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Free Bessel Laws

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Abstract. We introduce and study a remarkable family of real probability measures π_{st} that we call free Bessel laws. These are related to the free Poisson law π via the formulae $\pi_{s1} = \pi^{\boxtimes s}$ and $\pi_{1t} = \pi^{\boxplus t}$. Our study includes definition and basic properties, analytic aspects (supports, atoms, densities), combinatorial aspects (functional transforms, moments, partitions), and a discussion of the relation with random matrices and quantum groups.

Introduction

In this paper, we continue a program initiated by two of the authors, especially in [4], that aims to find quantum analogues for classical groups of unitary matrices and to study relations between these quantum analogues and free probability. We introduce a new class of quantum groups that we denote by $A_h^s(n)$, where s, n are two positive integers.

In turn, these quantum groups are closely related to a remarkable two-parameter family of real probability measures that we call free Bessel laws and study in detail. These appear naturally in the context of Voiculescu's free probability theory [27]. In free probability, the central role is played by Wigner's semicircle law:

$$\gamma = \frac{1}{2\pi}\sqrt{4 - x^2}\,dx.$$

This measure appears in the free version of the central limit theorem in the same way as the Gaussian law appears in the classical case. Moreover, Wigner's result can be understood in this way. See [27]. An alternative approach is based on the analogy between the Poisson law and the Marchenko–Pastur law, also called the free Poisson law:

$$\pi = \frac{1}{2\pi}\sqrt{4x^{-1} - 1}\,dx.$$

The free Bessel laws π_{st} are natural two-parameter generalizations of π. They can be introduced in several ways, depending on the values of the parameters. In the case $s \in (0,\infty)$ and $t \in (0,1]$, which is the most important, π_{st} appears as a free compression of $\pi^{\boxtimes s}$ by a projection of trace t: $\pi_{st} = (\pi^{\boxtimes s})_t$. An alternative formula, which actually works for a larger class of parameters, makes use of both of Voiculescu's free convolution operations:

$$\pi_{st} = \pi^{\boxtimes s-1} \boxtimes \pi^{\boxplus t}.$$

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This latter formula, while a bit less transparent than the first one, makes clear the relation with π. Indeed, we have the following particular cases:

$$\begin{cases} \pi_{s1} = \pi^{\boxtimes s}, \\ \pi_{1t} = \pi^{\boxplus t}. \end{cases}$$

The measure π_{st} with $s \in \mathbb{N}$ appears as a free analogue of the following measure, having as density a kind of s-dimensional Bessel function

$$p_{st} = e^{-t} \sum_{p_1=0}^{\infty} \cdots \sum_{p_s=0}^{\infty} \frac{1}{p_1! \cdots p_s!} \left(\frac{t}{s}\right)^{p_1+\cdots+p_s} \delta\left(\sum_{k=1}^{s} e^{2\pi ik/s} p_k\right)^{s}.$$

The analogy between Bessel laws and free Bessel laws can be understood in several ways. For instance, if $a_1, \ldots, a_s/\alpha_1, \ldots, \alpha_s$ are independent/free variables, each of them following the Poisson/free Poisson law of parameter $s^{-1}t$, then

$$p_{st} = \operatorname{law}\left(\sum_{k=1}^{s} e^{2\pi ik/s} a_k\right)^{s}, \qquad \pi_{st} = \operatorname{law}\left(\sum_{k=1}^{s} e^{2\pi ik/s} \alpha_k\right)^{s}.$$

Summarizing, the free Bessel laws are natural generalizations of the free Poisson law, in connection with Voiculescu's operations $\boxtimes$ and $\boxplus$, and with the Bessel functions. In this paper, we perform a systematic study of these laws.

The point is that the free Bessel laws have a number of remarkable combinatorial properties, coming from a subtle relation with several key objects:

(i) Poisson laws. We prove that the supports, atoms, and densities, as well as the various functional transforms of π_{st} are given by formulae similar to those for the free Poisson laws.

(ii) Noncrossing partitions. We prove that the combinatorics of π_{st} with $s \in \mathbb{N}$ is encoded by the set NC_s of noncrossing partitions having blocks of size multiple of s, studied by Edelman [14], Stanley [23], and Armstrong [1].

(iii) Random matrices. We prove that the moments of π_{s1} with $s \in \mathbb{N}$ are the asymptotic moments of the mean empirical distribution of the eigenvalues of $(DW)^s$, where W is a Wishart matrix, and D is a diagonal matrix formed by uniformly distributed s-roots of unity.

(iv) Quantum groups. We prove that p_{st} with $s \in \mathbb{N}$ is related to the finite group $H_n^s = \mathbb{Z}_s \wr S_n$, and that π_{st} is related to the free version of H_n^s. The relation is via asymptotic laws of truncated characters.

The quantum group results are part of a "representation theory correspondence", which is currently under construction. The following table collects various results from [4–6] and from the present paper.

Lie group	Classical law	Quantum law
O_n	Gaussian	Semicircular
U_n	Complex Gaussian	Circular
S_n	Poisson	Free Poisson
H_n	2-Bessel	Free 2-Bessel
H_n^s	Bessel	Free Bessel

The measures in this table are related by the general correspondence found by Bercovici and Pata in [9]. The table itself can be thought of as providing a concrete realization of the main particular cases of the correspondence.

The noncrossing partition and random matrix results also seem to be part of some general correspondences, extending fundamental results about π. We hope to come back with more results in this sense in some future work.

The paper is organized in four parts, as follows:

(i) In Sections 1–3, we discuss the construction of the free Bessel laws, and their basic analytic and combinatorial properties.
(ii) In Sections 4–6, we discuss the relation with noncrossing partitions, the moment formula, and the random matrix models.
(iii) In Sections 7–9, we discuss some free additivity properties, the classical analogues, and the compound Poisson law interpretations.
(iv) In 10–12 we discuss representation theory aspects, with the finite group model for p_{st}, and the free quantum group model for π_{st}.

Notations

Associated with a real probability measure having sequence of moments $m_1, m_2, m_3, \dots$ are the following functional transforms:

(i) Stieltjes transform: $f(z) = 1 + m_1 z + m_2 z^2 + \cdots$
(ii) ψ transform: $\psi(z) = f(z) - 1$.
(iii) χ transform: $\psi(\chi(z)) = z$.
(iv) S transform: $S(z) = (1 + z^{-1})\chi(z)$.
(v) Cauchy transform: $G(\xi) = \xi^{-1} f(\xi^{-1})$.
(vi) K transform: $G(K(z)) = z$.
(vii) R transform: $R(z) = K(z) - 1/z$.
(viii) η transform: $\eta(z) = 1 - 1/f(z)$.
(ix) Σ transform: $\Sigma(z) = S(z/(1 - z))$.

Here, all the notations, except maybe for that of the Stieltjes transform, are the standard ones from the free probability literature [17, 20, 26, 27].

1 Definition, Basic Properties

The origins of free probability theory go back to Voiculescu's noncommutative central limit theorem, where the Gaussian law is replaced by Wigner's semicircle law [24]. Since then, the analogy between the Gaussian law and the semicircle law has served as a guideline for the whole theory. See [27].

For the purposes of this paper, the guiding analogy will be that between the Poisson law and the Marchenko–Pastur law, also called free Poisson law.

We recall that the Poisson law is the following probability measure:

$$p = \frac{1}{e} \sum_{r=0}^{\infty} \frac{\delta_r}{r!}$$

According to general results in free probability, the free analogue of this measure can be introduced in the following way.

Definition 1.1 The free Poisson law is given by

$$\pi = \frac{1}{2\pi}\sqrt{4x^{-1}-1}\,dx.$$

The support of this measure is the interval where the square root is real, namely $[0,4]$. This measure is also called the Marchenko–Pastur law. See [26].

We denote by $\boxplus$ and $\boxtimes$ the free additive and multiplicative convolutions, and we use Voiculescu's R and S transforms, which linearize them. See [24, 25].

Given a real probability measure μ, one can ask whether the convolution powers $\mu^{\boxtimes s}$ and $\mu^{\boxplus t}$ exist for various values of $s,t > 0$. The problem makes sense because of the one-to-one correspondence between measures and their transforms. More precisely, the question is whether $S_s(z) = S(z)^s$ and $R_t(z) = tR(z)$ are the S and R transforms of some real probability measures.

For the free Poisson law, the answer to these questions is well known. We include here the precise statement, along with a complete proof. This will serve as a model for some subsequent generalizations.

Theorem 1.2 *The measures $\pi^{\boxtimes s}$, $\pi^{\boxplus t}$ exist for any $s,t > 0$.*

Proof The free Poisson law π, as introduced in Definition 1.1, is the $t = 1$ particular case of the free Poisson law of parameter t, given by

$$\pi_t = \max(1-t,0)\delta_0 + \frac{\sqrt{4t-(x-1-t)^2}}{2\pi x}\,dx.$$

The Cauchy transform of this measure is given by

$$G(\xi) = \frac{(\xi+1-t)+\sqrt{(\xi+1-t)^2-4\xi}}{2\xi}.$$

We can now compute the R transform by proceeding as follows:

$$\begin{aligned}
\xi G^2 + 1 = (\xi+1-t)G &\implies Kz^2+1 = (K+1-t)z\\
&\implies Rz^2+z+1 = (R+1-t)z+1\\
&\implies Rz = R-t\\
&\implies R = t/(1-z).
\end{aligned}$$

This expression being linear in t, the measures π_t form a semigroup with respect to free convolution. Thus we have $\pi_t = \pi^{\boxplus t}$, which proves the second assertion.

Regarding the measure $\pi^{\boxtimes s}$, there is no explicit formula for its density. However, we can prove that this measure exists by using some abstract results.

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We have the following computation for the S transform of π_t:

$$\begin{aligned}
\xi G^2 + 1 = (\xi + 1 - t)G &\implies z f^2 + 1 = (1 + z - zt) f \\
&\implies z(\psi + 1)^2 + 1 = (1 + z - zt)(\psi + 1) \\
&\implies \chi(z + 1)^2 + 1 = (1 + \chi - \chi t)(z + 1) \\
&\implies \chi(z + 1)(t + z) = z \\
&\implies S = 1/(t + z).
\end{aligned}$$

In particular, at $t = 1$ we have $S(z) = 1/(1 + z)$, so the Σ transform of π, which is by definition $\Sigma(z) = S(z/(1 - z))$, is given by $\Sigma(z) = 1 - z$.

The existence of $\pi^{\boxtimes s}$ follows now from general results in [10]. Indeed, it is shown there that the Σ transforms of the probability measures that are $\boxtimes$-infinitely divisible are the functions of the form $\Sigma(z) = e^{v(z)}$, where $v\colon \mathbb{C} - [0, \infty) \to \mathbb{C}$ is analytic, satisfying $v(\bar{z}) = \bar{v}(z)$ and $v(\mathbb{C}^+) \subset \mathbb{C}^-$ (here, and in what follows, we denote by $\mathbb{C}^+$ and $\mathbb{C}^-$ the upper and lower half-plane).

In the case of the free Poisson law, the function $v(z) = \log(1 - z)$ satisfies all the above properties, and this gives the result. ■

The starting point for the considerations in the present paper is the following remarkable identity.

Theorem 1.3 *For $s \geq 1$ and $t \in (0, 1]$ we have*

$$\pi^{\boxtimes s-1} \boxtimes \pi^{\boxplus t} = ((1 - t)\delta_0 + t\delta_1) \boxtimes \pi^{\boxtimes s}.$$

Proof We know from the previous proof that the S transform of π is given by $S(z) = 1/(1 + z)$, and that the S transform of $\pi^{\boxplus t}$ is given by $S(z) = 1/(t + z)$. Thus, the measure on the left has the following S transform:

$$S(z) = \frac{1}{(1 + z)^{s-1}} \cdot \frac{1}{t + z}.$$

The S transform of $\alpha_t = (1 - t)\delta_0 + t\delta_1$ can be computed as follows:

$$\begin{aligned}
f = 1 + tz/(1 - z) &\implies \psi = tz/(1 - z) \\
&\implies z = t\chi/(1 - \chi) \\
&\implies \chi = z/(t + z) \\
&\implies S = (1 + z)/(t + z).
\end{aligned}$$

This shows that the measure on the right has the following S transform:

$$S(z) = \frac{1}{(1 + z)^s} \cdot \frac{1 + z}{t + z}.$$

Thus, the S transforms of our two measures are the same, and we are done. ■

We are now in a position to introduce a remarkable two-parameter family of real probability measures. We call them free Bessel laws because of a certain relationship with the Bessel functions to be discussed later on.

Definition 1.4 The free Bessel law is the real probability measure π_t with $(s,t) \in (0,\infty) \times (0,\infty) - (0,1) \times (1,\infty)$, defined as follows:

(i) For $s \geq 1$, we set $\pi_{st} = \pi^{\boxtimes s-1} \boxtimes \pi^{\boxplus t}$.
(ii) For $t \leq 1$, we set $\pi_{st} = ((1-t)\delta_0 + t\delta_1) \boxtimes \pi^{\boxtimes s}$.

The compatibility between (i) and (ii) comes from Theorem 1.3.

We regard the free Bessel law π_{st} as being a natural two-parameter generalization of the free Poisson law π, in connection with Voiculescu's free convolution operations $\boxtimes$ and $\boxplus$. Observe that we have the following formulae:

$$\begin{cases} \pi_{s1} = \pi^{\boxtimes s}, \\ \pi_{1t} = \pi^{\boxplus t}. \end{cases}$$

Concerning the precise range of the parameters (s,t), the above results can probably be improved. The point is that the measure π_{st} still exists for certain points in the critical rectangle $(0,1) \times (1,\infty)$ but not for all of them.

We did a number of abstract or numeric checks in this sense, and the critical values of (s,t) seem to form an algebraic curve contained in $(0,1) \times (1,\infty)$, having $s = 1$ as an asymptote. However, the case we are the most interested in is $t \in (0,1]$, and here there is no problem: π_{st} exists for any $s > 0$.

2 The Measures π_{s1}

In this section and the next, we study the support, atoms, and density of π_{st}. As in the $s = 1$ case, the formulae depend on whether t is bigger, smaller, or equal to 1. We start with a complete study in the $t = 1$ case.

We will use a well-known result of Lindelöf several times, stating that an analytic function $g\colon \mathbb{C}^+ \to \mathbb{C}$ has nontangential limit a at a point x provided that $g(\mathbb{C}^+)$ omits at least two points of $\mathbb{C}$, and that we have $\lim_{t\to 1} g(\gamma(t)) = a$ for a certain path $\gamma \subset \mathbb{C}^+$, tending to x in the frontier of $\mathbb{C}^+$ as $t \to 1$.

Theorem 2.1 *The measure π_{s1} has the following properties:*

(i) *There are no atoms.*
(ii) *The support is $[0,K]$ where $K = (s+1)^{s+1}/s^s$.*
(iii) *The density is analytic on $(0,K)$.*
(iv) *The density is bounded at $x = K$, and is $\sim 1/(\pi x^{s/(s+1)})$ at $x = 0$.*

Proof We denote by G, η the Cauchy and η transforms of π_{s1}. We have

$$G(1/z) = \frac{z}{1-\eta(z)}.$$

We recall that at $s = 1$ the eta transform is

$$\eta_1(w) = \frac{1 - \sqrt{1 - 4w}}{2}.$$

In the case $s = 1$ the measure π_{11} is the free Poisson law, and all the assertions are clear from the explicit formula of the density. At $s \neq 1$ we have 2 cases.

Case 1: $s < 1$. We know that η is the right inverse of $\Phi(w) = w(1-w)^s$. The function Φ is analytic on $\mathbb{C} - [1, \infty)$, and the derivative is

$$\Phi'(w) = (1 - w)^{s-1}(1 - (s + 1)w).$$

This derivative vanishes at $w = 1/(s+1)$, and we have $\Phi(1/(s+1)) = 1/K$. Also, it follows that the right inverse η extends analytically from $-\infty$ up to $1/(s + 1)$. By [8], the restriction of η to $\mathbb{C}^+$ extends continuously and injectively to $\mathbb{R}$. Thus, $\eta(1/K, \infty)$ is a simple analytic curve in $\mathbb{C}^+$, tending to ∞ as $x \to \infty$. Now from the formula of G we get the assertions (i), (ii), and (iii). See [8].

By applying Lindelöf's theorem to the function $g(z) = \eta(z)/z^{1/(s+1)}$ we get

$$\lim_{x\to\infty} g(x) = e^{i\pi s/(s+1)}.$$

The formula of the Cauchy transform tells us that:

$$\lim_{x\downarrow 0} G(x)x^{s/(s+1)} = \lim_{x\to\infty} G(x^{-1})x^{-s/(s+1)} = \lim_{x\to\infty} \frac{x^{1/(s+1)}}{1 - \eta(x)} = e^{-i\pi s/(s+1)}$$

Thus, the negative imaginary part tends to $\sin(\pi s/(s + 1))$, and we are done.

Case 2: $s > 1$. It has been shown in the previous case ($s < 1$) that $\pi_{s1} = \pi^{\boxtimes s}$ has no atom. It is clear from the definition of the s-th convolution power that any probability π_{s1} for $s > 1$ belongs to a partially defined free convolution semigroup with respect to $\boxtimes$, starting at some $s < 1$. The statement referring to atoms follows now from [8, Proposition 5.2(1)]. It remains to show the corresponding statements for the density.

Consider the following function: $\omega_s(z) = \eta(z)\left(\frac{z}{\eta(z)}\right)^{1/s}$.

It has been shown in [8, Theorem 2.6] that ω_s is a subordination function (in the sense of Littlewood): the functions η and $\eta_1 := \eta_{\pi_{11}}$ satisfy the relation $\eta_1(\omega_s(z)) = \eta(z), z \in \mathbb{C} \setminus [0, +\infty)$. Several properties of this function ω_s have been studied in [8]; we will find useful the fact (proved in Theorem 4.9 of the above cited work) that the restriction of ω_s to $\mathbb{C}^+$ extends continuously and injectively to $\mathbb{R}$, and the left inverse of ω_s is

$$\Phi_s(z) = z\left(\frac{z}{\eta_1(z)}\right)^{s-1}.$$

The derivative of Φ_s is given by

$$\Phi_s'(w) = \left(\frac{w}{\eta_1(w)}\right)^{s-1}\left(1 + (s-1)\left(1 - \frac{w}{\eta_1(w)}\eta_1'(w)\right)\right).$$

As before, by using the formula of η_1, we get that

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(i) Φ_s is increasing on $(-\infty, s/(s+1)^2)$;
(ii) Φ_s has a maximum at $s/(s+1)^2$ of value $1/K$;
(iii) Φ_s is decreasing to $1/2^{s+1}$ on $(s/(s+1)^2, 1/4]$.

As in Case 1, one shows that $\omega_s(-\infty, 1/K) = (-\infty, s/(s+1)^2)$ and that $\omega_s(1/K, \infty)$ is a simple analytic curve in $\mathbb{C}^+$ tending to infinity when $x \to \infty$. The relation $\eta_1 \circ \omega_s = \eta$ and the formula of η_1 guarantees that $\eta(1/K, \infty) \subset \mathbb{C}^+$ and $\eta(-\infty, 1/K) \subset \mathbb{R}$. We conclude that $\operatorname{supp} \pi_{s1} = [0, K]$ and that the density is analytic on the interior of the support.

It remains to prove (iv). We use the same method as in the previous case, along with subordination functions as a tool. For $x \in \mathbb{R} - \{0\}$, we have

$$\begin{aligned}\frac{\omega_s(x)}{x^{2/(s+1)}} &= x^{1/s} \cdot \frac{\eta_1(\omega_s(x))^{(s-1)/s}}{\omega_s(x)^{(s-1)/(2s)}} \cdot \frac{\omega_s(x)^{(s-1)/(2s)}}{x^{2/(s+1)}} \\ &= x^{1/s} \cdot \frac{\omega_s(x)^{(s-1)/(2s)}}{x^{2/(s+1)}} \left(\frac{\eta_1(\omega_s(x))}{\sqrt{\omega_s(x)}} \right)^{(s-1)/s}.\end{aligned}$$

Recall that $\omega_s(x) \to \infty$ as $x \to \infty$, so that the third factor above has a finite limit in the closure of $\mathbb{C}^+$ (we use here the branch of the square root that is defined on $\mathbb{C} - \mathbb{R}^+$). From Lindelöf's theorem, we get

$$\lim_{x\to\infty} \frac{\eta_1(\omega_s(x))}{\sqrt{\omega_s(x)}} = \lim_{w\to-\infty} \frac{\eta_1(w)}{\sqrt{w}} = \lim_{w\to-\infty} \frac{1-\sqrt{1-4w}}{2i\sqrt{|w|}} = i$$

This gives the following formula:

$$\begin{aligned}\lim_{x\to\infty} \frac{\sqrt{\omega_s(x)}}{x^{1/(s+1)}} &= \lim_{x\to\infty} \left(\frac{\eta_1(\omega_s(x))}{\sqrt{\omega_s(x)}} \right)^{(s-1)/s} \left(\frac{\sqrt{\omega_s(x)}}{x^{1/(s+1)}} \right)^{(s-1)/(2s)} \\ &= i^{(s-1)/(2s)} \lim_{x\to\infty} \left(\frac{\sqrt{\omega_s(x)}}{x^{1/(s+1)}} \right)^{(s-1)/(2s)}.\end{aligned}$$

We get from this equation that the limit is $i^{(s-1)/(s+1)}$, so

$$\begin{aligned}\lim_{x\to\infty} \frac{\eta(x)}{x^{1/(s+1)}} &= \lim_{x\to\infty} \frac{\eta_1(\omega_s(x))}{\sqrt{\omega_s(x)}} \cdot \frac{\sqrt{\omega_s(x)}}{x^{1/(s+1)}} \\ &= i \cdot i^{(s-1)/(s+1)} \\ &= e^{i\pi s/(s+1)}.\end{aligned}$$

Together with the formula of G, this concludes the proof. ■

3 The Measures π_{st}

We discuss now the support, atoms, and density of π_{st} for general values of $s, t > 0$. Recall first that at $s = 1$ we have the following formula:

$$\pi_{1t} = \max(1-t, 0)\delta_0 + \frac{\sqrt{4t - (x-1-t)^2}}{2\pi x}\, dx.$$

For general s the density is no longer explicit, and the formula of the support is quite complicated. We fix $s, t > 0$, and we make the following notations:

$$\Phi(w) = tw\Big(\frac{1-w}{1-(1-t)w}\Big)^s,$$

$$w_\pm = \frac{ts - t + 2 \pm \sqrt{t^2(s-1)^2 + 4st}}{2(1-t)},$$

$$K_\pm = \frac{t}{\Phi(w_\mp)}.$$

With these notations, we have the following result.

Theorem 3.1 *The measure π_{st} with $t < 1$ has the following properties:*

(i) *The atomic part is $(1-t)\delta_0$.*
(ii) *The rest of the support is $[K_-, K_+]$.*
(iii) *The density is analytic on (K_-, K_+).*
(iv) *The density is 0 at both ends of the support.*

Proof The starting point is Definition 1.4(ii). By using general results in [20] about compressions with free projections, we get

$$\pi_{st} = (1-t)\delta_0 + t\Big[\big(\pi^{\boxtimes s}\big)^{\boxplus 1/t} \boxtimes \delta_t\Big].$$

We will first analyze the measure appearing on the right, namely, $\mu = (\pi^{\boxtimes s})^{\boxplus 1/t}$.

We denote by $R, S, \dots$ the various transforms of μ, and by $R_1, S_1, \dots$ the same functions in the particular case $t = 1$. With these notations, we have

$$S_1(z) = \frac{1}{(1+z)^s} \implies S(z) = \frac{t}{(1+tz)^s} \implies \Sigma(w) = t\Big(\frac{1-w}{1-(1-t)w}\Big)^s.$$

We recognize in the last formula the function $\Phi(w)/w$. Summarizing, we have proved that the η transform of μ is the right inverse of Φ.

Note that the restriction of Φ to $\mathbb{C}^+$ and to $\mathbb{C}^-$ extends continuously to $\mathbb{R}$, but the two extensions do not agree on $(1, 1/(1-t))$ when $s \notin \mathbb{N}$.

Let us analyse the derivative of Φ. This is given by

$$\Phi'(w) = \frac{t((1-t)w^2 + (t - st - 2)w + 1)}{(1-(1-t)w)^2}\Big(\frac{1-w}{1-(1-t)w}\Big)^{s-1}.$$

Thus, in the general case $s \notin \{2, 3, \dots\}$, the domain of analyticity Φ has exactly two singularities, namely at the points w_+ and w_-.

In the case $s \in \{2, 3, \dots\}$, the points 1 and $1/(1-t)$ become singularities as well, because the function becomes rational.

We have $w_- < 1 < 1/(1-t) < w_+$, and by direct computation, we get $\Phi(w_-) < \Phi(w_+)$. Moreover, Φ is analytic around infinity and is a local diffeomorphism. This follows indeed from

$$\Big(\frac{1}{\Phi(w^{-1})}\Big)' = \frac{w^2 - (2 + st - t)w + (1-t)}{t(w-1)^2}\Big(\frac{w-(1-t)}{w-1}\Big)^{s-1}.$$

Let us draw our conclusions about Φ from the above facts:

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(i) The restriction of Φ to $\mathbb{R}$ increases from $-\infty$ to $\Phi(w_-)$ on the interval $(-\infty, w_-)$, then decreases from $\Phi(w_-)$ to zero on $(w_-, 1)$.
(i) The restriction of Φ to $\mathbb{R}$ decreases from ∞ to $\Phi(w_+)$ on the interval $(1/(1-t), w_+)$, then increases back to ∞ on (w_+, ∞).
(i) The curve $\Phi(1, 1/(1-t))$ escapes in $\mathbb{C} - \mathbb{R}$ if $s \notin \mathbb{N}$, is $(-\infty, 0)$ if s is odd, and is $(0, \infty)$ if s is even.
(ii) Φ is invertible on a neighbourhood N of $\{\infty\} \cup (-\infty, w_-) \cup (w_+, \infty)$. Moreover, N can be chosen so that $\Phi(N \cap \mathbb{C}^{\pm}) \subset \mathbb{C}^{\pm}$.

From [8] we have $\eta(-\infty, 0) = (-\infty, 0)$, so we conclude that η is the unique inverse of Φ on N which carries ∞ into itself. By using the particular form of Φ and the equation $\Phi(\eta(z)) = z$ we get that the restriction to $\mathbb{C}^+$ of η extends continuously and injectively to $\mathbb{R}$. It is obvious that η extends analytically along the real line from infinity all the way to $\Phi(w_{\pm})$, and not to those points.

Claim We have $\eta(\Phi(w_-), \Phi(w_+)) \subset \mathbb{C}^+$.

We know that $\eta(\Phi(w_-), \Phi(w_+)) \subset \mathbb{C}^+ \cup \mathbb{R}$ (we know $\eta(\Phi(w_-), \Phi(w_+))$ is a bounded set because η is injective on a neighbourhood, in $\mathbb{C} \cup \{\infty\}$, of infinity). Thus, we only need to show that $\eta(x) \notin \mathbb{R}$ for any $x \in (\Phi(w_-), \Phi(w_+))$.

First, observe that the injectivity of η on the real line, together with (i) and (ii), forbids $\eta(x)$ to belong to $[w_-, 1] \cup [1/(1-t), w_+]$. Thus, we only need to worry about $(1, 1/(1-t))$. So assume that we have $\eta(x) \in (1, 1/(1-t))$.

Case 1: $s \in 2\mathbb{N}+1$. The same injectivity property together with (iii) provides a direct contradiction, since $\eta(-\infty, 0) = (-\infty, 0)$.

Case 2: $s \notin \mathbb{N}$. We denote by $\eta^-(x)$ the value of the extension of η from the lower half-plane. From $\eta(\bar{z}) = \overline{\eta(z)}$ we get $\eta(x) = \eta^-(x)$, so the following can happen only if s is even, and we are done:

$$\lim_{z \to x, z \in \mathbb{C}^+} \Phi(\eta(z)) = x = \lim_{z \to x, z \in \mathbb{C}^-} \Phi(\eta(z)).$$

Case 3: $s \in 2\mathbb{N}$. In this case, Φ has multiplicity s around 1 and $1/(1-t)$, *i.e.,* on some small neighbourhoods of 0 and ∞ respectively, Φ covers each point with s points in a neighbourhood of 0 and $1/(1-t)$, respectively. In particular, we claim that $(-\infty, 0)$ has in each half-plane $s/2$ bounded preimages via Φ, which are simple curves uniting 1 and $1/(1-t)$.

Indeed, we know that there is some $\varepsilon < 0$ so that $(\varepsilon, 0)$ has a preimage γ_ε starting from one and climbing into the upper half-plane. We will show that γ_ε extends to a bounded curve γ uniting 1 and $1/(1-t)$.

Choose $v > \Phi(w_+)$ and consider the intersection with $\mathbb{C}^+$ of the circle of diameter $(0, v)$. Denote it by C. Then $\eta(C)$ is a simple curve in the upper half-plane starting from zero and ending to a point $\eta(v) \in (w_+, \infty)$. Since μ has no atoms, we conclude that $\eta(C)$ stays bounded away from 1. We claim that γ lies between $\eta(C)$ and $(0, \eta(v))$. Indeed, otherwise it would have to intersect one of these two curves. If it were to intersect $(0, \eta(v))$, then we would have that Φ maps a point from $(0, \infty)$ into a negative number, a contradiction (recall s is even). If it were to intersect $\eta(C)$, then, since $\Phi(\eta(C)) = C$, we would have that C intersects the negative half-line, another contradiction.

Thus, γ is bounded. Since its image via Φ is unbounded, γ must end at $1/(1-t)$. It is easy to argue that γ is simple: it could fail to be so only by meeting a critical point of Φ. But there are only two critical points except 1 and $1/(1-t)$, namely w_- and w_+, and neither is mapped by Φ in $(-\infty, 0]$.

Now we complete our proof. We have seen that γ separates w_- and w_+ from $(1, 1/(1-t))$. Thus, if the simple curve $L = \eta(\Phi(w_-), \Phi(w_+))$ uniting w_- and w_+ were to touch $(1, 1/(1-t))$, then L would have to intersect γ, so that there were $a \in (\Phi(w_-), \Phi(w_+))$ with the property $a = \Phi(\eta(a)) \in \Phi(\gamma) = (-\infty, 0]$, an obvious contradiction, since we have seen that $\Phi(w_-), \Phi(w_+) > 0$.

The equation of G together with the formula of π_{st} in the beginning of the proof gives the result. ■

Theorem 3.2 *The measure π_{st} with $t > 1$ has the following properties:*

(i) *There are no atoms.*
(ii) *The support is $[0, K_+]$.*
(iii) *The density is analytic on $(0, K_+)$.*
(iv) *The density is $\sim 1/((t-1)x^{1/s})$ at the left endpoint of the support, and 0 at the right endpoint.*

Proof The absence of atoms follows from Theorem 2.1 and [7]. Indeed, it follows from Definition 1.4 that the condition $t > 1$ requires $s \geq 1$. If $s = 1$, then $\pi_{1t} = \pi^{\boxplus t}$, $t > 1$, so, as observed in Theorem 3.1(i), π_{1t} has no atoms. If $s > 1$, then, by Theorem 2.1(i) and the previous observation, neither of the two factors in the free multiplicative convolution $\pi_{st} = \pi_{s1} \boxtimes \pi_{1t}$ has an atom, so by [7, Theorem 4.1], π_{st} has no atoms.

By using the same method as in the previous proof, we get that the η transform of π_{st} is the right inverse of the following function:

$$\Phi_{st}(w) = \frac{w(1-w)^s}{t + (1-t)w}.$$

By taking the derivative we get that there is only one zero of Φ'_{st} in the domain of analyticity of Φ_{st} in the general case $s \notin \mathbb{N}$, namely,

$$w_1 = \frac{t(s+1) - \sqrt{t^2(s-1)^2 + 4st}}{2s(t-1)}.$$

The other root is in fact right of $t/(t-1)$. Thus, let us only worry about the value of $\Phi_{st}(w_1)$. This provides us with one over the right endpoint of the support. The left endpoint must now be zero. Indeed, for any $x > \Phi_{st}(w_1)$, $\eta_{\pi_{st}}(x) \notin [w_1, 1]$ (in fact $\notin (-\infty, 1]$) by injectivity of $\eta_{\pi_{st}}$. If $s \notin \mathbb{N}$, it follows immediately that $\eta_{\pi_{st}}(x) \notin (1, +\infty)$. If $s \in \{2, 3, \dots\}$, we observe that Φ_{st}, while being a rational function, has a singularity at infinity, because

$$\left(\frac{1}{\Phi_{st}(w^{-1})}\right)' = \left(\frac{w}{w-1}\right)^{s-1} \frac{tw^2 - (t+st)w + ts - t}{(w-1)^2}.$$

Thus, η is not analytic around ∞, so the support of π_{st} touches 0.

An argument similar to the one in proof of Theorem 2.1 shows that the support of π_{st} is connected. It remains to study the behaviour of the density near zero. Following the idea of the previous proof, we get

$$\begin{aligned}\lim_{x\to-\infty}\frac{\eta(x)}{x^{1/s}} &= \lim_{w\to-\infty}\frac{\eta(\Phi_{st}(w))}{\Phi_{st}(w)^{1/s}}\\ &= \lim_{w\to-\infty} w\left(\frac{w(1-w)^s}{t+(1-t)w}\right)^{-1/s}\\ &= -\frac{1}{e^{i\pi/s}}\lim_{w\to-\infty}|w|^{1-1/s}\frac{(t+(t-1)|w|)^{1/s}}{1+|w|}\\ &= (t-1)e^{i\pi(s-1)/s}.\end{aligned}$$

An application of Lindelöf's theorem concludes the proof. ∎

4 Noncrossing Partitions

In this section, we find a combinatorial model, in terms of noncrossing partitions, for the Stieltjes transform of π_{st} with $s \in \mathbb{N}$. This is obtained by generalizing a well-known result regarding the free Poisson laws, for which we refer to [20]. Let us also mention that the $s = 2$ case was already worked out in [4].

Proposition 4.1 *The Stieltjes transform of π_{st} satisfies $f = 1 + zf^s(f + t - 1)$.*

Proof We use the formula of the S transform in the proof of Theorem 1.3:

$$\begin{aligned}S = \frac{1}{(1+z)^{s-1}}\cdot\frac{1}{t+z} &\implies \chi = \frac{z}{(1+z)^s}\cdot\frac{1}{t+z}\\ &\implies z = \frac{\psi}{(1+\psi)^s}\cdot\frac{1}{t+\psi}\\ &\implies z = \frac{f-1}{f^s}\cdot\frac{1}{t+f-1}.\end{aligned}$$

This gives the equation in the statement. ∎

In order to find a combinatorial interpretation of f, we use the sets NC_s studied by Edelman [14], Stanley [23], and Armstrong [1].

Definition 4.2 We use the following notations.

(i) A partition of $\{1,\dots,k\}$ is called noncrossing if the following happens: if $a \sim b$ and $x \sim y$ with $a < x < b < y$, then $a \sim x \sim b \sim y$.

(ii) $NC_s(k)$ is the set of noncrossing partitions of $\{1,\dots,sk\}$ into blocks of size multiple of s, and NC_s is the disjoint union of the sets $NC_s(k)$.

(iii) The normalized length of a partition $p \in NC_s(k)$ is given by $k(p) = k$. Also, we denote by $b(p)$ the number of blocks of p.

(iv) In the above notations, we make the following convention: the value $k = 0$ is allowed, with $NC_s(0)$ consisting of one element $\emptyset$, having 0 blocks.

Perhaps most illuminating here is the following table, containing the diagrammatic description of the elements in $NC_s(k)$, for small values of s, k.

	NC_1	NC_2	NC_3
0	$*$	$*$	$*$
1	ǀ	⊓	┌┬┐
2	ǀǀ , ⊓	⊓⊓, ⋒ , ┌┬┬┐	┌┬┐┌┬┐, . . . , ┌┬┬┬┬┐ (4)
3	ǀǀǀ , ǀ⊓, ⊓ǀ, ⋂ , ┌┬┐	(12)	(22)
4	ǀǀǀǀ , ǀǀ⊓, ǀ⊓ǀ , . . . , ┌┬┐, ⋒ , ┌┬┬┐ (14)	(55)	(140)

In this table, the asterisks represent empty partitions, the dots represent partitions which are not shown, and the numbers count the partitions.

With these notations, we have the following result.

Theorem 4.3 *The Stieltjes transform of π_{s1} with $s \in \mathbb{N}$ is given by*

$$f(z) = \sum_{p \in NC_s} z^{k(p)},$$

where $k \colon NC_s \to \mathbb{N}$ is the normalized length.

Proof With the notation $C_k = \# NC_s(k)$, the sum on the right is $f(z) = \sum_k C_k z^k$.

For a given partition $p \in NC_s(k+1)$, we can consider the last s legs of the first block and make cuts at right of them (see [4] for $s = 2$). This gives a decomposition of p into $s + 1$ partitions in NC_s, and we get:

$$C_{k+1} = \sum_{\Sigma k_i = k} C_{k_0} \cdots C_{k_s}.$$

By multiplying by z^{k+1} then summing over k, we get that the generating series of these numbers satisfies $f - 1 = z f^{s+1}$. But this is the same as the equation $f = 1 + z f^{s+1}$ of the Stieltjes transform of π_{s1}, and we are done. ∎

Theorem 4.4 *The Stieltjes transform of π_{st} with $s \in \mathbb{N}$ is given by*

$$f(z) = \sum_{p \in NC_s} z^{k(p)} t^{b(p)},$$

where $k, b \colon NC_s \to \mathbb{N}$ are the normalized length and the number of blocks.

Proof We denote by F_{kb} the number of partitions in $NC_s(k)$ having b blocks, and we set $F_{kb} = 0$ for other integer values of k, b. All sums will be over integer indices ≥ 0. With these notations, the sum on the right in the statement is

$$f(z) = \sum_{kb} F_{kb} z^k t^b.$$

The recurrence formula for the numbers C_k in the previous proof becomes

$$\sum_b F_{k+1,b} = \sum_{\Sigma k_i = k} \sum_{b_i} F_{k_0 b_0} \cdots F_{k_s b_s}.$$

In this formula, each term contributes to $F_{k+1,b}$ with $b = \Sigma b_i$, except for those of the form $F_{00} F_{k_1 b_1} \cdots F_{k_s b_s}$, which contribute to $F_{k+1,b+1}$. We get

$$F_{k+1,b} = \sum_{\Sigma k_i = k} \sum_{\Sigma b_i = b} F_{k_0 b_0} \cdots F_{k_s b_s} + \sum_{\Sigma k_i = k} \sum_{\Sigma b_i = b-1} F_{k_1 b_1} \cdots F_{k_s b_s} - \sum_{\Sigma k_i = k} \sum_{\Sigma b_i = b} F_{k_1 b_1} \cdots F_{k_s b_s}.$$

This gives the following formula for the polynomials $P_k = \sum_b F_{kb} t^b$:

$$P_{k+1} = \sum_{\Sigma k_i = k} P_{k_0} \cdots P_{k_s} + (t-1) \sum_{\Sigma k_i = k} P_{k_1} \cdots P_{k_s}.$$

In terms of $f = \sum_k P_k z^k$, we get the following equation:

$$f - 1 = z f^{s+1} + (t-1) z f^s.$$

But this is the same as the equation $f = 1 + z f^s (f + t - 1)$ of the Stieltjes transform of π_{st}, and we are done. ■

5 Moments

We compute now the moments of π_{st} for arbitrary values of the parameters s, t. We use the following method:

(i) For $s \in \mathbb{N}$, the moments can be found by counting partitions.
(ii) For $s > 0$, we have the same formula by a complex variable argument.

The moments can be expressed in terms of generalized binomial coefficients. We recall that the coefficient corresponding to $\alpha \in \mathbb{R}$, $k \in \mathbb{N}$ is

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}.$$

We denote the moments of a given probability measure by $m_1, m_2, m_3, \ldots$.

Theorem 5.1 *The moments of π_{s1} with $s > 0$ are the Fuss–Catalan numbers*

$$m_k = \frac{1}{sk+1} \binom{sk+k}{k}.$$

Proof In the case $s \in \mathbb{N}$, we know from Theorem 4.3 that $m_k = \#NC_s(k)$. The formula in the statement follows by counting partitions, see [1].

In the general case $s > 0$, observe first that the Fuss–Catalan number in the statement is a polynomial in s

$$\frac{1}{sk+1}\binom{sk+k}{k} = \frac{(sk+2)(sk+3)\cdots(sk+k)}{k!}.$$

Thus, in order to pass from the case $s \in \mathbb{N}$ to the case $s > 0$, it is enough to check that the k-th moment of π_{s1} is analytic in s. But this is clear from the equation $f = 1 + zf^{s+1}$ of the Stieltjes transform of π_{s1}. ■

Theorem 5.2 *The moments of π_{st}, $s > 0$ are the Fuss–Narayana numbers*

$$m_k = \sum_{b=1}^{k} \frac{1}{b}\binom{k-1}{b-1}\binom{sk}{b-1} t^b.$$

Proof In the case $s \in \mathbb{N}$, we know from Theorem 4.4 that $m_k = \sum_b F_{kb} t^b$, where F_{kb} is the number of partitions in $NC_s(k)$ having b blocks. The formula in the statement follows by counting such partitions, see [14, 23].

This result can be extended to any $s > 0$ by using a complex variable argument, as in the proof of Theorem 5.1. ■

The Fuss–Catalan numbers are known to appear in several contexts, for instance as dimensions of the algebras introduced by Bisch and Jones in [13]. It is important here to understand the meaning of the Fuss–Narayana numbers in this context. As explained in [4], this can be done at $s = 1, 2$, due to a natural correspondence between partitions in NC_s and diagrams in FC_s. In the case $n = 3, 4, \dots$, the situation is quite unclear, and we do not have an answer.

In the case $s \notin \mathbb{N}$, the moments of π_{st} can be further expressed in terms of Gamma functions. We would like to work out here the case $s = 1/2$.

Proposition 5.3 *The moments of $\pi_{1/2,1}$ are given by*

$$m_{2p} = \frac{1}{p+1}\binom{3p}{p},$$
$$m_{2p-1} = \frac{2^{-4p+3}p}{(6p-1)(2p+1)} \cdot \frac{p!(6p)!}{(2p)!(2p)!(3p)!}.$$

Proof The even moments of π_{st} with $s = n - 1/2$, $n \in \mathbb{N}$, are given by

$$m_{2p} = \frac{1}{(n-1/2)(2p)+1}\binom{(n+1/2)(2p)}{2p} = \frac{1}{(2n-1)p+1}\binom{(2n+1)p}{2p}.$$

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With $n = 1$ we get the formula in the statement. Now for the odd moments, we can use here the well-known identity

$$\binom{m-1/2}{k} = \frac{4^{-k}}{k!} \cdot \frac{(2m)!}{m!} \cdot \frac{(m-k)!}{(2m-2k)!}.$$

With $m = 2np + p - n$ and $k = 2p - 1$, we get

$$\begin{aligned} m_{2p-1} &= \frac{1}{(n-1/2)(2p-1)+1} \binom{(n+1/2)(2p-1)}{2p-1} \\ &= \frac{2}{(2n-1)(2p-1)+2} \binom{(2np+p-n)-1/2}{2p-1} \\ &= \frac{2^{-4p+3}}{(2p-1)!} \cdot \frac{(4np+2p-2n)!}{(2np+p-n)!} \cdot \frac{(2np-p-n+1)!}{(4np-2p-2n+3)!}. \end{aligned}$$

In particular, with $n = 1$ we get

$$\begin{aligned} m_{2p-1} &= \frac{2^{-4p+3}}{(2p-1)!} \cdot \frac{(6p-2)!}{(3p-1)!} \cdot \frac{p!}{(2p+1)!} \\ &= \frac{2^{-4p+3}(2p)}{(2p)!} \cdot \frac{(6p)!(3p)}{(3p)!(6p-1)6p} \cdot \frac{p!}{(2p)!(2p+1)}. \end{aligned}$$

This gives the formula in the statement. ■

6 Random Matrices

In this section, we discuss two random matrix models for the measures π_{st} with $s \in \mathbb{N}$. We restrict attention to the case $t = 1$, since $\pi_{st} = \pi^{\boxtimes s-1} \boxtimes \pi^{\boxplus t}$ and therefore matrix models for π_{st} will follow from matrix models for $\pi^{\boxtimes s}$.

We first recall the definition of a Wishart matrix.

Let $Y_1, Y_2, \dots, Y_p$ be independent vectors in $\mathbb{C}^N$ with identical Gaussian distribution $N(0, \Sigma)$ and set $W = Y_1 Y_1^* + \cdots + Y_p Y_p^*$. Then the $N \times N$ Hermitian matrix W follows the complex Wishart distribution $W(N, p, \Sigma)$.

We also can write $G^* = (Y_1, \dots, Y_p)$, so that G is a $p \times N$ matrix and $W = G^* G$. When $\Sigma = \sigma^2 I_{N^2}$, then $G = (g_{ij})_{i=1\dots p, j=1\dots N}$ is a Gaussian random matrix with independent entries of variance σ^2 that is such that $\{\mathrm{Re}(g_{ij}), \mathrm{Im}(g_{ij})\}$ is a family of $2pN$ independent $N(0, \sigma^2/2)$ random variables.

When $\Sigma = I_{N^2}/N$ and $\lim_{N\to\infty} p/N = t$, the limiting spectral distribution of W is the free Poisson law of parameter t, *i.e.,* the measure $\pi_{1t} = \pi^{\boxplus t}$. See Haagerup and Thorbjørnsen [16].

Theorem 6.1 *Let $G_1, \dots, G_s$ be a family of $N \times N$ independent matrices formed by independent centered Gaussian variables of variance $1/N$. Then with $M = G_1 \cdots G_s$, the moments of the spectral distribution of (MM^*) converge to the corresponding moments of π_{s1}, as $N \to \infty$.*

Proof We proceed by induction. At $s = 1$, it is wellknown that MM^* is a model for π_{11}. So assume that the result holds for $s - 1 \geq 1$. We have:

$$\begin{aligned}\operatorname{tr}(MM^*)^k &= \operatorname{tr}(G_1 \cdots G_s G_s^* \cdots G_1^*)^k \\ &= \operatorname{tr}(G_1 (G_2 \cdots G_s G_s^* \cdots G_1^* G_1)^{k-1} G_2 \cdots G_s G_s^* \cdots G_1^*).\end{aligned}$$

We can pass the first G_1 matrix to the right, we get

$$\begin{aligned}\operatorname{tr}(MM^*)^k &= \operatorname{tr}((G_2 \cdots G_s G_s^* \cdots G_1^* G_1)^{k-1} G_2 \cdots G_s G_s^* \cdots G_1^* G_1) \\ &= \operatorname{tr}(G_2 \cdots G_s G_s^* \cdots G_1^* G_1)^k \\ &= \operatorname{tr}((G_2 \cdots G_s G_s^* \cdots G_2^*)(G_1^* G_1))^k.\end{aligned}$$

We know that $G_1^* G_1$ is a Wishart matrix, and hence is a model for π. Also, we know by the induction assumption that $G_2 \cdots G_s G_s^* \cdots G_2^*$ gives a matrix model for $\pi_{s-1,1}$. Since by [17], $G_1^* G_1$ and $G_2 \cdots G_s G_s^* \cdots G_2^*$ are asymptotically free, their product gives a matrix model for $\pi_{s-1,1} \boxtimes \pi_{11} = \pi_{s1}$, and we are done. ∎

Theorem 6.2 *If W is a $W(sN, sN, \frac{1}{sN} I_{(sN)^2})$ complex Wishart matrix and*

$$D = \begin{pmatrix} 1_N & 0 & \cdots & 0 \\ 0 & w1_N & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w^{s-1} 1_N \end{pmatrix}$$

with $w = e^{2\pi i/s}$, then the moments of the mean empirical distribution of the eigenvalues of $(DW)^s$ converge to the corresponding moments of π_{s1}, as $N \to \infty$.

Proof We use the formula of Graczyk, Letac, and Massam [15]:

$$E(\operatorname{Tr}(DW)^K) = \sum_{\sigma \in S_K} \frac{M^{\gamma(\sigma^{-1}\pi)}}{M^K} r_\sigma(D).$$

Here W is a $W(M, M, \frac{1}{M} I_{N^2})$ complex Wishart matrix and D is a deterministic $M \times M$. As for the right term, this is as follows:

(i) π is the cycle $(1, \ldots, K)$.
(ii) $\gamma(\sigma)$ is the number of disjoint cycles of σ.
(iii) If we denote by $C(\sigma)$ the set of such cycles and for any cycle c, by $|c|$ its length, then

$$r_\sigma(D) = \prod_{c \in C(\sigma)} \operatorname{Tr}(D^{|c|}).$$

In our situation, we have $K = sk$ and $M = sN$, and we get

$$E(\operatorname{Tr}(DW)^{sk}) = \sum_{\sigma \in S_{sk}} \frac{(sN)^{\gamma(\sigma^{-1}\pi)}}{(sN)^{sk}} r_\sigma(D).$$

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Now, since D is uniformly formed by s-roots of unity, we have

$$\mathrm{Tr}(D^p) = \begin{cases} sN & \text{if } s|p, \\ 0 & \text{if } s \not| p. \end{cases}$$

Thus, if we denote by S_{sk}^s the set of permutations $\sigma \in S_{sk}$ having the property that all the cycles of σ have length multiple of s, the above formula reads

$$E(\mathrm{Tr}(DW)^{sk}) = \sum_{\sigma \in S_{sk}^s} \frac{(sN)^{\gamma(\sigma^{-1}\pi)}}{(sN)^{sk}} (sN)^{\gamma(\sigma)}.$$

In terms of the normalized trace tr, we get

$$E(\mathrm{tr}(DW)^{sk}) = \sum_{\sigma \in S_{sk}^s} (sN)^{\gamma(\sigma^{-1}\pi)+\gamma(\sigma)-sk-1}.$$

The exponent on the right, say L_σ, can be estimated by using the distance on the Cayley graph of S_{sk}:

$$\begin{aligned} L_\sigma &= \gamma(\sigma^{-1}\pi) + \gamma(\sigma) - sk - 1 \\ &= (sk - d(\sigma,\pi)) + (sk - d(e,\sigma)) - sk - 1 \\ &= sk - 1 - (d(e,\sigma) + d(\sigma,\pi)) \\ &\le sk - 1 - d(e,\pi) \\ &= 0. \end{aligned}$$

Now, when taking the limit $N \to \infty$ in the above formula of $E(\mathrm{tr}(DW)^{sk})$, the only terms that count are those coming from permutations $\sigma \in S_{sk}^s$ having the property $L_\sigma = 0$, which each contribute with a 1 value. We get

$$\begin{aligned} \lim_{N\to\infty} E(\mathrm{tr}(DW)^{sk}) &= \#\{\sigma \in S_{sk}^s \mid L_\sigma = 0\} \\ &= \#\{\sigma \in S_{sk}^s \mid d(e,\sigma) + d(\sigma,\pi) = d(e,\pi)\} \\ &= \#\{\sigma \in S_{sk}^s \mid \sigma \in [e,\pi]\}. \end{aligned}$$

Now, by using Biane's correspondence in [11], this is the same as the number of noncrossing partitions of $\{1,\dots,sk\}$ having all blocks of size a multiple of s. Thus we have reached the sets $NC_s(k)$ from Section 4, and we are done. ∎

As a consequence of the above random matrix formula, we have the following alternative free probabilistic approach to the free Bessel laws.

Theorem 6.3 *The moments of the free Bessel law π_{s1} with $s \in \mathbb{N}$ coincide with those of $(\sum_{k=1}^s w^k \alpha_k)^s$, where $\alpha_1,\dots,\alpha_s$ are free random variables, each of them following the free Poisson law of parameter $1/s$, and $w = e^{2\pi i/s}$.*

Proof Let $G_1, \dots, G_s$ be a family of independent $sN \times N$ matrices formed by independent, centered, complex Gaussian variables, of variance $1/(sN)$. The following matrices $H_1, \dots, H_s$ are also complex Gaussian and independent:

$$H_k = \frac{1}{\sqrt{s}} \sum_{p=1}^{s} w^{kp} G_p.$$

Thus, the following matrix provides a model for $\sum w^k \alpha_k$:

$$\begin{aligned} M &= \sum_{k=1}^{s} w^k H_k H_k^* \\ &= \frac{1}{s} \sum_{k=1}^{s} \sum_{p=1}^{s} \sum_{q=1}^{s} w^{k+kp-kq} G_p G_q^* \\ &= \sum_{p=1}^{s} \sum_{q=1}^{s} \Big(\frac{1}{s} \sum_{k=1}^{s} \big(w^{1+p-q} \big)^k \Big) G_p G_q^* \\ &= G_1 G_2^* + G_2 G_3^* + \cdots + G_{s-1} G_s^* + G_s G_1^*. \end{aligned}$$

This matrix can be written as:

$$\begin{aligned} M &= \begin{pmatrix} G_1 & G_2 & \cdots & G_{s-1} & G_s \end{pmatrix} \begin{pmatrix} G_2^* \\ G_3^* \\ \vdots \\ G_s^* \\ G_1^* \end{pmatrix} \\ &= \begin{pmatrix} G_1 & G_2 & \cdots & G_{s-1} & G_s \end{pmatrix} \begin{pmatrix} 0 & 1_N & 0 & \cdots & 0 \\ 0 & 0 & 1_N & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1_N \\ 1_N & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} G_1^* \\ G_2^* \\ \vdots \\ G_{s-1}^* \\ G_s^* \end{pmatrix} \\ &= GOG^*. \end{aligned}$$

Here, $G = (G_1 \ \cdots \ G_s)$ is the $sN \times sN$ Gaussian matrix obtained by concatenating $G_1, \dots, G_s$, and O is the matrix in the middle. But this latter matrix is of the form $O = UDU^*$ with U unitary, so we have $M = GUDU^*G^*$. Now, since GU is a Gaussian matrix, M has the same law as $M' = GDG^*$, and we get

$$\begin{aligned} E\Big(\Big(\sum_{l=1}^{s} w^l \alpha_l \Big)^{sk} \Big) &= \lim_{N \to +\infty} E(\mathrm{tr}(M^{sk})) \\ &= \lim_{N \to +\infty} E(\mathrm{tr}(GDG^*)^{sk}) \\ &= \lim_{N \to +\infty} E(\mathrm{tr}(D(G^*G))^{sk}) \end{aligned}$$

Thus, with $W = G^*G$, we get the result. ■

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7 Free Additivity

In this section we investigate the free additivity property of π_{st}, in analogy with the well-known free additivity property of the free Poisson laws π_{1t}.

We begin with a generalization of Theorem 6.3.

Theorem 7.1 *The free Bessel law π_{st} with $s \in \mathbb{N}$ is given by*

$$\pi_{st} = \text{law}\Big(\sum_{k=1}^{s} w^k \alpha_k\Big)^s$$

where $\alpha_1, \dots, \alpha_s$ are free random variables, each of them following the free Poisson law of parameter t/s, and $w = e^{2\pi i/s}$.

We draw the attention of the reader to the fact that $(\sum_{k=1}^{s} w^k \alpha_k)^s$ is not a selfadjoint random variable, hence the equality between its law and that of π_{st} should be viewed strictly in the sense of coincidence of moments.

Proof Given a random variable α and a complex number q, we have the following relations between the functional transforms of law(α) and law$(q\alpha)$:

$$\begin{aligned} f_{q\alpha}(z) = f_\alpha(qz) &\implies G_{q\alpha}(z) = q^{-1} G_\alpha(q^{-1}z) \\ &\implies K_{q\alpha}(z) = qK_\alpha(qz) \\ &\implies R_{q\alpha}(z) = qR_\alpha(qz). \end{aligned}$$

Consider now the variable $\alpha = \sum w^k \alpha_k$. We have

$$R_\alpha(z) = \sum_{k=1}^{s} w^k R_{\alpha_k}(w^k z) = \sum_{k=1}^{s} w^k \cdot \frac{t}{s} \cdot \frac{1}{1 - w^k z}.$$

This gives the following formula:

$$R_\alpha(z) = t\Big(\frac{1}{s}\sum_{k=1}^{s} \frac{w^k}{1 - w^k z}\Big) = \frac{tz^{s-1}}{1 - z^s}.$$

Consider now the formal measure $\tilde{\pi}_{st}$ having Stieltjes transform $\tilde{f}(z) = f(z^s)$, where f is the Stieltjes transform of π_{st}. The R transform of $\tilde{\pi}_{st}$ can be computed by using the equation of f in Proposition 4.1:

$$\begin{aligned} f = 1 + zf^s(f + t - 1) &\implies \tilde{f} = 1 + (z\tilde{f})^s(\tilde{f} + t - 1) \\ &\implies \xi\tilde{G} = 1 + \tilde{G}^s(\xi\tilde{G} + t - 1) \\ &\implies \tilde{K}z = 1 + z^s(\tilde{K}z + t - 1) \\ &\implies \tilde{R}z + 1 = 1 + z^s(\tilde{R}z + t) \\ &\implies \tilde{R}(z) = tz^{s-1}/(1 - z^s). \end{aligned}$$

Thus, we have the equality of R transforms $\tilde{R} = R_\alpha$. In terms of measures, we get $\tilde{\pi}_{st} = \text{law}(\alpha)$, hence $\pi_{st} = \text{law}(\alpha^s)$, and we are done. ∎

It is convenient to introduce the following measures.

Definition 7.2 The modified free Bessel laws $\tilde{\pi}_{st}$ with $s \in \mathbb{N}$ are given by

$$\tilde{\pi}_{st} = \mathrm{law}\Big(\sum_{k=1}^{s} w^k \alpha_k \Big)$$

where $\alpha_1, \dots, \alpha_s$ are free random variables, each of them following the free Poisson law of parameter t/s, and $w = e^{2\pi i/s}$. Here, as in Theorem 7.1, the equality of law is not understood in the sense of $*$-moments but in the sense that the moments coincide.

We know from the previous section that the mean empirical distribution of the eigenvalues of DW converges towards $\tilde{\pi}_{s1}$, with the notations there. Also, we know from the previous proof that the family $\tilde{\pi}_{st}$ is freely additive with respect to t, the R transform being given by $\tilde{R}_{st} = tz^{s-1}/(1 - z^s)$.

These results show that we have $\tilde{\pi}_{st} = \pi_{t\rho}$, where ρ is the uniform measure on the s-roots of unity, and $\pi_{t\rho}$ is the corresponding compound free Poisson law.

For real measures ρ, the compound free Poisson laws $\pi_{t\rho}$ were introduced by Speicher in [22], and studied by Hiai and Petz in [17]. In our case, ρ is complex, but the main results (R-transform, matrix models) still hold. Our third main result is the Poisson limit one, which in our case is as follows.

Theorem 7.3 *We have the Poisson limit type convergence*

$$\big((1 - \tfrac{1}{n})\delta_0 + \tfrac{1}{n}\rho \big)^{\boxplus n} \to \tilde{\pi}_{s1},$$

where ρ is the uniform measure on the s-roots of unity.

Proof We compute first the R transform of the measure on the left:

$$\begin{aligned}
\mu = \Big(1 - \frac{1}{n}\Big)\delta_0 + \frac{1}{n}\rho \implies & f = \Big(1 - \frac{1}{n}\Big) + \frac{1}{n} \cdot \frac{1}{1 - z^s} \\
\implies & G(\xi) = \frac{1}{\xi} + \frac{1}{n} \cdot \frac{1}{\xi(\xi^s - 1)} \\
\implies & (K^s - 1)(zK - 1) = \frac{1}{n} \\
\implies & \Big(\Big(R + \frac{1}{z}\Big)^s - 1 \Big) zR = \frac{1}{n}.
\end{aligned}$$

This shows that the R transform of $\mu^{\boxplus n}$ satisfies

$$\Big(\Big(\frac{R}{n} + \frac{1}{z} \Big)^s - 1 \Big) z \frac{R}{n} = \frac{1}{n}.$$

We multiply by n, then we take the limit $n \to \infty$. We get $(\frac{1}{z^s} - 1)zR = 1$. Thus in the limit $n \to \infty$ we have $R = z^{s-1}/(1 - z^s)$, and we are done. ∎

8 Classical Analogues

We discuss now the classical analogues $p_{st}, \tilde{p}_{st}$ of the free Bessel laws $\pi_{st}, \tilde{\pi}_{st}$. There are several ways of introducing these laws. The most convenient is to start with the following formulae, similar to those in Theorem 7.1 and Definition 7.2.

Definition 8.1 The Bessel laws p_{st} and the modified Bessel laws $\tilde{p}_{st}$ with $s \in \mathbb{N}$ are given by

$$p_{st} = \mathrm{law}\Big(\sum_{k=1}^{s} w^k a_k\Big)^s \quad \text{and} \quad \tilde{p}_{st} = \mathrm{law}\Big(\sum_{k=1}^{s} w^k a_k\Big),$$

where $a_1, \ldots, a_s$ are independent random variables, each of which follows the Poisson law of parameter t/s, and $w = e^{2\pi i/s}$.

As a first remark, at $s = 1$, we get the Poisson law of parameter t

$$p_{1t} = \tilde{p}_{1t} = e^{-t} \sum_{r=0}^{\infty} \frac{t^r}{r!} \delta_r.$$

In what follows, we present a number of results that show that $p_{st}, \tilde{p}_{st}$ are indeed the classical analogues of $\pi_{st}, \tilde{\pi}_{st}$, for any $s \in \mathbb{N}$.

The first such interpretation comes from the Bercovici-Pata bijection [9]. Since this makes Poisson laws correspond to free Poisson laws, and convolution to free convolution, we get by linearity that it makes $\tilde{p}_{st}$ and $\tilde{\pi}_{st}$ correspond.

We discuss now the additivity property and the Poisson limit convergence for Bessel laws in analogy with the considerations from the previous section.

We use the level s exponential function $\exp_s z = \sum_{k=0}^{\infty} \frac{z^{sk}}{(sk)!}$.

We have the following formula, in terms of $w = e^{2\pi i/s}$:

$$\exp_s z = \frac{1}{s} \sum_{k=1}^{s} \exp(w^k z)$$

Observe that we have $\exp_1 = \exp$ and $\exp_2 = \cosh$.

Theorem 8.2 *The Fourier transform of $\tilde{p}_{st}$ is given by $\log \tilde{F}_{st}(z) = t(\exp_s z - 1)$, so, in particular, the measures $\tilde{p}_{st}$ are additive with respect to t.*

Proof Consider the variable $a = \sum w^k a_k$. For the Poisson law of parameter t, we have $\log F(z) = t(e^z - 1)$, and by using the identity $F_{qa}(z) = F_a(qz)$, we get

$$\log F_a(z) = \sum_{k=1}^{s} \log F_{a_k}(w^k z) = \sum_{k=1}^{s} \frac{t}{s}(\exp(w^k z) - 1).$$

This gives the following formula:

$$\log F_a(z) = t\left(\Big(\frac{1}{s}\sum_{k=1}^{s} \exp(w^k z)\Big) - 1\right) = t(\exp_s(z) - 1).$$

Now, since $\tilde{p}_{st}$ is the law of a, this gives the formula in the statement. ∎

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Theorem 8.3 *We have the Poisson limit type convergence* $((1-\frac{1}{n})\delta_0+\frac{1}{n}\rho)^{*n}\to\tilde{p}_{s1}$, *where* ρ *is the uniform measure on the s-roots of unity.*

Proof We compute first the Fourier transform of the measure on the left:

$$\mu=\left(1-\tfrac{1}{n}\right)\delta_0+\tfrac{1}{n}\rho\implies F=\left(1-\tfrac{1}{n}\right)+\tfrac{1}{n}\exp_s(z).$$

This shows that the Fourier transform of μ^{*n} is given by

$$F=\left(\left(1-\frac{1}{n}\right)+\frac{1}{n}\exp_s(z)\right)^n=\left(1+\frac{\exp_s(z)-1}{n}\right)^n\simeq\exp(\exp_s(z)-1).$$

Thus in the limit $n\to\infty$ we have $\log F=\exp_s z-1$, and we are done. ∎

9 Bessel Functions

In this section, we study the densities of $p_{st},\tilde{p}_{st}$. At $s=2$ this will lead to Bessel functions, which will justify the general terminology for $\pi_{st},\tilde{\pi}_{st}$.

Theorem 9.1 *We have the formulae*

$$p_{st}=e^{-t}\sum_{p_1=0}^{\infty}\cdots\sum_{p_s=0}^{\infty}\frac{1}{p_1!\cdots p_s!}\left(\frac{t}{s}\right)^{p_1+\cdots+p_s}\delta\left(\sum_{k=1}^{s}w^kp_k\right)^s,$$

$$\tilde{p}_{st}=e^{-t}\sum_{p_1=0}^{\infty}\cdots\sum_{p_s=0}^{\infty}\frac{1}{p_1!\cdots p_s!}\left(\frac{t}{s}\right)^{p_1+\cdots+p_s}\delta\left(\sum_{k=1}^{s}w^kp_k\right),$$

where $w=e^{2\pi i/s}$, *and the* δ *symbol is a Dirac mass.*

Proof It is enough to prove the formula for $\tilde{p}_{st}$. For this purpose, we compute the Fourier transform of the measure on the right. This is given by

$$\begin{aligned}F(z)&=e^{-t}\sum_{p_1=0}^{\infty}\cdots\sum_{p_s=0}^{\infty}\frac{1}{p_1!\cdots p_s!}\left(\frac{t}{s}\right)^{p_1+\cdots+p_s}F\delta\left(\sum_{k=1}^{s}w^kp_k\right)(z)\\&=e^{-t}\sum_{p_1=0}^{\infty}\cdots\sum_{p_s=0}^{\infty}\frac{1}{p_1!\cdots p_s!}\left(\frac{t}{s}\right)^{p_1+\cdots+p_s}\exp\left(\sum_{k=1}^{s}w^kp_kz\right)\\&=e^{-t}\sum_{r=0}^{\infty}\left(\frac{t}{s}\right)^r\sum_{\Sigma p_i=r}\frac{\exp\left(\sum_{k=1}^{s}w^kp_kz\right)}{p_1!\cdots p_s!}.\end{aligned}$$

We multiply by e^t, and we compute the derivative with respect to t:

$$\begin{aligned}(e^tF(z))'&=\sum_{r=1}^{\infty}\frac{r}{s}\left(\frac{t}{s}\right)^{r-1}\sum_{\Sigma p_i=r}\frac{\exp\left(\sum_{k=1}^{s}w^kp_kz\right)}{p_1!\cdots p_s!}\\&=\frac{1}{s}\sum_{r=1}^{\infty}\left(\frac{t}{s}\right)^{r-1}\sum_{\Sigma p_i=r}\left(\sum_{l=1}^{s}p_l\right)\frac{\exp\left(\sum_{k=1}^{s}w^kp_kz\right)}{p_1!\cdots p_s!}\\&=\frac{1}{s}\sum_{r=1}^{\infty}\left(\frac{t}{s}\right)^{r-1}\sum_{\Sigma p_i=r}\sum_{l=1}^{s}\frac{\exp\left(\sum_{k=1}^{s}w^kp_kz\right)}{p_1!\cdots p_{l-1}!(p_l-1)!p_{l+1}!\cdots p_s!}.\end{aligned}$$

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By using the variable $u = r - 1$, we get

$$
\begin{aligned}
(e^t F(z))' &= \frac{1}{s} \sum_{u=0}^{\infty} \Big(\frac{t}{s}\Big)^u \sum_{\Sigma q_i = u} \sum_{l=1}^{s} \frac{\exp\big(w^l z + \sum_{k=1}^{s} w^k q_k z\big)}{q_1! \cdots q_s!} \\
&= \Big(\frac{1}{s} \sum_{l=1}^{s} \exp(w^l z)\Big) \Big(\sum_{u=0}^{\infty} \Big(\frac{t}{s}\Big)^u \sum_{\Sigma q_i = u} \frac{\exp\big(\sum_{k=1}^{s} w^k q_k z\big)}{q_1! \cdots q_s!}\Big) \\
&= (\exp_s z)(e^t \tilde{F}_{st}(z)).
\end{aligned}
$$

On the other hand, the function $\Phi(t) = \exp(t \exp_s z)$ also satisfies the equation $\Phi'(t) = (\exp_s z)\Phi(t)$. Thus, we have $e^t F(z) = \Phi(t)$, which gives

$$
\log F = \log(e^{-t} \exp(t \exp_s z)) = \log(\exp(t(\exp_s z - 1))) = t(\exp_s z - 1).
$$

This gives the formulae in the statement. ∎

Recall now that the Bessel function of the first kind is given by

$$
\varphi_r(t) = \sum_{p=0}^{\infty} \frac{t^{2p+r}}{p!(p+r)!}.
$$

The following result justifies the terminology used in this paper.

Theorem 9.2 *We have the formulae*

$$
p_{2t} = e^{-t} \sum_{r=-\infty}^{\infty} \varphi_{|r|}\Big(\frac{t}{2}\Big) \delta_{r^2}, \quad \tilde{p}_{2t} = e^{-t} \sum_{r=-\infty}^{\infty} \varphi_{|r|}\Big(\frac{t}{2}\Big) \delta_r,
$$

where φ_r is the Bessel function of the first kind.

Proof At $s = 2$, the primitive root of unity is $w = -1$, and we get

$$
\begin{aligned}
\tilde{p}_{2t} &= e^{-t} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(t/2)^{p+q}}{p!q!} \delta_{p-q} \\
&= e^{-t} \sum_{r=-\infty}^{\infty} \sum_{p-q=r} \frac{(t/2)^{p+q}}{p!q!} \delta_r \\
&= e^{-t} \Big(\sum_{r=0}^{\infty} \sum_{q=0}^{\infty} \frac{(t/2)^{r+2q}}{(r+q)!q!} \delta_r + \sum_{r=-\infty}^{-1} \sum_{p=0}^{\infty} \frac{(t/2)^{2p-r}}{p!(p-r)!} \delta_r\Big).
\end{aligned}
$$

Thus the density of $\tilde{p}_{2t}$ is given indeed by the Bessel function

$$
\begin{aligned}
\tilde{p}_{2t} &= e^{-t} \Big(\sum_{r=0}^{\infty} \sum_{q=0}^{\infty} \frac{(t/2)^{r+2q}}{(r+q)!q!} \delta_r + \sum_{r=-\infty}^{-1} \sum_{p=0}^{\infty} \frac{(t/2)^{2p+|r|}}{p!(p+|r|)!} \delta_r\Big) \\
&= e^{-t} \sum_{r=-\infty}^{\infty} \sum_{p=0}^{\infty} \frac{(t/2)^{|r|+2p}}{(|r|+p)!p!} \delta_r.
\end{aligned}
$$

This gives the formulae in the statement. ∎

We know that p_{1t}, p_{2t} are supported by $\mathbb{N}, \mathbb{Z}$. In the general case, the situation is a bit more complicated: the support is formed by the s powers of certain elements in $\mathbb{Z}[w]$, so we can only say that it is contained in $\mathbb{Z}[w]$. As for the density, this should be thought of as being a kind of s-dimensional Bessel function.

10 Quantum Groups

We discuss now the representation theory approach to p_{st}, π_{st}. The material presented in this and the next two sections is organized as follows:

(i) We first discuss the case $s = 1, 2$, by surveying some previously known results, from [3, 4, 6]. In order to simplify the presentation, we use the formalism of compact quantum groups in this introductory part.

(ii) Then we discuss the case of arbitrary $s \in \mathbb{N}$, with all the definitions and results written by using Woronowicz's Hopf algebra formalism in [29].

Consider a compact group $G \subset U_n$. The character of the fundamental representation $\chi\colon G \to \mathbb{C}$ is by definition the restriction to G of the usual trace $\chi(g) = \sum_{i=1}^n g_{ii}$.

In functional analytic terms, the character $\chi \in C(G)$ can be defined, starting with the n^2 matrix coordinates $u_{ij} \in C(G)$, as being the trace of $u = (u_{ij})$: $\chi = \sum_{i=1}^n u_{ii}$.

The law of χ with respect to the Haar functional of $C(G)$ is a fundamental object in representation theory, because of the following formula:

$$\int \chi^k = \#\{1 \in u^{\otimes k}\}$$

Here we regard $C(G)$ as a Hopf C^*-algebra, and the number of the right is the multiplicity of the trivial corepresentation 1 into the k-th tensor power $u^{\otimes k} = u_{1,k+1}u_{2,k+1}\cdots u_{k,k+1}$. This corepresentation has character χ^k, and the above formula comes from the well-known fact that the number of copies of 1 can be obtained by integrating the character. See, *e.g.,* Woronowicz [29].

The following statement from [3] is a representation theory interpretation of the relationship between the Poisson law p and the free Poisson law π.

Proposition 10.1 *We have the following formulae.*

(i) *For $G = S_n$, with $n \to \infty$ we have* $\mathrm{law}(\chi) \to p$.

(ii) *For $G = S_n^+$, with $n \geq 4$ we have* $\mathrm{law}(\chi) = \pi$.

Here S_n^+ is Wang's quantum permutation group [28]. This quantum group does not exist as a concrete object, but the character under investigation exists as an element of the associated Hopf algebra $A_s(n)$. See [3] for details.

Observe that there is a slight problem with the above statement, which is not fully symmetric in terms of convergences. As pointed out in [6], a uniform statement can be obtained in terms of truncated characters, given by $\chi_t = \sum_{i=1}^{[tn]} u_{ii}$.

Here, $t \in (0, 1]$ is a parameter.

The laws of truncated characters can be computed by using the Weingarten formula, and we have the following result [6].

Theorem 10.2 *We have the following formulae.*

(i) *For $G = S_n$, with $n \to \infty$ we have* $\mathrm{law}(\chi_t) \to p_{1t}$.
(ii) *For $G = S_n^+$, with $n \to \infty$ we have* $\mathrm{law}(\chi_t) \to \pi_{1t}$.

The second result of this type concerns the hyperoctahedral group H_n. This is the symmetry group of the cube in $\mathbb{R}^n$.

The hyperoctahedral group has a wreath product decomposition $H_n = \mathbb{Z}_2 \wr S_n$, and its free version H_n^+ has a free wreath product decomposition $H_n^+ = \mathbb{Z}_2 \wr_* S_n^+$.

The laws of truncated characters can be computed by using wreath product techniques and the Weingarten formula, and we have the following result [4].

Theorem 10.3 *We have the following formulae.*

(i) *For $G = H_n$, with $n \to \infty$ we have* $\mathrm{law}(\chi_t) \to \tilde{p}_{2t}$.
(ii) *For $G = H_n^+$, with $n \to \infty$ we have* $\mathrm{law}(\chi_t) \to \tilde{\pi}_{2t}$.

Summarizing, the groups S_n, H_n and their free analogues S_n^+, H_n^+ provide models for the Bessel and free Bessel laws $\tilde{p}_{st}, \tilde{\pi}_{st}$, at $s = 1, 2$.

In what follows, we will generalize these results with a single two-fold statement (classical and quantum) that works for any $s \in \mathbb{N}$.

Together with the additional results in [5], concerning the groups O_n, U_n and their free versions O_n^+, U_n^+, this will justify the table in the introduction.

11 The Group H_n^s

We discuss here the generalization of Theorem 10.2(i) and Theorem 10.3. We discuss as well the generalization of (ii), with some preliminary facts.

A matrix is called monomial if it has exactly one nonzero entry in each row and each column. The basic examples are the permutation matrices.

Definition 11.1 $H_n^s = \mathbb{Z}_s \wr S_n$ is the group of monomial $n \times n$ matrices having as entries the s-roots of unity.

In other words, an element of H_n^s is a permutation matrix, with each 1 entry replaced by an s-th root of unity. When identifying the group of s-roots of unity with $\mathbb{Z}_s$, this gives the wreath product decomposition in the above definition.

Observe that we have $H_n^s \subset U_n$, and that $H_n^1 = S_n, H_n^2 = H_n$.

Theorem 11.2 *For H_n^s, with $n \to \infty$ we have* $\mathrm{law}(\chi_t) \to \tilde{p}_{st}$.

Proof For $s = 1, 2$, this is known from [4,6], and we will use the same method here. We denote by ρ the uniform measure on the s-roots of unity.

First, we work out the case $t = 1$. Since the limit probability for a random permutation to have exactly k fixed points is $e^{-1}/k!$, we get

$$\lim_{n\to\infty} \mathrm{law}(\chi_1) = e^{-1} \sum_{k=0}^{\infty} \frac{1}{k!} \rho^{*k}.$$

On the other hand, we get from Theorem 8.3

$$\begin{aligned}\tilde{p}_{s1} &= \lim_{n\to\infty}\left(\left(1-\frac{1}{n}\right)\delta_0+\frac{1}{n}\rho\right)^{*n}\\ &= \lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}\left(1-\frac{1}{n}\right)^{n-k}\frac{1}{n^k}\rho^{*k}\\ &= e^{-1}\sum_{k=0}^{\infty}\frac{1}{k!}\rho^{*k}.\end{aligned}$$

This gives the assertion for $t=1$. Now in the case $t>0$ arbitrary, we can use the same method by performing the following modifications:

$$\lim_{n\to\infty}\operatorname{law}(\chi_t)=e^{-t}\sum_{k=0}^{\infty}\frac{t^k}{k!}\rho^{*k}=\lim_{n\to\infty}\left(\left(1-\tfrac{1}{n}\right)\delta_0+\tfrac{1}{n}\rho\right)^{*[tn]}=\tilde{p}_{st}$$

This finishes the proof. ■

It remains to discuss the quantum group model for π_{st}. The quantum group will be a suitably chosen free version of H_n^s.

Definition 11.3 The universal C^*-algebra $A_h^s(n)$ is defined with n^2 generators u_{ij} and with the following relations:

(i) $u=(u_{ij})$ and $\bar{u}=(u_{ij}^*)$ are unitaries.
(ii) $u_{ij}u_{ij}^*=u_{ij}^*u_{ij}=p_{ij}$ (projection).
(iii) $u_{ij}^s=p_{ij}$.

In this definition, the meaning of the second condition is that each u_{ij} is normal, and that the elements $p_{ij}=u_{ij}u_{ij}^*$ are idempotents: $p_{ij}^2=p_{ij}$. Observe that each u_{ij} is a normal partial isometry in the C^*-algebra sense.

We use the symmetric and hyperoctahedral Hopf algebras $A_s(n), A_h(n)$, associated with the free quantum groups S_n^+, H_n^+. See [4,6].

Proposition 11.4 *The algebra $A_h^s(n)$ has the following properties.*

(i) *It is a Hopf C^*-algebra, with u being a corepresentation.*
(ii) *Its maximal commutative quotient is $C(H_n^s)$.*
(iii) $A_h^1(n)=A_s(n)$, $A_h^2(n)=A_h(n)$.

Proof We use the notation $A=A_h^s(n)$.

(i) The comultiplication, counit and antipode can be constructed by using the universal property of A, according to the following formulae:

$$\Delta(u_{ij})=\sum_{k=1}^{n}u_{ik}\otimes u_{kj},\quad \varepsilon(u_{ij})=\delta_{ij},\quad S(u_{ij})=u_{ji}^*.$$

Indeed, the matrices $(\Delta u)_{ij}=\Sigma_k u_{ik}\otimes u_{kj}$, $(\varepsilon u)_{ij}=\delta_{ij}$, $(Su)_{ij}=u_{ji}^*$ satisfy the conditions in Definition 11.3, so we can define Δ,ε,S as above. This shows that we have a Hopf C^*-algebra in the sense of Woronowicz [29], and we are done.

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(ii) The matrix coordinates of $H_n^s \subset U_n$ satisfy the relations in Definition 11.3, so we have a surjective morphism $A \to C(H_n^s)$.

Consider the ideal $J \subset A$ generated by the relations $[u_{ij}, u_{kl}] = 0$. It is routine to check that A/J is a Hopf algebra, with Δ, ε, S defined as above. Thus, we have $A/J = C(G)$, where $G \subset U_n$ is a certain compact group, containing H_n^s.

From the fact that u is unitary, we get that the matrix of projections $p = (p_{ij})$ has sum 1 on each row and each column. It follows that the entries p_{ij} are pairwise orthogonal on rows and columns, and we get that G consists of monomial matrices. Now, from the relation $u_{ij}^s = p_{ij}$ we get $G \subset H_n^s$, and we are done.

(iii) This follows from a routine comparison between Definition 11.3 and the definitions in [4] by using the above-mentioned fact that when a number of projections sum up to 1, they are pairwise orthogonal. ■

Summarizing, $A_h^s(n)$ appears to be the natural candidate for a model for π_{st}. We will prove in the next section that indeed it is so.

We should mention that $A_h^s(n)$ has a number of other interesting properties not to be investigated here. One can prove, for instance, that we have a decomposition $A_h^s(n) = C(\mathbb{Z}_s) *_w A_s(n)$, analogous to the decomposition $H_n^s = \mathbb{Z}_s \wr S_n$. Here, $*_w$ is a free wreath product in the sense of Bichon [12], and the proof is as in [4] by using a suitable reformulation of Definition 11.3 as a sudoku type condition.

12 Integration over $A_h^s(n)$

We now compute the asymptotic laws of truncated characters for the algebra $A_h^s(n)$. The integration is with respect to the Haar functional, known to exist by general results of Woronowicz in [29].

Theorem 12.1 *For $A_h^s(n)$, with $n \to \infty$ we have* $\mathrm{law}(\chi_t) \to \tilde{\pi}_{st}$.

Proof We use a standard method developed in [2, 4–6]. The general idea is that at $s = 1, 2$ the result is known from [4, 6], so we can use an extension of the proof there. The main problem at $s \geq 3$ comes from the fact that the fundamental corepresentation u is no longer self-adjoint, so the underlying combinatorial objects will be indexed by elements of $\mathbb{N} * \mathbb{N}$ rather than by numbers in $\mathbb{N}$. In order to deal with this problem, we use methods from [2, 5].

The proof uses tensor categories, denoted C, and has 9 steps:

1. We introduce three auxiliary algebras: $A_k(n), A_h^\infty(n), A_s(n)$.
2. We discuss the relation between the associated categories.
3. We describe the passage $CA_k(n) \to CA_h^\infty(n)$.
4. We compute $CA_h^\infty(n)$.
5. We describe the passage $CA_h^\infty(n) \to CA_h^s(n)$.
6. We compute $CA_h^s(n)$.
7. We discuss the integration formula for characters.
8. We prove the result for $t = 1$.
9. We prove the result for any $t > 0$.

As already mentioned, the method is quite standard, so we will insist on technical details only. In fact, the only problem comes from the overall level of complexity, which is higher than in the previous papers [2, 4–6]. Let us also mention that in the last part of the proof, the cumulant computations can probably be deduced as well from the general results of Lehner in [18].

Step 1. Let $A_h^\infty(n)$ be the algebra defined as $A_h^s(n)$, but with the condition (iii) in Definition 11.3 missing. That is, $A_h^\infty(n)$ is the algebra generated by n^2 normal partial isometries u_{ij}, such that $u = (u_{ij})$ and $\bar{u} = (u_{ij}^*)$ are unitaries.

Observe that we have an arrow $A_h^\infty(n) \to A_h^s(n)$ for any $s \in \mathbb{N}$.

By arguing as in the proof of Proposition 11.4 we get that $A_h^\infty(n)$ is a Hopf algebra having as maximal commutative quotient the algebra of functions on the group $H_n^\infty = \mathbb{T} \wr S_n$ consisting of unitary monomial matrices.

Consider also the algebra $A_k(n)$ generated by n^2 variables u_{ij}, having the property that $u = (u_{ij})$ and $\bar{u} = (u_{ij}^*)$ are unitaries, and that the relation $ab^* = a^*b = 0$ holds for a, b distinct entries on the same row or column of u.

Once again by arguing as in the proof of Proposition 11.4, we get that $A_k(n)$ is a Hopf algebra having $C(H_n^\infty)$ as maximal commutative quotient. See [2].

Observe that we have an arrow $A_k(n) \to A_h^\infty(n)$.

This follows from the orthogonality condition on the supporting projections p_{ij} obtained in the proof of Proposition 11.4.

The fact that $A_k(n), A_h^\infty(n)$ have the same maximal commutative quotient might seem a bit surprising. The point is that when trying to liberate the commutative Hopf algebra $C(H_n^\infty)$, the normality condition of the elements u_{ij} can be kept or not. This is why we end up with two different algebras.

Finally, consider the algebra $A_s(n) = A_h^1(n)$. This is Wang's quantum permutation algebra [28], which can be also described as being the quotient of $A_k(n)$ by the relations making each u_{ij} a projection.

Observe that we have an arrow $A_h^s(n) \to A_s(n)$.

Summarizing, the algebra $A_h^s(n)$ under investigation and its $s = \infty$ version are part of the sequence $A_k(n) \to A_h^\infty(n) \to A_h^s(n) \to A_s(n)$.

The point is that the algebras on the left and on the right are well understood, and this can be used for studying the algebras in the middle.

Step 2. We denote by CA the tensor category associated with a pair (A, u) as in Woronowicz's paper [29]. That is, the objects are the tensor products between u and $\bar{u}$, and the arrows are the intertwiners between them. Observe that in the case $u = \bar{u}$, the objects are just the tensor powers of u.

Since applying morphisms increases the spaces of intertwiners, we have embeddings of tensor categories as follows:

$$CA_k(n) \subset CA_h^\infty(n) \subset CA_h^s(n) \subset CA_s(n)$$

The result that we want to prove is of an asymptotic nature, so we can make the assumption that $n \geq 4$. Now with this condition in hand, it is well known that $CA_s(n)$ is the Temperley–Lieb category of index n. That is, for any $k, l \in \mathbb{N}$, the space $\mathrm{Hom}(u^{\otimes k}, u^{\otimes l})$ can be identified with the abstract vector space $D_s(k, l)$ spanned by the Temperley–Lieb diagrams between $2k$ points and $2l$ points, and the categorical operations in $CA_s(n)$ are the usual planar operations in D_s (with the rule that a closed loop corresponds to a multiplicative factor n). See, *e.g.*, [4].

Also, it is known that the subcategory $CA_k(n) \subset CA_s(n)$ is spanned by a certain subset of diagrams $D_k \subset D_s$ constructed in the following way. For a, b tensor words in $u, \bar{u}$ and for a diagram $T \in D_s(|a|, |b|)$, where $|\cdot|$ is the length of words, we have $T \in D_k(a, b)$ provided that the following happens: when putting a, b on the two rows of points of T, with the replacements $u \to xy$, $\bar{u} \to yx$, where x, y are two colors, the strings of T have to match the colors.

Summarizing, we have a diagrammatic description of the categories on the left and on the right, and we want to compute the categories in the middle:

$$\operatorname{span}(D_k) \subset CA_h^\infty(n) \subset CA_h^s(n) \subset \operatorname{span}(D_s).$$

The idea would be to prove that the categories in the middle are spanned by certain sets of diagrams, say D_h^∞ and D_h^s.

Step 3. In order to compute D_h^∞, we use Woronowicz's Tannakian duality [30], which allows one to translate algebraic relations into categorical relations. We know that $A_h^\infty(n)$ is the quotient of $A_k(n)$ by the relations making the elements u_{ij} normal, so we must first find a diagrammatic formulation of these normality conditions. Consider the following diagram in $D_s(2,2)$: $P = |\,{}^{\cup}_{\cap}\,|$.

According to the general rules in [4] for diagrammatic calculus for $A_s(n)$, we have the following formula for this diagram, viewed as an operator: $P = \sum_i e_{ii} \otimes e_{ii}$.

Now, let u be the fundamental corepresentation of $A_k(n)$. We have

$$\begin{aligned}(P \otimes 1)(u \otimes \bar{u}) &= \Big(\sum_i e_{ii} \otimes e_{ii} \otimes 1\Big)\Big(\sum_{ijkl} e_{ij} \otimes e_{kl} \otimes u_{ij}u_{kl}^*\Big)\\ &= \sum_{ijl} e_{ij} \otimes e_{il} \otimes u_{ij}u_{il}^* = \sum_{ij} e_{ij} \otimes e_{ij} \otimes u_{ij}u_{ij}^*.\end{aligned}$$

Once again by using the defining relations for $A_k(n)$, we have

$$\begin{aligned}(\bar{u} \otimes u)(P \otimes 1) &= \Big(\sum_{ijkl} e_{ij} \otimes e_{kl} \otimes u_{ij}^*u_{kl}\Big)\Big(\sum_j e_{jj} \otimes e_{jj} \otimes 1\Big)\\ &= \sum_{ijk} e_{ij} \otimes e_{kj} \otimes u_{ij}^*u_{kj} = \sum_{ij} e_{ij} \otimes e_{ij} \otimes u_{ij}^*u_{ij}.\end{aligned}$$

We conclude that the n^2 normality conditions for the generators $u_{ij} \in A_k(n)$ are equivalent to the following condition: $P \in \operatorname{Hom}(u \otimes \bar{u}, \bar{u} \otimes u)$.

Now, by applying Tannakian duality, we get $CA_h^\infty(n) = \operatorname{span}\langle D_k, P\rangle$.

Thus, we have proved a result announced in Step 2, namely that $CA_h^\infty(n)$ is spanned by diagrams. In the sense of [4], this means that $A_h^\infty(n)$ is free.

Step 4. We now compute $CA_h^\infty(n)$ explicitly.

We define a subset $D_h^\infty \subset D_s$ in the following way. For a, b tensor words in $u, \bar{u}$ and for a diagram $T \in D_s(|a|, |b|)$, where $|\cdot|$ is the length of words, we have $T \in D_h^\infty(a, b)$ provided that the following happens: when collapsing consecutive neighbors of T, as to get a noncrossing partition $\tilde{T}$, then putting the words a, b on the points of $\tilde{T}$, each block has the same number of u and $\bar{u}$.

Here the collapsed diagram $\tilde{T}$ is best thought of as being a noncrossing partition of points on a circle (with a marked point). Another approach is to use the formalism of annular noncrossing partitions of Mingo and Nica [19].

We have, by definition, embeddings $D_k \subset D_h^\infty \subset D_s$.

Consider now the diagram $P = |\,{}^{\cup}_{\cap}\,|$. When performing the collapsing operation we get the diagram H, having a single block. Now when putting the tensor words $u \otimes \bar{u}$ and $\bar{u} \otimes u$ on the points of H, this unique block contains two u's and two $\bar{u}$'s, so the above condition is satisfied. That is, we have $P \in D_h^\infty(u \otimes \bar{u}, \bar{u} \otimes u)$.

It is routine to check that each diagram in D_h^∞ decomposes as a product of diagrams in D_k and of diagrams of the following type: $\| \cdots |\,{}^{\cup}_{\cap}\,| \cdots \|$.

These latter diagrams being tensor products of P with the identity, we have $D_h^\infty = \langle D_k, P \rangle$. Now by combining this with the result in Step 3, we get $CA_h^\infty(n) = \operatorname{span}(D_h^\infty)$.

Summarizing, we now have a fully satisfactory description of $CA_h^\infty(n)$.

Step 5. We compute now the category CA_h^s, for arbitrary values of s. We use the same idea as in Step 3. Consider the following diagram in $D_s(0, s+2)$: $Z = |\overline{\sqcap\sqcap\cdots\sqcap\sqcap}|$.

When viewed as an operator, this diagram is uniquely determined by its value $\xi = Z(1)$, which is a vector in $(\mathbb{C}^n)^{\otimes s+2}$, given by the following formula: $\xi = \sum_i e_i^{\otimes s+2}$.

Now let u be the fundamental corepresentation of $A_h^\infty(n)$. We have:

$$(u^{\otimes s+1} \otimes \bar{u})(\xi \otimes 1) = \sum_{i_1 \cdots i_n j} e_{i_1} \otimes \cdots \otimes e_{i_{s+2}} \otimes u_{i_1 j} \cdots u_{i_{s+1} j} u^*_{i_{s+2} j}$$

$$= \sum_{ij} e_i \otimes \cdots \otimes e_i \otimes u_{ij}^{s+1} u_{ij}^* = \sum_i e_i^{\otimes s+2} \otimes \Big(\sum_j u_{ij}^{s+1} u_{ij}^* \Big).$$

This shows that ξ is a fixed vector of the corepresentation $u^{s+1} \otimes \bar{u}$ if and only if the following condition is satisfied. For any i, $\sum_j u_{ij}^{s+1} u_{ij}^* = 1$.

Now, since for i fixed the supporting projections $p_{ij} = u_{ij} u_{ij}^*$ are pairwise orthogonal and sum up to 1, it is routine to check that this condition is equivalent to $u_{ij}^s = p_{ij}$ for any j. Thus, by getting back to the diagram Z, the collection of conditions $u_{ij}^s = p_{ij}$ on the generators $u_{ij} \in A_h^\infty(n)$ is equivalent to

$$Z \in \operatorname{Hom}(1, u^{\otimes s+1} \otimes \bar{u}).$$

Now by applying Tannakian duality, we get

$$CA_h^s(n) = \operatorname{span}\langle D_h^\infty, Z \rangle.$$

Thus, we have reached a similar conclusion to the one at the end of Step 3.

Step 6. We now compute $CA_h^s(n)$ explicitly. Let us point out first that this category is already known at $s = 1, 2, \infty$, from [4, 6] and Step 4.

We define a subset $D_h^s \subset D_s$ in the following way. For a, b tensor words in $u, \bar{u}$ and for a diagram $T \in D_s(|a|, |b|)$, where $|\cdot|$ is the lenght of words, we have $T \in D_h^s(a, b)$

provided that the following happens: when putting the words a, b on the points of the corresponding noncrossing partition $\tilde{T}$, each block has the same number of u and $\bar{u}$, modulo s.

We have, by definition, embeddings $D_h^\infty \subset D_h^s \subset D_s$.

Consider now the diagram Z. When performing the collapsing operation, we get the diagram I_{s+2} having $s+2$ legs and a single block. Now when putting the word $u^{\otimes s+1} \otimes \bar{u}$ on the points of I_{s+2}, this unique block contains $s+1$ copies of u and one copy of $\bar{u}$, so the above condition is satisfied. That is, we have

$$Z \in D_h^s(1, u^{\otimes s+1} \otimes \bar{u}).$$

It is routine to check that each diagram in D_h^s decomposes as a product of diagrams in D_h^∞ and of diagrams of the following type:

$$\| \cdots |\overline{\cap\cap\cdots\cap\cap}| \cdots \| .$$

These latter diagrams being tensor products of Z with the identity, we have $D_h^s = \langle D_h^\infty, Z\rangle$. Now by combining this with the result in Step 5, we get $CA_h^s(n) = \operatorname{span}(D_h^s)$.

This finishes the categorial computations of the present proof.

Step 7. We are now in a position to prove the integration results.

We recall from the previous step that the space of fixed vectors of a k-fold tensor product a between $u, \bar{u}$ can be identified with the abstract vector space spanned by the set $P_h^s(a)$ of noncrossing partitions of $\{1, \ldots, k\}$, having the following property: when putting the word a on the points of the partition, each block has to contain the same number of u and $\bar{u}$, modulo s.

By [29], the $*$-moments of χ_1 are the number of fixed points of the tensor products between u and $\bar{u}$, which are in turn equal to the number of diagrams in P_h^s. That is, if $e_1, \ldots, e_k \in \{1, *\}$ are exponents and $a = (u_{ij}^{e_1}) \otimes \cdots \otimes (u_{ij}^{e_k})$ is the corresponding tensor product between $u, \bar{u}$, then

$$\int \chi_1^{e_1} \cdots \chi_1^{e_k} = \int \chi(a) = \dim \operatorname{Hom}(1, a) = \# P_h^s(a).$$

The idea will be that, by performing a computation using free cumulants, these numbers will turn to be as well the $*$-moments of $\tilde{\pi}_{s1}$.

Step 8. We now complete the proof at $t = 1$.

We recall from Section 7 that if $\alpha_1, \ldots, \alpha_s$ are free Poisson variables of parameter $1/s$ and $w = e^{2\pi i/s}$, then the following variable has law $\tilde{\pi}_{s1}$: $\alpha = \sum_{l=1}^s w^l \alpha_l$.

We can compute the $*$-moments of this variable by using free cumulants. With

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standard notations from [20], we have

$$\begin{aligned}\int \alpha^{e_1}\cdots\alpha^{e_k} &= \sum_{p\in NC(k)} K_p(\alpha^{e_1},\dots,\alpha^{e_k})\\ &= \sum_{p\in NC(k)}\sum_{i_1\cdots i_k=1}^{s} K_p((w^{i_1}\alpha_{i_1})^{e_1},\dots,(w^{i_k}\alpha_{i_k})^{e_k})\\ &= \sum_{p\in NC(k)}\sum_{i_1\cdots i_k=1}^{s} w^{i_1\varepsilon_1+\cdots+i_k\varepsilon_k} K_p(\alpha_{i_1},\dots,\alpha_{i_k}).\end{aligned}$$

Here the signs $\varepsilon_i \in \{1,-1\}$ come from the exponents $e_i \in \{1,*\}$.

We now use Speicher's result that the mixed cumulants vanish [21]. This shows that for a nonzero term in the above sum, the corresponding indices $i_1,\dots,i_k$ must be constant over the blocks of p. Now by factoring each cumulant on the right as a product over the blocks $b=\{b_1,\dots,b_r\}$ of p, we get

$$\begin{aligned}\int \alpha^{e_1}\cdots\alpha^{e_k} &= \sum_{p\in NC(k)}\prod_{b\in p}\sum_{i=1}^{s} w^{i\varepsilon_{b_1}+\cdots+i\varepsilon_{b_r}} K_b(\alpha_1,\dots,\alpha_1)\\ &= \sum_{p\in NC(k)}\prod_{b\in p}\sum_{i=1}^{s} (w^{\varepsilon_{b_1}+\cdots+\varepsilon_{b_r}})^i K_b(\alpha_1,\dots,\alpha_1)\\ &= \sum_{p\in NC(k)}\prod_{b\in p}(s|\varepsilon_{b_1}+\cdots+\varepsilon_{b_r}) sK_b(\alpha_1,\dots,\alpha_1).\end{aligned}$$

Here the symbol $(s|m)$ is given by $(s|m)=1$ if $s|m$, and $(s|m)=0$ if not.

Now, given a partition p, in order for its contribution to the above $*$-moment to be nonzero, we must have $s|\varepsilon_{b_1}+\cdots+\varepsilon_{b_r}$ for any block $b\in p$. But this is the same as saying that when putting the word a on the points of p, each block of p contains the same number of u's and $\bar{u}$'s, modulo s, which is, by definition, equivalent to $p\in P_h^s(a)$. Thus we have

$$\int \alpha^{e_1}\cdots\alpha^{e_k} = \sum_{p\in P_h^s(a)}\prod_{b\in p} sK_b(\alpha_1,\dots,\alpha_1).$$

Now by general results in [20], each of the numbers on the right is $s(1/s)=1$, so the above $*$-moment equals $\#P_h^s(a)$. This finishes the proof at $t=1$.

Observe that we have $\mathrm{law}(\chi_1)=\tilde{\pi}_{s1}$, independently of $n\geq 4$. The fact that the convergence is stationary is not surprising, in view of Proposition 10.1(ii).

Step 9. We now discuss the general case $t>0$. Here the convergence $\mathrm{law}(\chi_t)\to\tilde{\pi}_{st}$ will no longer come from a stationary sequence, and we will have to use a more technical argument, based on the Weingarten formula:

$$\int u_{i_1j_1}^{e_1}\cdots u_{i_kj_k}^{e_k} = \sum_{p,q\in P_h^s(a)} \delta_{pi}\delta_{qj}W_{an}(p,q).$$

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Here we use exponents $e_1, \dots, e_k \in \{1, *\}$, and $a = (u_{ij}^{e_1}) \otimes \cdots \otimes (u_{ij}^{e_k})$ is the corresponding tensor product between $u, \bar{u}$. The delta symbols, equal to 0 or 1, represent the couplings between diagrams and multi-indices, and W_{an} is the Weingarten matrix, obtained as inverse of the Gram matrix. See [4, 5].

Now once again by general arguments developed in [4–6], the Weingarten formula leads to the following formula for the asymptotic $*$-moments of χ_t:

$$\lim_{n\to\infty} \int \chi_t^{e_1} \cdots \chi_t^{e_k} = \sum_{p \in P_h^s(a)} t^{|p|}.$$

Here we use, as above, exponents $e_1, \dots, e_k \in \{1, *\}$, along with the corresponding tensor product $a = (u_{ij}^{e_1}) \otimes \cdots \otimes (u_{ij}^{e_k})$ between $u, \bar{u}$. As for the exponent $|p|$ on the right, this is the number of blocks of p. See [4].

At the level of modified free Bessel laws now, what changes when making the replacement $\tilde{\pi}_{s1} \to \tilde{\pi}_{st}$ is that the variables $\alpha_1, \dots, \alpha_s$ now become free Poisson variables of parameter t/s. Thus, in the cumulant computation in Step 8, what changes is the contribution of the partitions: instead of a product of numbers $s(1/s) = 1$, we now have a product of numbers $s(t/s) = t$. We get

$$\int \alpha^{e_1} \cdots \alpha^{e_k} = \sum_{p \in P_h^s(a)} t^{|p|}.$$

Summarizing, in both computations, each partition now contributes an additive factor t^b, where b is the number of blocks, and we are done. ∎

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