A STRONG CONVERSE OF THE WIENER-LEVY THEOREM

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Introduction. Let A_1 denote the class of all complex functions on the unit circle which are sums of absolutely convergent trigonometric series; that is to say, the class of all f of the form

$$f(e^{i\theta}) = \sum_{-\infty}^{\infty} c_n e^{in\theta}, \qquad \sum_{-\infty}^{\infty} |c_n| < \infty.$$

Similarly, if $1 < p < \infty$, let A_p be the class of all complex functions f on the circle whose Fourier coefficients

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} d\theta$$

satisfy the condition

$$\sum_{-\infty}^{\infty} |\hat{f}(n)|^p < \infty.$$

If $p < q$, then A_p is obviously a proper subclass of A_q. The Wiener-Lévy theorem (**3**, p. 245) asserts that if F is analytic on the range of some $f \in A_1$, then $F(f) \in A_1$. Katznelson (**2**; **1**) has established the converse: if F is defined (for instance) on the interval $[-1, 1]$ of the real axis and if $F(f) \in A_1$ for all $f \in A_1$ whose range is in $[-1, 1]$, then F is analytic on $[-1, 1]$.

In the present paper we show that the same conclusion follows from weaker hypotheses; we assume only that for each $f \in A_1$ there is a $p < 2$ such that $F(f) \in A_p$, and we prove that F must then be analytic. For fixed $p < 2$, this result was conjectured by Malliavin.

In its major outlines, our proof is similar to that of Katznelson's theorem. One difference lies in the fact that we now have to prove the continuity of F. If F carries A_1 to A_1, then F is obviously continuous, since A_1 contains only continuous functions, and the analyticity of F may therefore be established by showing that the Fourier coefficients of F are sufficiently small. But if $p > 1$, then A_p contains discontinuous functions; hence F is not *a priori* continuous, and the knowledge of the Fourier coefficients of F determines F only, almost everywhere.

Instead of considering this problem on the unit circle, we can (and do) study it on an arbitrary infinite compact abelian group. The solution turns out

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to be the same, and except for Theorem 2, the greater generality causes very little additional work.

1. The spaces $A_p(G)$. Throughout this paper, abelian groups will be written additively, with 0 as identity, G will be an infinite compact abelian group, Γ will be the dual group of G, and the symbol (x, γ) will denote the value of the character $\gamma \in \Gamma$ at the point $x \in G$. The Fourier coefficients $\hat{f}(\gamma)$ of a Haar integrable function f on G are defined by

$$\hat{f}(\gamma) = \int_G f(x)(-x, \gamma)dx \qquad (\gamma \in \Gamma)$$

where dx denotes the Haar measure of G, normalized so that $\int_G dx = 1$.

For $1 \leqslant p < \infty$, we define

$$A_p[f] = \left\{ \sum_\gamma |\hat{f}(\gamma)|^p \right\}^{1/p}$$

and we let $A_p(G)$ be the set of all f on G for which the norm $A_p[f]$ is finite. For $p = 1$, we also stipulate that all $f \in A_1(G)$ are continuous; that is, these functions are defined everywhere, not just almost everywhere on G.

Hölder's inequality shows that if $f \in A_p(G)$ and $g \in A_1(G)$, then $fg \in A_p(G)$. In fact

$$A_p[fg] \leqslant A_p[f]A_1[g]. \tag{1}$$

THEOREM 1. *Suppose $p < 2$. There exists a constant $K_p > 1$, independent of G, such that, setting*

$$g_{\gamma_0}(x) = \exp\{i\mathrm{Re}(x, \gamma_0)\} \qquad (x \in G,\ \gamma_0 \in \Gamma)$$

we have

$$A_p[g_{\gamma_0}] \geqslant K_p \qquad (\gamma_0 \neq 0). \tag{2}$$

Proof. The left side of (2) depends only on the order of γ. Thus $A_p[g_{\gamma_0}] = K(p, m)$ if γ_0 has order m $(2 \leqslant m \leqslant \infty)$. Since $|g_{\gamma_0}| = 1$, $A_2[g_{\gamma_0}] = 1$. Since g_{γ_0} is not a constant multiple of a character, it follows that $|\hat{g}_{\gamma_0}| < 1$, hence $|\hat{g}_{\gamma_0}|^2 < |\hat{g}_{\gamma_0}|^p$ at all points where $\hat{g}_{\gamma_0} \neq 0$. Hence $K(p, m) > 1$ for all m. As $m \to \infty$, $K(p, m) \to K(p, \infty)$. Thus, setting $K_p = \inf_m K(p, m)$, we have $K_p > 1$, and (2) holds.

THEOREM 2. *Suppose $p < 2$. For each positive integer r,*

$$\sup_Q A_p[e^{iQ}] \geqslant K_p^r.$$

Here Q ranges over all real trigonometric polynomials on G with $A_1[Q] \leqslant r$.

Proof. We shall show that to each number $\mu < 1$ and to each positive integer r there exist characters $\gamma_1, \ldots, \gamma_r \in \Gamma$ such that

$$A_p[g_{\gamma_1}g_{\gamma_2}\ldots g_{\gamma_r}] > (\mu K_p)^r. \tag{3}$$

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Setting $Q(x) = \text{Re}[(x, \gamma_1) + \ldots + (x, \gamma_r)]$, the theorem then follows from (3). On the other hand, Theorem 1 shows that (3) is a consequence of

$$A_p[g_{\gamma_1}g_{\gamma_2}\ldots g_{\gamma_r}] > \mu^r A_p[g_{\gamma_1}]\ldots A_p[g_{\gamma_r}]. \tag{4}$$

Clearly (4) is true for $r = 1$. Assume (4) holds for some r, and put $f = g_{\gamma_1}\ldots g_{\gamma_r}$. If we can find γ_{r+1} such that

$$A_p[fg_{\gamma_{r+1}}] > \mu A_p[f]A_p[g_{\gamma_{r+1}}], \tag{5}$$

the proof is complete, by induction.

Since $f \in A_1(G)$, f can be approximated in the norm of $A_1(G)$ (hence also in the norm of $A_p(G)$) by partial sums of its Fourier series. The inequality (1) then shows that it suffices to prove (5) under the assumption that f is a trigonometric polynomial. Let E' be the support of $\hat{f}$, that is, the set of all γ for which $\hat{f}(\gamma) \neq 0$. Then E' is a finite set.

The remainder of this proof is motivated by the following easily proved remark: if $h \in A_p(G)$, if E'' is the support of $\hat{h}$, and if no $\gamma \in \Gamma$ has more than one representation of the form $\gamma = \gamma' + \gamma''$, then $A_p[fh] = A_p[f]A_p[h]$.

It is now convenient to consider three cases.

(a) If Γ contains an element γ_0 of infinite order, we can choose an integer m so large that the cyclic group E'' generated by $\gamma_{r+1} = m\gamma_0$ satisfies the above condition; then

$$A_p[fg_{\gamma_{r+1}}] = A_p[f]A_p[g_{\gamma_{r+1}}]. \tag{6}$$

(b) If Γ is of bounded order, then Γ is a direct sum of finite cyclic groups, hence we can find γ_{r+1} such that the cyclic group E'' generated by γ_{r+1} has only 0 in common with the finite group generated by E'; this again gives (6).

(c) In the remaining case, Γ is not of bounded order, but every element of Γ has finite order, and the group Λ generated by E' is thus finite. To each integer N we can find γ_{r+1} such that $m\gamma_{r+1} \notin \Lambda$ if $|m| < 2N$ (otherwise Γ would be of bounded order). We have

$$g_{\gamma_{r+1}}(x) = \sum_m a_m(x, m\gamma_{r+1}), \qquad \sum_m |a_m| < \infty. \tag{7}$$

Let $h(x)$ be the partial sum of (7), consisting of those terms which have $|m| < N$. Then $A_p[fh] = A_p[f]A_p[h]$. Taking N large enough, $A_1[g_{\gamma_{r+1}} - h]$ can be made as small as desired, and thus (5) can be achieved.

THEOREM 3. *Suppose* $\{f_n\}$ *is a sequence of functions on* G, $|f_n(x)| \leqslant M_1$, *and* $A_p[f_n] \leqslant M_2$ *for* $n = 1, 2, 3, \ldots, x \in G$. *If* $f_n(x) \to f(x)$ *almost everywhere on* G, *then* $A_p[f] \leqslant M_2$.

Proof. By Lebesgue's convergence theorem, $\hat{f}_n(\gamma) \to \hat{f}(\gamma)$ for all $\gamma \in \Gamma$, and hence

$$\sum_\gamma |\hat{f}(\gamma)|^p \leqslant \liminf_{n\to\infty} \sum_\gamma |\hat{f}_n(\gamma)|^p \leqslant M_2^p.$$

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THEOREM 4. *If $p < 2$, then*

$$\sup_E A_p[\chi_E] = \infty.$$

Here E ranges over all closed subsets of G, and χ_E is the characteristic function of E.

Proof. Suppose, on the contrary, that $A_p[\chi_E] \leqslant M < \infty$ for all closed sets E in G. The same is then true for all measurable sets E, by Theorem 3. If f is a simple measurable function, $0 \leqslant f \leqslant 1$, then $f = \sum \alpha_i \chi_{E_i}$ with $\sum \alpha_i \leqslant 1$, and so $A_p[f] \leqslant M$. Since every complex bounded measurable function on G is a uniform limit of simple functions, we conclude that

$$A_p[f] \leqslant 4M\|f\|_\infty \quad (f \in L^\infty(G)). \tag{8}$$

Now choose $\gamma_1, \ldots, \gamma_N \in \Gamma$ such that no $\gamma \in \Gamma$ has more than one representation of the form $\gamma = \gamma_i + \gamma_j$, and put

$$g(x) = \sum_1^N (x, \gamma_i).$$

If $\|f\|_\infty \leqslant 1$, (8) implies

$$\left|\int_G f(x)g(-x)dx\right| = |\sum \hat{f}(\gamma)\hat{g}(\gamma)| \leqslant A_p[f]A_q[g] \leqslant 4M \cdot N^{1/q},$$

where $1/p + 1/q = 1$, and hence

$$\int_G |g| \leqslant 4M \cdot N^{1/q}. \tag{9}$$

On the other hand, the Fourier coefficients of g^2 are 1 in N places, 2 in $N(N-1)/2$ places, so that

$$\int |g|^4 = N + 2N(N-1) < 2N^2.$$

By Hölder's inequality, we therefore have

$$N = \int |g|^2 \leqslant (\int |g|)^{2/3} (\int |g|^4)^{1/3} < (\int |g|)^{2/3}\, 2^{1/3} N^{2/3},$$

and hence

$$\int |g| > (N/2)^{1/2}. \tag{10}$$

Since $q > 2$, (9) and (10) are in contradiction if N is large.

Remark. More is true: there is a closed set E in G such that $\chi_E \notin A_p(G)$. This can be deduced from Theorem 4, and also follows from our main theorem: take $F(s) = 0$ except at one point s_0 where $F(s_0) = 1$; for $f \in A_1(G)$, $F(f)$ is then the characteristic function of the set on which $f = s_0$. The proof of Theorem 4 can also be adapted to show that there are continuous functions on G which are not in $A_p(G)$.

2. The main theorem.

THEOREM 5. *Suppose G is an infinite abelian group. Suppose F is a complex* *function defined on the interval* $[-1, 1]$, *with the following property: if* $f \in A_1(G)$ *and if* $-1 \leqslant f(x) \leqslant 1$ *for all* $x \in G$, *then* $F(f) \in A_p(G)$ *for some* $p < 2$; *a priori*, p *may depend on* f. *Then* F *is analytic on* $[-1, 1]$.

It will be convenient to reduce this to a slightly different version:

THEOREM 5′. *Suppose* G *is as above,* F *is defined on the real line* R, *and for each real* $f \in A_1(G)$, *we have* $F(f) \in A_p(G)$ *for some* $p < 2$. *Then* F *is analytic at the origin.*

Suppose Theorem 5′ is proved and F_1 satisfies the hypotheses of Theorem 5. Put $F(s) = F_1(\sin s)$. Since $\sin f \in A_1$ if $f \in A_1$, F is analytic on R, and so $F_1(s) = F(\arcsin s)$ is analytic in a neighbourhood of $s = 0$. By translation, F_1 is analytic on the open segment $(-1, 1)$. To deal with the endpoints, put $F_2(t) = F_1(1 - t^2)$. The preceding work shows that F_2 is analytic at $t = 0$; being even, $F_2(t) = \sum_0^\infty a_n t^{2n}$ for small t, and so $F_1(1 - s) = \sum_0^\infty a_n s^n$ for small positive s. The other endpoint is treated similarly.

We thus have to prove Theorem 5′. Since constants can be added to F without affecting the hypotheses, we shall assume that

$$F(0) = 0. \tag{11}$$

The proof is divided into four steps:

I. *Under the above hypotheses, there exist*
(a) *a neighbourhood* V *of* 0 *in* G,
(b) *numbers* $p < 2$, $\delta > 0$, $M < \infty$,
such that $A_p[F(f)] < M$ *for all real* $f \in A_1(G)$ *which vanish outside* V *and which have* $A_1[f] < \delta$.

II. *With* p *as in* I, *there exist numbers* $\eta > 0$, $B < \infty$, *such that* $A_p[F(f)] < B$ *for all real* f *with* $A_1[f] < \eta$.

III. F *is continuous.*

IV. F *is analytic at the origin.*

Proof of I. Let $V_1, V_2, V_3, \ldots,$ be non-empty open sets in G such that the closure of V_i does not intersect the closure of $\cup V_j$ $(j \neq i)$. Then there are real functions $\phi_i \in A_1(G)$ such that $\phi_i(x) = 1$ on V_i but $\phi_i(x) = 0$ on V_j if $j \neq i$.

Suppose I is false. Since all norms under consideration are translation invariant, it follows that there exist sequences $\{p_i\}$, $\{f_i\}$ such that $p_i < 2$, $\lim p_i = 2$, f_i is real, f_i has its support in V_i, and

$$A_1[f_i] < 1/i^2 \text{ but } A_{p_i}[F(f_i)] > iA_1[\phi_i].$$

Put $f = \sum_1^\infty f_i$. Then $f \in A_1(G)$, hence $F(f) \in A_p(G)$ for some $p < 2$. For $i = 1, 2, 3, \ldots,$ we have

$$F(f_i(x)) = \phi_i(x) F(f(x)) \qquad (x \in G); \tag{12}$$

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for if $x \in V_i$, then $\phi_i = 1$ and $f_i = f$; at all other points, both sides of (12) are 0, by (11). Hence

$$iA_1[\phi_i] < A_{p_i}[F(f_i)] \leqslant A_1[\phi_i]A_{p_i}[F(f)],$$

by (1), and so

$$(13) \qquad A_{p_i}[F(f)] > i \qquad (i = 1, 2, 3, \ldots).$$

But for all large enough i, we have $p < p_i < 2$, so that

$$A_{p_i}[F(f)] \leqslant \max \{A_p[F(f)], A_2[F(f)]\},$$

which contradicts (13).

Proof of II. Let p, V, δ, M be as in I. Let U, W be neighbourhoods of 0 in G such that $\bar{W} \subset U \subset \bar{U} \subset V$. There are finitely many translates of W, say $W_1, \ldots, W_n$, whose union covers G. Let $U_1, \ldots, U_n$, $V_1, \ldots, V_n$ be the corresponding translates of U and V, and choose non-negative functions $\alpha_i, \beta_i \in A_1(G)$ such that $\alpha_i = 0$ outside V_i, $\alpha_i = 1$ on U_i, $\beta_i = 0$ outside U_i, $\beta_i = 1$ on W_i $(1 \leqslant i \leqslant n)$. Define

$$\psi_i = \frac{\beta_i}{\beta_1 + \ldots + \beta_n}$$

and put

$$\eta = \inf_{1 \leqslant i \leqslant n} \frac{\delta}{A_1[\alpha_i]}, \qquad B = M \sum_{i=1}^{n} A_1[\psi_i].$$

(Note that $\psi_i \in A_1(G)$, since $\beta_1 + \ldots + \beta_n > 0$.)

If now f is real and $A_1[f] < \eta$, each of the functions $\alpha_i f$ satisfies the conditions of Step I (modulo a translation), so that $A_p[F(\alpha_i f)] < M$. If $\alpha_i(x) \neq 1$ then $\psi_i(x) = 0$; this shows that

$$\psi_i F(f) = \psi_i F(\alpha_i f).$$

Since $\sum \psi_i = 1$, we finally have therefore

$$A_p[F(f)] = A_p\left[\sum_i \psi_i F(f)\right] = A_p\left[\sum_i \psi_i F(\alpha_i f)\right]$$
$$\leqslant \sum_i A_p[\psi_i F(\alpha_i f)] \leqslant \sum_i A_1[\psi_i] A_p[F(\alpha_i f)] < B.$$

Proof of III. Suppose F has a discontinuity at the origin. Applying Step II to constant functions, we see that $|F(s)| < B$ if $|s| < \eta$. Hence there exists $\{s_i\}$, $s_i \to 0$, such that $F(s_i) \to \mu \neq 0$.

Let E be a closed set in G, let $\{C_n\}$ be an increasing sequence of closed sets in the complement of E, such that $m(E \cup C_n) \to 1$, and let $\{f_n\}$ be a sequence of real function in $A_1(G)$ which are 1 on E, 0 on C_n. To each n we can find i_n such that $|s_{i_n}| A_1[f_n] < \eta$. By Step II, we then have

$$A_p[F(s_{i_n} f_n)] < B \qquad (n = 1, 2, 3, \ldots).$$

But $F(s_{i_n}f_n) = F(s_{i_n})$ on E, 0 on C_n. Since $|F(s_{i_n}f_n(x))| < B$ for all n, x, and since $F(s_{i_n}f_n)(x) \to \mu \cdot \chi_E(x)$ almost everywhere, Theorem 3 shows that $A_p[\chi_E] \leqslant B/|\mu|$. Since we placed no restrictions on the closed set E, Theorem 4 is contradicted.

Thus F is continuous at the origin. Considering $F(s_0 + s) - F(s_0)$ in place of $F(s)$, we conclude that F is continuous on the whole line.

Proof of IV. Fix p, η, B as in Step II. Put

$$\Phi(s) = F\left(\frac{\eta}{e}\sin s\right) \qquad (s \text{ real}).$$

If $A_1[f] \leqslant 1$ and α is any real number, then

$$A_1[\sin(f+\alpha)] \leqslant |\cos\alpha| \,.\, A_1[\sin f] + |\sin\alpha| A_1[\cos f]$$

$$\leqslant \sum_{n=1}^{\infty} \frac{A_1[f]^{2n-1}}{(2n-1)!} + \sum_{n=0}^{\infty} \frac{A_1[f]^{2n}}{(2n)!} = \exp\{A_1[f]\} \leqslant e.$$

Hence if Q is a real trigonometric polynomial on G, with $A_1[Q] \leqslant 1$, Step II shows that

$$A_p[\Phi(Q+\alpha)] \leqslant B \qquad (\alpha \text{ real}). \tag{14}$$

Now fix Q as above, fix an integer n, and define

$$\phi_N(x) = \frac{1}{N}\sum_{j=1}^{N} \Phi\left(Q(x) + \frac{2\pi j}{N}\right)\exp\left\{-\frac{2\pi i n j}{N}\right\}.$$

By III, Φ is continuous; hence

$$\lim_{N\to\infty} \phi_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi(Q(x)+s)e^{-ins}ds$$

for all $x \in G$. This integral is equal to

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi(s)e^{in(Q(x)-s)}ds = \hat{\Phi}(n)e^{inQ(x)}.$$

By (14), Theorem 3 now shows that

$$|\hat{\Phi}(n)|A_p[e^{inQ}] \leqslant B\,; \tag{15}$$

taking the supremum of the left side of (15), with respect to Q, Theorem 2 gives

$$|\hat{\Phi}(n)| \leqslant BK_p^{-|n|} \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{16}$$

The series

$$\sum_{-\infty}^{\infty} \hat{\Phi}(n)e^{in(s+it)} \tag{17}$$

therefore converges absolutely if $|t| < \log K_p$; its sum is an analytic function of $s + it$ in a horizontal strip containing the real axis, and the continuity of Φ shows that Φ coincides with this sum on the whole real axis.

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Thus Φ is analytic on the real axis, and since $F(s) = \Phi(\text{arc sin } (es/\eta))$, F is analytic at the origin.

This completes the proof.

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