J. Austral. Math. Soc. Ser. B 31(1989), 97–107

AN EVOLUTIONARY MONOTONE FOLLOWER PROBLEM IN [0, 1]

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(Received 13 January 1988; revised 23 June 1988)

Abstract

We consider in this article an evolutionary monotone follower problem in [0,1]. State processes under consideration are controlled diffusion processes $y_x(t) = y_x^v(t)$, solutions of $dy_x(t) = g(y_x(t), t)\,dt + \sigma(y_x(t), t)\,dw_t + dv_t$ with $y_x(0) = x \in [0, 1]$, where the control processes v_t are increasing, positive, and adapted. The cost functional is of integral type, with certain explicit cost of control action including the cost of jumps. We shall present some analytic results of the value function, mainly its characterisation, by standard dynamic programming arguments.

1. Introduction

This article deals with the evolutionary version of a monotone follower problem considered in Sun [6]. Monotone follower problems originally arose from the control of spaceships (cf. Bather and Chernoff [1], [2]), where the control variable represented the cumulative amount of fuel used up to a certain time. Great progress has been made since 1980 (cf. Beneš, Shepp, and Witsenhausen [3]). One is referred to [6] and the references therein for detailed background and relevant results.

We shall study a class of time-dependent monotone follower problems in the bounded interval [0, 1] over a finite horizon T. The state processes under investigation are given by $y_x(t) = y_x^v(t)$, solutions of

$$dy_x(t) = g(y_x(t), t)\,dt + \sigma(y_x(t), t)\,dw_t + dv_t$$
$$y_x(0) = x \in [0, 1]$$
$$v_t, \text{increasing adapted control processes with } v_0 = 0.$$

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Our aim is to minimise

$$J_x(v) = E\left\{\int_0^{\tau\wedge T} f(y_x(t),t)\,dt + \int_0^{\tau\wedge T}\phi(y_x(t),t)\,dv^c(t) + \psi(y_x(\tau),\tau)I_{(\tau<T)} + \xi(y_x(T))I_{(\tau\geq T)} + \sum_{0<s\leq\tau\wedge T}\int_{y_x(s-)}^{y_x(s)}\phi(z,s)\,dz\right\},$$

over all such controls v_t, where f is holding cost, ϕ is cost of control, ψ is exit cost, ξ is terminal cost, and τ is the first exit time of $y_x(t)$ from $[0,1]$.

Note that the last integral in the cost functional represents a class of explicit cost of jumps at time s. As in [6], our value function can be characterised as the maximum solution to a variational inequality, which is the main result of this article.

In Section 2, we give the precise formulation of the problem and state basic assumptions we need to develop the main result. Section 3 concerns the derivation of the main result.

Since the ideas of approach will be similar to those used in [6], some details will be skipped for brevity of presentation. However, let us point out that the new feature of our derivation here lies in the fact that the *a priori* estimates of $\partial_t u^\varepsilon$, $\partial_x u^\varepsilon$, $\partial_x^2 u^\varepsilon$, and $(1/\varepsilon)\{-\partial_x u^\varepsilon - \phi\}^+$ are obtained in a manner different from that in [6]. As before, u^ε is the value function when the admissible controls are taken from a subclass of regularised ones. Details for deriving these estimates will be provided.

Finally, let us mention that in Chow *et al.* [4] a related singular control problem in the whole space R^1 with a finite horizon was investigated by means of the linearity and convexity arguments. However, we do not use any of these properties. Instead, we shall use a P.D.E. technique to derive our estimates with fairly strong regularity assumptions.

2. Statement of problem and basic assumptions

Let a bounded time interval $[0,T]$ and a measurable space (Ω, F) be given. Let $\{F_t\}_{t\in[0,T]}$ be an increasing right continuous family of (completed) sub σ-algebras of F. We consider for a triple $(P, w(\cdot), v(\cdot))$ the one-dimensional controlled diffusion process given by

$$dy_x(t) = g(y_x(t),t)\,dt + \sigma(y_x(t),t)\,dw_t + dv_t, \qquad y_x(0) = x\in[0,1], \quad (2.1)$$

where P is a probability measure on (Ω, F), $\{w(t)\colon 0\leq t\leq T\}$ a standard Wiener process in R^1 with respect to $(\Omega, \{F_t\}, P)$ and $\{v(t)\colon 0\leq t\leq T\}$ a progressively measurable random process from $[0, T]$ into $[0, \infty)$, right continuous, having left limits and nondecreasing with $v(0) = 0$, which will be denoted by $v(\cdot) \in V$.

Let $G = (0, 1) \times (0, T)$. Let us use the following definition of distance in R^2 (cf. e.g. Friedman [5], Chapter 3)

$$d(P, Q) = [|x - x'|^2 + |t + t'|]^{1/2}$$

where $P = (x, t)$ and $Q(x', t')$. The concept of Hölder continuity will be defined below with respect to this distance.

We adopt the following notations:

$$|u|_0 = \operatorname{l.u.b.}_{G} |u|, \qquad \bar{H}_r(u) = \operatorname{l.u.b.}_{P,Q\in G} \frac{|u(P) - u(Q)|}{d(P, Q)^r}, \qquad (r > 0)$$

$$\overline{|u|}_r = |u|_0 + \overline{H}_r(u), \qquad \overline{|u|}_{2+r} = \overline{|u|}_r + \overline{|\partial_x u|}_r + \overline{|\partial_x^2 u|}_r + \overline{|\partial_t u|}_r,$$

where $\partial_y u$ (or simply u_y) denotes the partial derivative of u with respect to y. $\partial_y^2 u$ or u_{yy} will denote the second partial derivative of u with respect to y, and similarly for u_{xy}, etc.

We say that u is uniformly Hölder continuous with exponent r in G if $\bar{H}_r(u) < \infty$. By the local Hölder continuity of u in G, we mean the uniform Hölder continuity of u on small compact subdomains. Then we introduce the following function spaces:

$$\overline{C}_{2+r}(G) = \{u \in C(G) \colon \partial_t u, \partial_x u \text{ and } \partial_x^2 u \text{ exist in } G, \text{ and } \overline{|u|}_{2+r} < \infty\}$$

$$\overline{C}_{1+r}(G) = \{u \in C(G) \colon \partial_t u, \partial_x u \in C(G) \text{ and they are uniformly Hölder continuous with exponent } r \text{ in } G\}.$$

We adopt the following assumptions $(0 < r \leq 1)$.

g and σ are in $\overline{C}_{1+r}(G)$ such that $\partial_x^2 g$ and $\partial_x^2 \sigma$ are locally Hölder continuous with exponent r in G. (2.2)

$$\sigma(x) \geq \sigma_0 > 0, \quad \text{for some constant } \sigma_0. \tag{2.3}$$

$\phi \geq 0$, $f \in \overline{C}_{1+r}(G)$ such that $\partial_x^2 \phi$ and $\partial_x^2 f$ are locally Hölder continuous with exponent r in G. (2.4)

We also introduce the following compatibility conditions:

There exists a $\overline{C}_{2+r}(G)$ extension Ψ of $\Psi(0, \cdot), \Psi(1, \cdot)$, and $\xi(\cdot)$ to G. (2.5.a)

$$\xi'(z) + \phi(z, T) \geq 0, \quad \text{for } z = 0 \text{ and } 1. \tag{2.5.b}$$

$$-\Psi'(z, T) - \sigma^2(z, T)/2\xi''(z) - g(z, T)\xi'(z) = f(z, T) \quad \text{for } z = 0 \text{ and } 1. \tag{2.5.c}$$

We want to minimise the following expected cost

$$J_x(v) = E\left\{\int_0^{\tau\wedge T} f(y_x(s),s)\,ds + \int_0^{\tau\wedge T} \phi(y_x(s),s)\,dv^c(s) + \Psi(y_x(\tau),\tau)I_{(\tau<T)} + \xi(y_x(T))I_{(\tau\geq T)} + \sum_{0\leq s\leq \tau\wedge T}\int_{y_x(s-)}^{y_x(s)} \phi(z,s)\,dz\right\},$$

where

$$\tau = \tau_x = \inf\{0 \leq s < \infty : y_x(s) \in \{0,1\}\}.$$

3. Characterisation of optimal cost

As usual, we are going to minimise the following expected running cost

$$J_{x,t}(v) = E\left\{\int_t^{\tau\wedge T} f(y(s),s)\,ds + \int_t^{\tau\wedge T} \phi(y(s),s)\,dv^c(s-t) + \Psi(y(\tau),\tau)I_{(\tau<T)} + \xi(y(T))I_{(\tau\geq T)} + \sum_{t<s\leq \tau\wedge T}\int_{y(s-)}^{y(s)} \phi(z,s)\,dz\right\}, \tag{3.1}$$

where

$$\begin{aligned} &\tau = \tau_{x,t} = \inf\{t \leq s < \infty : y(s) \in \{0,1\}\},\\ &y(\cdot) = y^v_{x,t}(\cdot) \text{ satisfying}\\ &dy(s) = g(y(s),s)\,ds + \sigma(y(s),s)\,dw(s-t) + dv(s-t) \qquad (3.2)\\ &y(t) = x \in [0,1]. \end{aligned}$$

We denote

$$u(x,t) = \inf\{J_{x,t}(v) : v \in V\}. \tag{3.3}$$

Our main results are given in Theorem 3.2 which characterises $u(x,t)$ as the maximum solution to variational inequality (3.12). In order to prove the theorem, we need to study a penalised version of the original problem, as we did in [6]. Let us define a penalised version of V by ($\varepsilon > 0$)

$$V^\varepsilon = \{v \in V : 0 \leq \dot{v}_s \leq 1/\varepsilon,\ 0 \leq s \leq T\}. \tag{3.4}$$

We then set

$$u^\varepsilon(x,t) = \inf\{J_{x,t}(v) : v \in V^\varepsilon\}. \tag{3.5}$$

The main results concerning u^ε are given as follows.

Theorem 3.1. *We assume* (2.2)–(2.5). *Then the optimal expected cost* u^ε *given in* (3.5) *is the unique solution of* ($\{z\}^+$ *standing for the positive part of* z)

$$\begin{aligned}
&-\partial_t u^\varepsilon - (\sigma^2/2)\partial_x^2 u^\varepsilon - g\partial_x u^\varepsilon + (1/\varepsilon)\{-\partial_x u^\varepsilon - \phi\}^+ = f, \quad \text{in } (0,1)\times[0,T),\\
&u^\varepsilon(0,t) = \Psi(0,t), \qquad u^\varepsilon(1,t) = \Psi(1,t),\ \forall t \in [0,T),\\
&u^\varepsilon(x,T) = \xi(x),\ \forall x \in [0,1], \qquad u^\varepsilon \in \overline{C}_{2+r}(G).
\end{aligned} \tag{3.6}$$

Moreover, $\|u^\varepsilon\|_{C(G)}$ *is bounded in* ε.

Proof. First of all, we prove the existence of a solution to (3.6). Standard results in P.D.E. (e.g., [5, Theorem 3.7]) imply that the problem (for Hölder continuous h with $h(z,T) = f(z,T)$ for $z = 0$ and 1)

$$\begin{aligned}
&-\partial_t u - (\sigma^2/2)\partial_x^2 u - g\partial_x u = h, \quad \text{in } (0,1)\times[0,T),\\
&u(0,t) = \Psi(0,t), \qquad u(1,t) = \Psi(1,t),\ \forall t \in [0,T),\\
&u(x,T) = \xi(x),\ \forall x \in [0,1],\\
&u \in \overline{C}_{2+r}(G),
\end{aligned} \tag{3.7}$$

has one and only one solution, which is given explicitly by

$$u(x,t) = E\left\{\int_t^{\tau\wedge T} h(y_{x,t}(s),s)\,ds + \Psi(y_{x,t}(\tau),\tau)I_{(\tau<T)} + \xi(y_{x,t}(T))I_{(\tau\geq T)}\right\},$$

where $y_{x,t}(\cdot)$ is given by (3.2) with $v(\cdot) = 0$, and $\tau = \tau_{x,t}$ is the corresponding first exit time. Moreover, we have (cf. [5, Theorem 3.6])

$$\overline{|u|}_{2+r} \leq C(\overline{|\Psi|}_{2+r} + \overline{|h|}_r) \text{ with } C \text{ independent of } h.$$

Then the standard argument of fixed point shows the existence.

We can show the uniqueness of solution by the stochastic interpretation, which is similar to that given in [6].

The second part comes from the stochastic interpretation, too. Thus the proof is complete.

So far we have not seen any significant difference between the stationary problem studied in [6] and its evolutionary version considered in this article. As pointed out in the introduction, the crucial difference lies in the establishment of estimates of $\partial_t u^\varepsilon$, $\partial_x u^\varepsilon$, $\partial_x^2 u^\varepsilon$ and $(1/\varepsilon)\{-\partial_x u^\varepsilon - \phi\}^+$. In the stationary case, the penalised Hamilton-Jacobi-Bellman equation is actually an ordinary differential equation, while in the present case, we have to deal with partial differential equations. We need to overcome this technical difficulty in order to adopt the standard technique used in [6] to characterise u. In what follows, we discuss these estimates in several lemmas.

We need to introduce the following assumption first:
There exists $w \in C^{2,1}(G) \cap C^{1,1}(\overline{G})$ such that

$$\begin{gathered} -\partial_t w - (\sigma^2/2)\partial_x^2 w - g a_x w \le f, \text{ and } -\partial_x w - \phi \le 0, \text{ in } (0,1)\times[0,T), \\ w(0,t) = \Psi(0,t), \qquad w(1,t) = \Psi(1,t),\ \forall t \in [0,T), \\ w(x,T) = \xi(x),\ \forall x \in [0,1]. \end{gathered} \tag{3.8}$$

Lemma 3.1. *The condition* (3.8) *will be satisfied if one of the following holds.*

$$\begin{aligned} &\Psi(0,T) = \Psi(1,T), \qquad \Psi(0,t) \le \Psi(1,t),\ \forall t, \\ &|\partial_t[\Psi(1,t) - \Psi(0,t)]| \le C[\Psi(1,t) - \Psi(0,t)],\ \forall t, \\ &\xi'(x) \ge -\phi(x,t), \qquad \xi(0) = \xi(1) = \Psi(0,T), \\ &-\partial_t\Psi(0,t) - (\sigma^2/2)\xi''(x) - g\xi'(x) \le f(x,t); \end{aligned} \tag{a}$$

$$\begin{aligned} &\Psi(0,T) = \Psi(1,T), \\ &\xi'(x) \ge -\phi(x,t) + \Psi(0,t) - \Psi(1,t), \\ &\xi(0) = \xi(1) = \Psi(0,T), \end{aligned} \tag{b}$$

$$\begin{aligned} -\partial_t\Psi(0,t) &- [\partial_t\Psi(1,t) - \partial_t\Psi(0,t)]x \\ &-(\sigma^2/2)\xi''(x) - g\xi'(x) - g[\Psi(1,t) - \Psi(0,t)] \le f(x,t). \end{aligned}$$

Proof. (a) Consider

$$w = \Psi(0,t) + (\Psi(1,t) - \Psi(0,t))x^p + \xi(x) - \Psi(0,T)$$

for $p \gg 1$. The boundary and initial conditions are seen to be satisfied. One can also check:

$$\begin{aligned} -\partial_x w &= -\partial_x\xi - p(\Psi(1,t) - \Psi(0,t))x^{p-1} \le -\partial_x\xi \le \phi; \\ -\partial_t w &- (\sigma^2/2)\partial_x^2 w - g\partial_x w \\ &= -\partial_t\Psi(0,t) - [\partial_t\Psi(1,t) - \partial_t\Psi(0,t)]x^p \\ &\quad - \sigma^2/2\{p(p-1)[\Psi(1,t) - \Psi(0,t)]x^{p-2} + \xi''(x)\} \\ &\quad - gp(\Psi(1,t) - \Psi(0,t))x^{p-1} - g\xi'(x) \\ &= -\partial_t\Psi(0,t) - (\sigma^2/2)\xi''(x) - g\xi'(x) - x^{p-2}\{[\sigma^2 p(p-1)/2 \\ &\quad + gpx][\Psi(1,t) - \Psi(0,t)] + x^2[\partial_t\Psi(1,t) - \partial_t\Psi(0,t)]\} \\ &\le -\partial_t\Psi(0,t) - (\sigma^2/2)\xi''(x) - g\xi''(x). \end{aligned}$$

The last inequality holds as long as $p \gg 1$. Thus we can get (3.8) from (a).

(b) Consider

$$w = \Psi(0,t) + (\Psi(1,t) - \Psi(0,t))x + \xi(x) - \Psi(0,T).$$

Then similar arguments prove the desired result.

Remark 3.1 The sufficient conditions (a) and (b) for (3.8) to hold are very special. We do not intend to get any more general sufficient conditions.

Lemma 3.2. *For the optimal cost* u^ε, $\|\partial_x u^\varepsilon\|_{C(G)} \le C$ *(independent of* ε*).*

Proof. Let us consider a regularised version of (3.6)

$$\begin{aligned}
&-\partial_t u^{\varepsilon,\delta} - \sigma^2\partial_x^2 u^{\varepsilon,\delta}/2 - g\partial_x u^{\varepsilon,\delta} + h_\delta(-\partial_x u^{\varepsilon,\delta} - \phi)/\varepsilon = f, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0,1)\times[0,T), \\
&u^{\varepsilon,\delta}(0,t) = \Psi(0,t), \qquad u^{\varepsilon,\delta}(1,t) = \Psi(1,t)\ \forall t\in[0,T), \\
&u^{\varepsilon,\delta}(x,T) = \xi(x)\ \forall x\in[0,1], \\
&u^{\varepsilon,\delta}\in \overline{C}_{2+r}(G) \quad \text{such that } \partial_x^3 u^{\varepsilon,\delta}, \partial_t^2 u^{\varepsilon,\delta}, \partial_{xt}^2 u^{\varepsilon,\delta}, \text{ and } \partial_{xxt}^3 u^{\varepsilon,\delta} \\
&\text{are all locally Hölder continuous with exponent } r \text{ in } G,
\end{aligned} \tag{3.9}$$

where

$$h_\delta(t) = \begin{cases} 0, & t\in(-\infty,0], \\ (t-\delta\sin(t/\delta))/2, & t\in(0,\delta\pi], \\ t-\delta\pi/2 & t\in(\delta\pi,\infty). \end{cases}$$

Let us observe that

$$h_\delta\in C^2, \qquad h_\delta \to \{\cdot\}^+ \text{ uniformly in } R^1 \text{ as } \delta\to 0,$$
$$0\le h_\delta(t)\le\{t\}^+, \qquad h'_\delta(t)\ge 0.$$

The existence of a solution to (3.9) is guaranteed, e.g., by [5, Theorems 3.7 and 3.10]. In fact, the $\overline{C}_{2+r}$ regularity readily follows from [5, Theorem 3.7] as in the proof of Theorem 3.1. Let $\hat f = f - h_\delta(-\partial_x u^{\varepsilon,\delta} - \phi)/\varepsilon$. Then $\partial_x \hat f$ is locally Hölder continuous in G. Hence, [5, Theorem 3.10] implies that $\partial_x^3 u^{\varepsilon,\delta}$ and $\partial_{xt}^2 u^{\varepsilon,\delta}$ are locally Hölder continuous in G. Consequently, $\partial_x^2 \hat f$ is locally Hölder continuous in G. Using [5, Theorem 3.10] once again, we get that $\partial_x^4 u^{\varepsilon,\delta}$ and $\partial_{xxt}^3 u^{\varepsilon,\delta}$ are locally Hölder continuous in G. Now it follows from (3.9) that $\partial_t u^{\varepsilon,\delta}$, given by

$$-\sigma^2\partial_x^2 u^{\varepsilon,\delta}/2 - g\partial_x u^{\varepsilon,\delta} + h_\delta(-\partial_x u^{\varepsilon,\delta} - \phi)/\varepsilon - f,$$

has the partial derivative with respect to t, and $\partial_t^2 u^{\varepsilon,\delta}$ is locally Hölder continuous. Thus, we have proved Hölder continuity of $\partial_t^2 u^{\varepsilon,\delta}$, $\partial_x^3 u^{\varepsilon,\delta}$, $\partial_{xxt}^3 u^{\varepsilon,\delta}$, $\partial_{xt}^2 u^{\varepsilon,\delta}$ and $\partial_x^2 u^{\varepsilon,\delta}$.

We also see that (cf. [5, P. 81]) $u^{\varepsilon,\delta}\to u^\varepsilon$ in $\overline{C}_{2+0}(G^*)$, for any closed subset G^* of G. Thus we need only show the boundedness of $\partial_x u^{\varepsilon,\delta}$ uniformly in (δ,ε,x,t).

(a) estimation of $\partial_x u^{\varepsilon,\delta}$ on $\{0,1\} \times [0,T]$:

To simplify the notation, let us write $u = u^{\varepsilon,\delta}$. Consider $\hat{u}$ satisfying

$$\begin{aligned} &-\partial_t \hat{u} - \sigma^2 \partial_x^2 \hat{u}/2 - g\partial_x \hat{u} = f, \quad \text{in } (0,1) \times [0,T), \\ &\hat{u}(0,t) = \Psi(0,t), \qquad \hat{u}(1,t) = \Psi(1,t)\ \forall t \in [0,T), \\ &\hat{u}(x,T) = \xi(x) \quad \forall x \in [0,1], \qquad \hat{u} \in \overline{C}_{2+r}(G). \end{aligned} \tag{3.10}$$

Then we have $w(x,t) \leq u(x,t) \leq \hat{u}(x,t)$, where $w(\cdot,\cdot)$ is given by (3.8). Hence we get the required estimate since $w(\cdot,\cdot)$ and $\hat{u}(\cdot,\cdot)$ are independent of (δ,ε).

(b) estimation of $\partial_x u^{\varepsilon,\delta}$ in G:

Let $\Phi(x,t) = \int_0^x \phi(y,t)\,dy$. Then setting $w = u + \Phi$, we get

$$-\partial_t w - \sigma^2 \partial_x^2 w/2 - g\partial_x w + h_\delta(-\partial_x w)/\varepsilon = f - g\phi - \partial_t \Phi - \sigma^2 \partial_x \phi/2 = \hat{f}. \tag{3.11}$$

Set $v = e^{\lambda t}(w_x^2 + 1)$ with $\lambda > 0$ to be fixed later (w_x^2 standing for $w_x w_x$). Then

$$\begin{aligned} v_t &= \lambda e^{\lambda t}(w_x^2 + 1) + 2e^{\lambda t} w_x w_{xt}, \\ v_x &= 2e^{\lambda t} w_x w_{xx}, \\ v_{xx} &= 2e^{\lambda t}(w_{xx}^2 + w_x w_{xxx}). \end{aligned}$$

Consider next

$$\begin{aligned} Lv &\equiv -v_t - (\sigma^2/2)v_{xx} - gv_x - (1/\varepsilon)h'_\delta(-w_x)v_x \\ &= -\lambda e^{\lambda t}(w_x^2 + 1) - 2e^{\lambda t} w_x w_{xt} - \sigma^2 e^{\lambda t}(w_{xx}^2 + w_x w_{xxx}) \\ &\quad - 2ge^{\lambda t} w_x w_{xx} - (2/\varepsilon)h'_\delta(-w_x)e^{\lambda t} w_x w_{xx} \\ &= 2e^{\lambda t} w_x[-w_{xt} - (\sigma^2/2)w_{xxx} - gw_{xx} - (1/\varepsilon)h'_\delta w(-w_x)w_{xx}] \\ &\quad - \lambda e^{\lambda t}(w_x^2 + 1) - \sigma^2 e^{\lambda t} w_{xx}^2 \\ &= 2e^{\lambda t} w_x(\hat{f}_x + \sigma\sigma_x w_{xx} + g_x w_x) - \lambda e^{\lambda t}(w_x^2 + 1) - \sigma^2 e^{\lambda t} w_{xx}^2 \\ &\leq -\lambda e^{\lambda t}(w_x^2 + 1)/2 \\ &\leq -\lambda v/2, \end{aligned}$$

when λ is large enough. Then the classical maximum principle implies that

$$\begin{aligned} \max_{(x,t)\in\overline{G}} v(x,t) &= \max\left\{\max_{0\leq x\leq 1} v(x,T), \max_{0\leq t\leq T} v(0,t), \max_{0\leq t\leq T} v(1,t)\right\} \\ &\leq \max\left\{\max_{0\leq x\leq 1}[(\xi'(x) + \phi(x,T))^2 + 1],\right. \\ &\qquad \max_{0\leq t\leq T}[(\partial_x u(0,t) + \phi(0,t))^2 + 1], \\ &\qquad \left.\max_{0\leq t\leq T}[(\partial_x u(1,t) + \phi(1,t))^2 + 1]\right\} e^{\lambda T} \\ &\leq C(\text{independent of } \delta \text{ and } \varepsilon), \end{aligned}$$

which completes the proof of the lemma.

Lemma 3.3. *For the optimal penalised expected cost u^ε, $\|\partial_t u^\varepsilon\|_{C(G)} \le C$ (independent of ε).*

Proof. As in the proof of Lemma 3.2, it suffices to show that $\|\partial_t u^{\varepsilon,\delta}\|_{C(G)} \le C$, uniformly in (δ,ε). Recalling in view of (3.8) and (3.10) that $w(x,t) \le u(x,t) \le \hat{u}(x,t)$, we get a uniform estimate of $\partial_t u|_{t=T}$.

To get the interior estimate of $\partial_t u$, let us consider $v = e^{\lambda t}(w_t^2 + kw_x^2 + 1)$, where w is the same as that in (3.11). Then

$$\begin{aligned} v_t &= \lambda e^{\lambda t}(w_t^2 + kw_x^2 + 1) + 2e^{\lambda t}(w_t w_{tt} + kw_x w_{xt}), \\ v_x &= 2e^{\lambda t}(w_t w_{xt} + kw_x w_{xx}), \\ v_{xx} &= 2e^{\lambda t}(w_{xt}^2 + w_t w_{xxt} + kw_{xx}^2 + kw_x w_{xxx}). \end{aligned}$$

Consider, as before,

$$\begin{aligned} Lv = &-\lambda e^{\lambda t}(w_t^2 + kw_x^2 + 1) - 2e^{\lambda t}(w_t w_{tt} + kw_x w_{xt}) \\ &- \sigma^2 e^{\lambda t}(w_{xt}^2 + w_t w_{xxt} + kw_{xx}^2 + kw_x w_{xxx}) \\ &- 2ge^{\lambda t}(w_t w_{xt} + kw_x w_{xx}) - (2/\varepsilon)h_\delta'(-w_x)e^{\lambda t}(w_t w_{xt} + kw_x w_{xx}) \\ = &-\lambda e^{\lambda t}(w_t^2 + kw_x^2 + 1) - \sigma^2 e^{\lambda t}(w_{xt}^2 + kw_{xx}^2) \\ &+ 2e^{\lambda t}w_t[-w_{tt} - (\sigma^2/2)w_{xxt} - gw_{xt} - (1/\varepsilon)h_\delta'(-w_x)w_{xt}] \\ &+ 2ke^{\lambda t}w_x[-w_{xt} - (\sigma^2/2)w_{xxx} - gw_{xx} - (1/\varepsilon)h_\delta'(-w_x)w_{xx}] \\ = &-\lambda e^{\lambda t}(w_t^2 + kw_x^2 + 1) - \sigma^2 e^{\lambda t}(w_{xt}^2 + kw_{xx}^2) \\ &+ 2e^{\lambda t}w_t[\hat{f}_t + \sigma\sigma_t w_{xx} + g_t w_x] + 2ke^{\lambda t}w_x(\hat{f}_x + \sigma\sigma_x w_{xx} + g_x w_x) \\ \le &-\lambda v/2, \end{aligned}$$

as long as $\lambda(>0)$ and $k(>0)$ are sufficiently large. Consequently, the maximum principle implies that

$$\begin{aligned} \max_{(x,t)\in\overline{G}} v(x,t) &= \max\left\{\max_{0\le x\le 1} v(x,T), \max_{0\le t\le T} v(0,t), \max_{0\le t\le T} v(1,t)\right\} \\ &\le \max\Big\{\max_{0\le x\le 1}[(\partial_t u(x,T) + \partial_t\Phi(x,T))^2 + kw_x^2(x,T) + 1], \\ &\qquad \max_{0\le t\le T}[(\partial_t\Psi(0,t) + \partial_t\Phi(0,t))^2 + kw_x^2(0,t) + 1], \\ &\qquad \max_{0\le t\le T}[(\partial_t\Psi(1,t) + \partial_t\phi(1,t))^2 + kw_x^2(1,t) + 1]\Big\} e^{\lambda T} \\ &\le C \quad (\text{independent of } \delta \text{ and } \varepsilon), \end{aligned}$$

from which we easily get the desired estimate.

Lemma 3.4. *For the optimal penalised expected cost u^ε,*

$$(1/\varepsilon)\{-\partial_x u^\varepsilon(x,t) - \phi(x,t)\}^+ \le C \quad (\textit{independent of } x,t,\varepsilon).$$

https://doi.org/10.1017/S0334270000006500 Published online by Cambridge University Press

Proof. (3.8) and (3.10) imply that

$$w(x,t) \le u^\varepsilon(x,t) \le \hat{u}(x,t).$$

Hence, $-\partial_x u^\varepsilon(0,t) - \phi(0,t) \le 0$.

Let $v = -\partial_x u^\varepsilon - \phi$. Then (3.6) implies that

$$\sigma^2/2\partial_x v + 1/\varepsilon\{v\}^+ = f + \partial_t u^\varepsilon - \sigma^2/2\partial_x \phi + g\partial_x u^\varepsilon \equiv \hat{f}.$$

Let $v(x') = \max_{0\le x\le 1} v(x,t)$ for any fixed $t \in [0,T]$. If $v(x') \le 0$, then $(1/\varepsilon)\{v\}^+ \le 0$ for all x. If $v(x') > 0$, then $x'' \in (0,1]$. Consequently, $\partial_x v(x',t) \ge 0$, and for all x

$$\begin{aligned}(1/\varepsilon)\{v(x,t)\}^+ &\le (1/\varepsilon)v(x',t) = (1/\varepsilon)\{v(x',t)\}^+ \\ &\le \hat{f}(x',t) \le \max\{\hat{f}(x,t)\colon x\} < \infty.\end{aligned}$$

Here we have used the results in Lemmas 3.2 and 3.3. So we get the desired inequality.

Remark 3.2. An easy consequence of Lemmas 3.2–3.4 is the uniform boundedness of $\partial_x^2 u^\varepsilon$, which was established by studying an auxiliary deterministic control problem in the stationary case (cf. [6]).

Let $L^{1,0}(\overline{G})$ be a subclass of functions in $C^{1,0}(\overline{G})$, defined by

$$\begin{aligned}L^{1,0}(\overline{G}) = \{u \in C^{1,0}(\overline{G})\colon &\partial_x u \text{ and } u \text{ are Lipschitz-continuous} \\ &\text{(in the usual sense) w.r.t. } x \text{ and } t, \text{ respectively}\}.\end{aligned}$$

We are now able to prove the main theorem in this article.

Theorem 3.2. *Under all the assumptions made so far,* $u(x,t) = \lim_{\varepsilon\to 0} u^\varepsilon(x,t)$ *exists, which is in* $L^{1,0}(\overline{G})$. *Moreover, the limiting function* $u(\cdot,\cdot)$ *is the value function with respect to* V *and is the maximum solution of the following variational inequality*

$$\begin{aligned}&-\partial_t u - \sigma^2\partial_x^2 u/2 - g\partial_x u \le f, \text{ and } -\partial_x u - \phi \le 0, \quad \text{a.e. in } (0,1)\times[0,T),\\ &(-\partial_t u - \sigma^2\partial_x^2 u/2 - g\partial_x u - f)(-\partial_x u - \phi) = 0, \quad \text{a.e. in } (0,1)\times[0,T),\\ &u(0,t) = \Psi(0,t), \qquad u(1,t) = \Psi(1,t)\ \forall t \in [0,T),\\ &u(x,t) = \xi(x)\ \forall x\in[0,1], \qquad u \in L^{1,0}(\bar{G}).\end{aligned} \tag{3.12}$$

Proof. In view of *a priori* estimates of $\partial_t u^\varepsilon$, $\partial_x u^\varepsilon$, $\partial_x^2 u^\varepsilon$, and $\{-\partial_x u^\varepsilon - \phi\}^+/3$, and the monotonicity of u^ε in ε, we should be able to take the limit in (3.6) as $\varepsilon \to 0$ to get (3.12) for some u in $L^{1,0}(\overline{G})$. By the stochastic interpretation of u^ε, we easily see that $u(x,t) \ge \inf\{J_{x,t}(v)\colon v \in V\}$. On the other hand, for any $v \in V$, let us apply a generalised Itô's formula to the semi-martingale $u(y(s),s)$ with $y(s) = y_{x,t}(s)$ and τ given by (3.2) in order to get

$$\begin{aligned} u(x,t) = E\Bigg\{ \int_t^{\tau\Lambda T} &[-\partial_t u - (\sigma^2/2)\partial_x^2 u - g\partial_x u](y(s),s)\,ds + u(y(\tau\Lambda T),\tau\Lambda T) \\ &+ \int_t^{\tau\Lambda T} -\partial_x u(y(s),s)\,dv^c(s-t) \\ &- \sum_{t<s\leq\tau\Lambda T} [u(y(s),s) - u(y(s-),s)] \Bigg\} \\ \leq E\Bigg\{ \int_t^{\tau\Lambda T} &f(y(s),s)\,ds + \Psi(y(\tau),\tau)I_{(\tau<T)} + \xi(y(T))I_{(\tau\geq T)} \\ &+ \int_t^{\tau\Lambda T} \phi(y(s),s)\,dv^c(s-t) + \sum_{t<s\leq\tau\Lambda T} \int_{y_x(s-)}^{y_x(s)} \phi(z,s)\,dz \Bigg\}. \end{aligned}$$

Consequently, $u(x,t) \leq \inf\{J_{x,t}(v) \colon v \in V\}$. Thus, we have proved that $u(x,t) = \inf\{J_{x,t}(v) \colon v \in V\}$.

The maximality of $u(x,t)$ as a solution to (3.12) follows from the arguments above.

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