \subset A(2;\mathbf{1})$ is an H-module isomorphic to H. Put $G := H \oplus U$ the semidirect sum of H with the abelian ideal U, $M = A(2;\mathbf{1})$. For $u \in U$ and $h \in M$ let uh be the associative product in $A(2;\mathbf{1})$. Define a map $\rho\colon G \longrightarrow \mathrm{gl}(M)$ by

$$\rho(g+u)h := \{g,h\} + \lambda(uh)1.$$

As $\lambda(\{A(2;\mathbf{1}), A(2;\mathbf{1})\}) = 0$ we obtain

$$\begin{aligned}
&\rho(g_1+u_1)\rho(g_2+u_2)h - \rho(g_2+u_2)\rho(g_1+u_1)h \\
&\quad = \{g_1,\{g_2,h\}\} + \lambda(u_1\{g_2,h\}) + \lambda\big(\lambda(u_2h)u_1\big) \\
&\qquad - \{g_2,\{g_1,h\}\} - \lambda(u_2\{g_1,h\}) - \lambda\big(\lambda(u_1h)u_2\big) \\
&\quad = \{\{g_1,g_2\},h\} + \lambda(\{g_2,u_1h\}) - \lambda(\{g_2,u_1\}h) - \lambda(\{g_1,u_2h\}) \\
&\qquad + \lambda(\{g_1,u_2\}h) \\
&\quad = \{\{g_1,g_2\},h\} + \lambda(\{g_1,u_2\}h) - \lambda(\{g_2,u_1\}h) \\
&\quad = \rho(\{g_1+u_1, g_2+u_2\})h.
\end{aligned}$$

Therefore ρ is a representation of G of dimension p^2. This strange module M has the one-dimensional submodule $F1$, but M does not split , since every submodule contains $F1$. This representation has structural features completely different from those mentioned in the theorem.

7. **Tensor products.** *We presuppose the assumptions of* § 4. For future applications in the classification theory of simple Lie algebras we are interested in the situation that a Lie algebra has a subalgebra G such that $G/\operatorname{rad} G \cong H$, and G-invariant subspaces U, V, W of dimension $\leq p^2$, such that $[U,V] \subset W$. More specifically we consider in the beginning of this section the following setting: U, V, W are induced L-modules of dimension p^2 with representations $\rho_1\colon L \longrightarrow \mathrm{gl}(U)$, $\rho_2\colon L \longrightarrow \mathrm{gl}(V)$, $\rho_3\colon L \longrightarrow \mathrm{gl}(W)$, characters μ_i ($i = 1,2,3$), and the respective one-dimensional $L_{(0)}$-submodules Fu, Fv, Fw. $\varphi\colon U \otimes V \longrightarrow W$ denotes an L-module homomorphism.

LEMMA 7.1. *If $g \in W$ is annihilated by $\rho_3(x^3)$, $\rho_3(xy^2)$, $\rho_3(y^3)$ then there are $\alpha,\beta,\gamma,\delta \in F$ with $g = \alpha w + \beta\rho_3(x)w + \gamma\rho_3(y)w + \delta\rho(x)^{p-1}\rho(y)^{p-1}w$. If $\mu_3 \neq 0$, then $\delta = 0$. In either case $\rho_3(x^2y)g = 0$.*

PROOF. A basis of W is given by $\{\rho_3(x)^a\rho_3(y)^b w \mid 0 \leq a,b \leq p-1\}$. Put $g =$

$\sum_{0\le i,j\le p-1} \alpha_{ij}\rho_3(x)^i\rho_3(y)^j w$. Then by (3.6) and (3.5)

$$\begin{aligned}
0 &= \rho_3(x^3)g \\
&= \sum \alpha_{ij}\rho_3(x)^i[\rho_3(x^3), \rho_3(y)^j]w \\
&= \sum \alpha_{ij}3j(j-1)\rho_3(x)^i\rho_3(y)^{j-2}\rho_3(x)w \\
&= \sum \alpha_{ij}3j(j-1)\rho_3(x)^{i+1}\rho_3(y)^{j-2}w \\
&= \sum_{\substack{i\neq p-1\\ j\ge 2}} 3j(j-1)\alpha_{ij}\rho_3(x)^{i+1}\rho_3(y)^{j-2}w \\
&\qquad + \sum_{j\ge 2} 3j(j-1)\alpha_{p-1,j}\rho_3(x)^p\rho_3(y)^{j-2}w \\
&= \sum_{\substack{i\neq p-1\\ j\ge 2}} 3j(j-1)\alpha_{ij}\rho_3(x)^{i+1}\rho_3(y)^{j-2}w \\
&\qquad + \mu_3(\Gamma)\sum_{j\ge 2} 3j(j-1)\alpha_{p-1,j}\rho_3(y)^{j-2}w.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\alpha_{ij} &= 0 \text{ for } i\neq p-1,\, j\ge 2, \\
\mu_3(\Gamma)\alpha_{p-1,j} &= 0 \text{ for } j\ge 2.
\end{aligned}$$

Symmetrically, we obtain

$$\begin{aligned}
\alpha_{ij} &= 0 \text{ for } i\ge 2,\, j\neq p-1, \\
\mu_3(\Theta)\alpha_{i,p-1} &= 0 \text{ if } i\ge 2.
\end{aligned}$$

Thus the only nonvanishing summands correspond to indices $(i,j)\in\{(0,0),(0,1),(1,0),(1,1),(p-1,p-1)\}$

Applying $\rho_3(xy^2)$ we obtain by (3.4)

$$0 = \rho_3(xy^2)g = \sum \alpha_{ij}i(-2j+i-1)\rho_3(x)^{i-1}\rho_3(y)^j w.$$

Hence $\alpha_{1,1} = 0$. This proves the first part.

Under these conditions on the coefficients a similar computation yields $\rho_3(x^2y)g = 0$. If $\mu_3(\Gamma)\neq 0$ or $\mu_3(\Theta)\neq 0$, then $\alpha_{p-1,p-1} = 0$. ■

LEMMA 7.2. *1)* $\varphi\big(\rho_1(x)^a u\otimes\rho_2(y)^s v\big) = 0$ *for* $1\le a, s\le p-1,\ a+s\le 2p-4$

2) $\varphi\big(\rho_1(y)^a u\otimes\rho_2(x)^s v\big) = 0$ *for* $1\le a,\ s\le p-1,\ a+s\le 2p-4$

3) $\varphi\big(\rho_1(x)^a u\otimes\rho_2(x)^s v\big) = 0$ *for* $1\le a,\ s\le p-1,\ a+s\le 2p-4$

4) $\varphi\big(\rho_1(y)^a u\otimes\rho_2(y)^s v\big) = 0$ *for* $1\le a,\ s\le p-1,\ a+s\le 2p-4$.

PROOF. We proceed by induction on $a+s$.

$a = s = 1$: as

$$\begin{aligned}
&\rho_3(xy^2)\varphi\big(\rho_1(x)u\otimes\rho_2(x)v\big) \\
&\quad = \varphi\big(\rho_1(xy^2)\rho_1(x)u\otimes\rho_2(x)v\big) + \varphi\big(\rho_1(x)u\otimes\rho_2(xy^2)\rho_2(x)v\big) \\
&\quad = \varphi\big([\rho_1(xy^2),\rho_1(x)]u\otimes\rho_2(x)v\big) + \varphi\big(\rho_1(x)u\otimes[\rho_2(xy^2),\rho_2(x)]v\big) \\
&\quad = 0,
\end{aligned}$$

(3.4) implies that $\varphi\big(\rho_1(x)u\otimes\rho_2(x)v\big) = \alpha w+\beta\,\rho_3(x)w+\gamma\,\rho_3(y)w+\delta\,\rho_3(x)^{p-1}\rho_3(y)^{p-1}w$ for suitable $\alpha,\beta,\gamma,\delta$. The eigenvalue with respect to $\rho_3(xy)$ associated with $\varphi\big(\rho_1(x)u\otimes\rho_2(x)v\big)$ is -2, which is only possible if $\alpha=\beta=\gamma=\delta=0$.

We treat the case $\varphi\big(\rho_1(y)u\otimes\rho_2(y)v\big)$ similarly. It is then a direct computation that the assumptions of Lemma 7.1 are satisfied by $g := \varphi\big(\rho_1(x)u\otimes\rho_2(y)^2v\big)$. Thus

$$\begin{aligned}0 &= \rho_3(x^2y)\varphi\big(\rho_1(x)u\otimes\rho_2(y)^2v\big)\\ &= \varphi\big(\rho_1(x^2y)\rho_1(x)u\otimes\rho_2(y)^2v\big)+\varphi\big(\rho_1(x)u\otimes\rho_2(x^2y)\rho_2(y)^2v\big)\\ &= 2\varphi\big(\rho_1(x)u\otimes\rho_2(y)v\big).\end{aligned}$$

The fourth equation of the assertion is derived analogously.

$a+s>2$: By symmetry we may assume that $a\geq s\geq 1$, $a\geq 2$. Note that our assumptions imply $s<p-1$. Then the induction hypothesis yields

$$\rho_3(xy^2)\varphi\big(\rho_1(x)^au\otimes\rho_2(y)^sv\big) = a(a-1)\varphi\big(\rho_1(x)^{a-1}u\otimes\rho_2(y)^sv\big)=0.$$

Similarly, if $s\neq 1$, we obtain inductively

$$\rho_3(x^2y)\varphi\big(\rho_1(x)^au\otimes\rho_2(y)^sv\big) = s(s-1)\varphi\big(\rho_1(x)^au\otimes\rho_2(y)^{s-1}v) = 0.$$

This equation however is also true for $s=1$, since then the scalar factor vanishes. Due to Proposition 3.4 we obtain

$$\varphi\big(\rho_1(x)^au\otimes\rho_2(y)^sv\big) = \alpha w+\beta\,\rho_3(x)w+\gamma\,\rho_3(y)w+\delta\,\rho_3(x)^{p-1}\rho_3(y)^{p-1}w.$$

Next (3.5) and the induction hypothesis yield

$$\begin{aligned}&\rho_3(y^2)\varphi\big(\rho_1(x)^au\otimes\rho_2(y)^sv\big)\\ &\quad= \varphi\big(\rho_1(y^2)\rho_1(x)^au\otimes\rho_2(y)^sv\big)+\varphi\big(\rho_1(x)^au\otimes\rho_1(y^2)\rho_2(y)^sv\big)\\ &\quad= -2a\varphi\big(\rho_1(x)^{a-1}\rho_1(y)u\otimes\rho_2(y)^sv\big)\\ &\quad= -2a\varphi\big(\rho_1(y)\rho_1(x)^{a-1}u\otimes\rho_2(y)^sv\big)\\ &\quad= -2a\rho_3(y)\varphi\big(\rho_1(x)^{a-1}u\otimes\rho_2(y)^sv\big)+2a\varphi\big(\rho_1(x)^{a-1}u\otimes\rho_2(y)^{s+1}v\big)\\ &\quad= 2a\varphi\big(\rho_1(x)^{a-1}u\otimes\rho_2(y)^{s+1}v\big),\\ &\rho_3(x^2)\rho_3(y^2)\varphi\big(\rho_1(x)^au\otimes\rho_2(y)^sv\big)\\ &\quad= 2a\rho_3(x^2)\varphi\big(\rho_1(x)^{a-1}u\otimes\rho_2(y)^{s+1}v\big)\\ &\quad= 4a(s+1)\varphi\big(\rho_1(x)^{a-1}u\otimes\rho_2(y)^s\rho_2(x)v\big)\\ &\quad= -4a(s+1)\varphi\big(\rho_1(x)^au\otimes\rho_2(y)^sv).\end{aligned}$$

On the other hand, the above result yields

$$\begin{aligned}&\rho_3(x^2)\rho_3(y^2)\varphi\big(\rho_1(x)^au\otimes\rho_2(y)^sv\big)\\ &\quad= \rho_3(x^2)\rho_3(y^2)\big(\alpha w+\beta\,\rho_3(x)w+\gamma\,\rho_3(y)w+\delta\,\rho_3(x)^{p-1}\rho_3(y)^{p-1}w\big)\\ &\quad= \rho_3(x^2)(-2\beta\,\rho_3(y)w+2\delta\,\rho_3(x)^{p-2}\rho_3(y)^pw\big) = -4\beta\,\rho_3(x)w.\end{aligned}$$

Hence $\alpha = 0, \gamma = 0, \delta = 0, -4a(s+1)\beta = -4\beta$. Considering eigenvectors with respect to $\rho_3(xy)$ we obtain

$$-\beta\,\rho_3(x)w = \rho_3(xy)\varphi\big(\rho_1(x)^a u \otimes \rho_2(y)^s v\big) = (s-a)\varphi\big(\rho_1(x)^a u \otimes \rho_2(y)^s v\big),$$

hence $(s-a)\beta = -\beta$. Thus if $\beta \neq 0$, then $a = s+1 = a^{-1}$, and $a = 1$ or $a = p-1$. The first case contradicts our choice of a, while in the second case $s = p-2$, contradicting our assumption $a+s \leq 2p-4$. Thus $\beta = 0$, proving the induction step in one case.

Next we conclude

$$0 = \rho_3(x^2)^s\varphi\big(\rho_1(x)^a u \otimes \rho_2(y)^s v\big) = 2^s s!\,\varphi\big(\rho_1(x)^a u \otimes \rho_2(x)^s v\big).$$

2) and 4) are just the symmetric statements. ∎

LEMMA 7.3. *1) Assume that* $0 \leq a+r,\ b+s \leq p-1,\ 1 \leq a+b,\ r+s,\ a+b+r+s \leq 2p-4$. *Then* $\varphi\big(\rho_1(x)^a\rho_1(y)^b u \otimes \rho_2(x)^r\rho_2(y)^s v\big) = 0$.

2) Assume that $\varphi\big(\rho_1(x)u \otimes v\big) = 0$ *or* $\varphi\big(\rho_1(y)u \otimes v\big) = 0$. *Then* $\varphi\big(\rho_1(x)^a\rho_1(y)^b u \otimes v\big) = 0$ *whenever* $0 \leq a,\ b \leq p-1,\ 1 \leq a+b \leq 2p-4$.

PROOF. 1) If a and b both are nonzero we may, proceeding by induction on $a+b+r+s$, shift x to the right side of the expression. Thus we may assume that $a = 0$ or $b = 0$. Similarly, we obtain inductively $s = 0$ if $a = 0$ and $r = 0$ if $b = 0$. Lemma 7.2 yields the result.

2) As $\varphi\big(\rho_1(y)u \otimes v\big) = -(1/2)\varphi\big(\rho_1(y^2)\rho_1(x)u \otimes v\big) = -(1/2)\rho_3(y^2)\varphi\big(\rho_1(x)u \otimes v\big)$, $\varphi\big(\rho_1(x)u \otimes v\big) = (1/2)\varphi(\rho_3(x^2)\rho_1(y)u \otimes v\big) = (1/2)\rho_3(x^2)\varphi\big(\rho_1(y)u \otimes v\big)$ the assumption implies that both of these elements vanish. This proves the assertion for $a+b = 1$. The general result follows easily by induction on $a+b$, by shifting one of the factors on the left to the right and then using 1). ∎

THEOREM 7.4. *Let U, V, W be induced L-modules of dimension p^2 or irreducible of dimension p^2-1, with representations $\rho_1 : L \longrightarrow \mathrm{gl}(U)$, $\rho_2 : L \longrightarrow \mathrm{gl}(V)$, $\rho_3 : L \longrightarrow \mathrm{gl}(W)$, and characters μ_i $(i = 1,2,3)$ with $\mu_i(H) = 0$. Fu, Fv denote the respective one-dimensional $L_{(0)}$-submodules of U and V. Assume that $\varphi : U \otimes V \longrightarrow W$ is an L-module homomorphism.*

1) $\varphi\big(\rho_1(x)^a\rho_1(y)^b u \otimes \rho_2(y)v\big) = 0$ *whenever* $0 \leq a \leq p-1$, $0 \leq b \leq p-2$, $1 \leq a+b \leq 2p-5$.

2) If $\mu_2 \neq 0$, *then* $\varphi\big(\rho_1(x)^a\rho_1(y)^b u \otimes v\big) = 0$ *whenever* $0 \leq a,\ b \leq p-1$, $1 \leq a+b \leq 2p-4$.

3) If $\mu_1 \neq 0$, $\mu_2 \neq 0$, *then* $\varphi\big(\rho_1(x)^a\rho_1(y)^b u \otimes v\big) = 0$ *whenever* $0 \leq a,\ b \leq p-1$, $0 \leq a+b \leq 2p-4$.

PROOF. a) Assume first that U, V, W are induced. 1) is a direct consequence of Lemma 7.3.(1). In order to prove 2) we assume that $\mu_2(\Gamma) \neq 0$ (the case $\mu_2(\Theta) \neq 0$ is similar). Lemma 7.2 yields

$$\mu_2(\Gamma)\varphi\big(\rho_1(x)u \otimes v\big) = \varphi\big(\rho_1(x)u \otimes \rho_2(x)^p v\big) =$$
$$\rho_3(x)\varphi\big(\rho_1(x)u \otimes \rho_2(x)^{p-1}v\big) - \varphi\big(\rho_1(x)^2 u \otimes \rho_2(x)^{p-1}v\big) = 0.$$

Hence $\varphi\big(\rho_1(x)u \otimes v\big) = 0$, and Lemma 7.3.(2) gives the result.

3) Using the preceding result and Lemma 7.2 we obtain

$$\mu_1(\Gamma)\varphi(u \otimes v) = \varphi\big(\rho_1(x)^p u \otimes v\big) =$$
$$\rho_3(x)\varphi\big(\rho_1(x)^{p-1}u \otimes v\big) - \varphi\big(\rho_1(x)^{p-1}u \otimes \rho_2(x)v\big) = 0,$$

and similarly $\mu_1(\Theta)\varphi(u \otimes v) = 0$. As $\mu_1 \neq 0$ this gives the result.

b) If some of the U, V, W are not induced but irreducible, we observe that in either case U, V, W are direct summands of induced modules $\tilde{U}$, $\tilde{V}$, $\tilde{W}$ of dimension p^2 ((4.9) and (4.8)). Thus U, V are homomorphic images of $\tilde{U}$ and $\tilde{V}$, and W is a submodule of $\tilde{W}$. There is a pull back $\tilde{\varphi}\colon \tilde{U} \otimes \tilde{V} \longrightarrow \tilde{W}$ of φ. a) applies to $\tilde{\varphi}$ and gives the result. ∎

Motivated by the classification of simple Lie algebras we finally consider the case that G is a Lie algebra with radical $\operatorname{rad} G \not\subset C(G)$, and U, V are faithful irreducible G-modules of dimension $\leq p^2$, while W is any module of dimension $\leq p^2$. We recall (Theorem 6.5) that U and V are induced by a subalgebra K of some p-envelope G_p of G, which has codimension 2 in G_p. $\operatorname{rad} G_p$ is abelian. Let Fu, Fv be the one-dimensional K-submodules of U and V, respectively, and μ_1, μ_2 the corresponding characters.

THEOREM 7.5. *Let $G = H \oplus \operatorname{rad} G$ be the semidirect sum of H and the radical of G. Assume $\operatorname{rad} G \not\subset C(G)$. Let U, V be faithful irreducible G-modules of dimension $\leq p^2$ and W an arbitrary G-module of dimension $\leq p^2$. ρ_i $(i = 1,2,3)$ denote the respective representations. Then*

1) $\tilde{G} := L \oplus \operatorname{rad} G$ carries naturally the structure of a restricted Lie algebra, such that G is a subalgebra and $\tilde{G}$ is a p-envelope of G.

2) $\rho_i (i = 1,2,3)$ can be extended to representations of $\tilde{G}$. These are faithful irreducible with characters μ_1, μ_2 in case of ρ_1, ρ_2. Put $K := L_{(0)} + \operatorname{rad} G$. U and V are induced

$$U \cong u(\tilde{G}, \mu_1) \otimes_{u(K,\mu_1|K)} Fu$$
$$V \cong u(\tilde{G}, \mu_2) \otimes_{u(K,\mu_2|K)} Fv.$$

3) Let $\varphi\colon U \otimes V \longrightarrow W$ denote a $\tilde{G}$-module homomorphism. Then

$$\varphi\Big(\sum_{0 \leq a+b \leq 2p-4} F\rho_1(x)^a \rho_1(y)^b u \otimes Fv\Big) = 0.$$

PROOF. 1) Choose a p-envelope $(G_p, [p]')$ of G of minimal dimension, such that ρ_1 extends to a faithful representation. Let $\tilde{H}$ be the restricted subalgebra generated by H and $[p]'$. $C(\tilde{H}) + \operatorname{rad} G$ is invariant under G, and hence is an ideal of G_p. Therefore $C(\tilde{H}) + \operatorname{rad} G \subset \operatorname{rad} G_p$, which is abelian (6.5). In particular,

$$[C(\tilde{H}), \operatorname{rad} G] = 0, \quad [\operatorname{rad} G, \operatorname{rad} G] = 0.$$

As $(L, [p])$ is a minimal p-envelope of H (and is centerless) there is an algebra isomorphism $\tilde{H} \cong L \oplus C(\tilde{H})$, such that $g^{[p]'} - g^{[p]} \in C(\tilde{H})$ for all $g \in \tilde{H}$ ([SF-88], II.5.8). Therefore

$$[g^{[p]}, f] = [g^{[p]'}, f] = (\operatorname{ad} g)^p(f) \text{ for all } g \in \tilde{H},\ f \in \operatorname{rad} G.$$

According to ([SF-88], II.2.5) $L \oplus \operatorname{rad} G \subset G_p$ now is restrictable as well. The minimality of G_p shows that $G_p = L \oplus \operatorname{rad} G = \tilde{G}$.

2) In the course of 1) we additionally proved that ρ_1 can be extended to a faithful representation of $\tilde{G}$. By symmetry the same holds for ρ_2. Clearly, these representations are still irreducible. Since $\tilde{G}$ is a p-envelope of G, also ρ_3 can be extended to $\rho_3 \colon \tilde{G} \longrightarrow \mathrm{gl}(W)$. We apply Theorem 6.5. The subalgebra K mentioned there contains $\operatorname{rad} G$ and has codimension 2 in $\tilde{G}$. Therefore $K = \operatorname{rad} G + K \cap L$ and $K \cap L$ has codimension 2 in L. Thus $K \cap L = L_{(0)}$ and $K = L_{(0)} + \operatorname{rad} G$ (for both, ρ_1 and ρ_2 we obtain the same subalgebra). (6.5) shows that U and V have the asserted form. In particular, U and V are, considered just as L-modules, induced.

3) a) Assume first, that W, considered an L-module, is induced by a one-dimensional $L_{(0)}$-submodule. Since U and V are also induced as L-modules, Lemma 7.3 applies proving that

$$\varphi\big(\rho_1(x)u \otimes \rho_2(y)v\big) = 0.$$

Recall the construction of K: there are eigenvalue functions $\lambda_1,\ \lambda_2 \in (\operatorname{rad} G)^*$, such that $K = \{h \in \tilde{G} \mid \lambda_i([h, \operatorname{rad} G]) = 0\}$ $(i = 1, 2)$. Choose $f,\ g \in \operatorname{rad} G$ such that $\lambda_1([f, x]) = 1$, $\lambda_1([f, y]) = 0$, $\lambda_2([g, x]) = 0$, $\lambda_2([g, y]) = 1$.

$$\begin{aligned} 0 &= \Big(\rho_3(g) - \big(\lambda_1(g) + \lambda_2(g)\big)\,\mathrm{id}_W\Big)\Big(\varphi\big(\rho_1(x)u \otimes \rho_2(y)v\big)\Big) \\ &= \varphi\big(\rho_1([g, x])u \otimes \rho_2(y)v\big) + \varphi\big(\rho_1(x)u \otimes \rho_2([g, y])v\big) \\ &= \lambda_1([g, x])\varphi\big(u \otimes \rho_2(y)v\big) + \varphi\big(\rho_1(x)u \otimes v\big). \end{aligned}$$

Considering eigenvalues with respect to xy we obtain $\varphi\big(\rho_1(x)u \otimes v\big) = 0$. The application of $\rho_3(f) - \big(\lambda_1(f) + \lambda_2(f)\big)\,\mathrm{id}_W$ now yields $\varphi(u \otimes v) = 0$. Lemma 7.3 gives the result.

b) If W is not induced but irreducible, it is a direct summand of an induced module W' (4.8). Then φ gives rise to an $\tilde{G}$-module homomorphism $\varphi' \colon U \otimes V \longrightarrow W'$. a) applies and proves the assertion in this case.

c) Let W have a one-dimensional trivial H-module Fc. If W/Fc is irreducible we obtain (using b)) that

$$\varphi\Big(\sum_{0 \le a+b \le 2p-4} F\rho_1(x)^a \rho_1(y)^b u \otimes Fv\Big) \subset Fc.$$

If W is a trivial H-module we use induction on $\dim W$ to obtain the same conclusion.

Observe then, that

$$\begin{aligned}\varphi\Big(\sum_{1\le a+b\le 2p-4} F\rho_1(x)^a\rho_1(y)^b u\otimes Fv\Big) &\subset \varphi\Big(\sum_{0\le a+b\le 2p-4} F\rho_1(xy)\rho_1(x)^a\rho_1(y)^b u\otimes Fv\Big)\\ &\quad+\varphi\Big(\sum_{0\le a+b\le 2p-4} F\rho_1(x^2)\rho_1(x)^a\rho_1(y)^b u\otimes Fv\Big)\\ &\subset \rho_3(xy)\varphi\Big(\sum_{0\le a+b\le 2p-4} F\rho_1(x)^a\rho_1(y)^b u\otimes Fv\Big)\\ &\quad+\rho_3(x^2)\varphi\Big(\sum_{0\le a+b\le 2p-4} F\rho_1(x)^a\rho_1(y)^b u\otimes Fv\Big)\\ &\subset \rho_3(xy)Fc+\rho_3(x^2)Fc=0.\end{aligned}$$

This gives the result for $(a,b)\neq(0,0)$. To prove $\varphi(u\otimes v)=0$ we consider eigenvalues. Using the element f determined above and observing that $\varphi\big(\rho_1(x)u\otimes v\big)=0$ we obtain

$$\begin{aligned}0&=\Big(\rho_3(f)-\big(\lambda_1(f)+\lambda_2(f)\big)\,\mathrm{id}_W\Big)\varphi\big(\rho_1(x)u\otimes v\big)\\ &=\varphi\big(\rho_1([f,x])u\otimes v\big)=\varphi(u\otimes v).\end{aligned}$$

d) There is only one case left: W has a submodule of codimension 1. Then $W\cong H+F(\alpha\Gamma+\beta\Theta)$ for suitable α,β. As in c) we conclude that

$$\varphi\Big(\sum_{1\le a+b\le 2p-3} F\rho_1(x)^a\rho_1(y)^b u\otimes Fv\Big)\subset \rho_3(xy)W+\rho_3(x^2)W\subset H.$$

Observe that for all $h\in\operatorname{rad}G$ the transformation $\rho_3(h)$ acts as a scalar multiple on the irreducible module H, $\rho_3(h)|H=\rho_3(h)\,\mathrm{id}_H$. If $\lambda_3(h)\neq\lambda_1(h)+\lambda_2(h)$ for some $h\in\operatorname{rad}G$ then $\rho_3(h)-\rho_3(h)\,\mathrm{id}$ would act invertibly on $H\cap\varphi(U\otimes V)$, i.e. $H\cap\varphi(U\otimes V)=0$. Thus assume that $\lambda_3(h)=\lambda_1(h)+\lambda_2(h)$ for all $h\in\operatorname{rad}G$. Choose $f,f'\in\operatorname{rad}G$ such that $\lambda_1([f,x])=1$, $\lambda_1([f,y])=0$, $\lambda_1([f',x])=0$, $\lambda_1([f',y])=1$. Then

$$\begin{aligned}&\varphi\Big(\sum_{0\le a+b\le 2p-4} F\rho_1(x)^a\rho_1(y)^b u\otimes Fv\Big)\\ &\quad\subset\big(\rho_3(f)-\lambda_3(f)\,\mathrm{id}\big)\varphi\Big(\sum_{1\le a+b\le 2p-3} F\rho_1(x)^a\rho_1(y)^b u\otimes Fv\Big)\\ &\qquad+\big(\rho_3(f')-\lambda_3(f')\,\mathrm{id}\big)\varphi\Big(\sum_{1\le a+b\le 2p-3} F\rho_1(x)^a\rho_1(y)^b u\otimes Fv\Big)\\ &\quad\subset\big(\rho_3(f)-\lambda_3(f)\,\mathrm{id}\big)H+\big(\rho_3(f')-\lambda_3(f')\,\mathrm{id}\big)H=0.\end{aligned}$$

■

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