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# STABLE PLUMBING FOR HIGH ODD-DIMENSIONAL FIBRED KNOTS

BY

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ABSTRACT. Plumbing a Hopf band on the fibre-surface of a simple fibred knot is a geometric operation that produces another such knot. We show by algebraic methods that every high odd-dimensional simple fibred knot is obtained from the unknot by using this operation and its inverse.

In [3], J. Harer has shown that every classical fibred link bounds a fibre-surface obtained by a sequence of Hopf plumbings, Stallings twists and deplumbings. P. Melvin and H. Morton ([6]) have given examples of genus 2 classical fibred knots that cannot be obtained by Hopf plumbings only. They have nevertheless remarked that their examples are stably obtained by Hopf plumbings (i.e. by using Hopf plumbings and deplumbings) and J. Harer ([3], §5) asks whether the same holds for all classical fibred links.

We have generalized in [4] and [5] to high-dimensional knots the concepts of plumbing and Stallings twist and shown that there exist for all $k \geqslant 3$ simple spherical fibred $(2k - 1)$-dimensional knots that cannot be obtained by Hopf plumbings and Stallings twists only ([4], Theorem 5.5).

The aim of this note is to prove by algebraic methods the following theorem:

THEOREM 1. *Let $k$ be an integer, $k \geqslant 3$. Every simple fibred $(2k - 1)$-dimensional knot is stably obtained by Hopf plumbings.*

We recall below a few facts about high-dimensional knots (see [4] for details).

DEFINITION. *A $(2k - 1)$-dimensional knot $K$ is an oriented differentiable closed $(k - 2)$-connected codimension 2 submanifold of $S^{2k+1}$ ($K$ is therefore not necessarily spherical). The knot $K$ is simple if it bounds a $(k - 1)$-connected Seifert surface.*

In [2], A. Durfee has shown that for $k \geqslant 3$ there is a one-to-one correspondence between:

i) isotopy classes of simple fibred $(2k - 1)$-dimensional knots,

ii) isotopy classes of $(k - 1)$-connected $(2k)$-dimensional fibre-surfaces,

iii) congruence classes of integral unimodular matrices.

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(Two matrices $A$ and $A'$ are *congruent* if there exists an integral unimodular matrix $P$ such that $P^T AP = A'$).

The correspondence associates to a simple fibred knot its fibre-surface and a Seifert matrix for its fibre-surface respectively.

For $k = 1$, the two *Hopf bands* are the fibre-surfaces of the two Hopf links (see [3], fig. 1); for $k \geq 3$ they are by definition the two fibre-surfaces corresponding via the classification above to the two rank 1 matrices $(+1)$ and $(-1)$. (For a topological characterization of these fibre-surfaces, see [4], §2).

DEFINITION. *Let $F$ be a $(2k)$-dimensional $(k - 1)$-connected fibre-surface and $H$ be a $(2k)$-dimensional Hopf band. Divide $S^{2k+1}$ into two hemispheres $B_1$ and $B_2$ intersecting in a $(2k)$-dimensional sphere $S$. Let $\psi : D^k \times D^k \hookrightarrow S$ be an embedding and suppose that:*

i) *$F \subseteq B_1$ and $H \subseteq B_2$,*

ii) *$F \cap S = H \cap S = F \cap H = \psi(D^k \times D^k)$,*

iii) $\psi(\partial D^k \times D^k) = \partial F \cap \psi(D^k \times D^k)$ and $\psi(D^k \times \partial D^k) = \partial H \cap \psi(D^k \times D^k)$,

iv) *the orientations of $F$ and $H$ match on $\psi(D^k \times D^k)$.*

$F' = F \cup H$ is again a $(k - 1)$-connected fibre-surface and we say that $F'$ is *obtained by plumbing together* $F$ and the Hopf band $H$.

DEFINITION. *Let $A$ and $A'$ be integral unimodular matrices. We say that $A'$ is an extension of $A$ if $A'$ is congruent to*

$$\left[\begin{array}{c|c} & 0 \\ A & \vdots \\ & 0 \\ \hline x_1 \ldots x_n & \epsilon \end{array}\right]$$

*where $n = \operatorname{rank} A$, $x_i \in \mathbb{Z}$, $i = 1, \ldots, n$ and $\epsilon = \pm 1$. We say that it is a positive extension if $\epsilon = +1$.*

The proof of the following lemma is contained in the proof of [4], proposition 2.4.

LEMMA 2. *Let $k \geq 3$, $F$ and $F'$ be two $(k - 1)$-connected $(2k)$-dimensional fibre-surfaces in $S^{2k+1}$, let $A$ and $A'$ be two Seifert matrices for $F$ and $F'$ respectively. The fibre-surface $F'$ is obtained by plumbing together $F$ and a Hopf band if and only if $A'$ is an extension of $A$.*

LEMMA 3. *Let $F$ be a $(k - 1)$-connected $(2k)$-dimensional fibre-surface in $S^{2k+1}$. If $F_1$ and $F_2$ are obtained by plumbing together $F$ and a Hopf band, there is a fibre-surface $\bar{F}$ obtained by plumbing together $F_1$ and a Hopf band on one hand, $F_2$ and a Hopf band on the other.*

PROOF. This lemma can be proved by geometric arguments for all dimensions; for $k \geq 3$ it results from the following fact: If $A$ is a Seifert matrix for $F$,

$$A_1 = \left[\begin{array}{c|c} A & \begin{matrix}0\\ \cdot\\ \cdot\\ \cdot\\ 0\end{matrix} \\ \hline x_1 \ldots x_n & \epsilon_1 \end{array}\right]$$

is a Seifert matrix for $F_1$,

$$A_2 = \left[\begin{array}{c|c} A & \begin{matrix}0\\ \cdot\\ \cdot\\ \cdot\\ 0\end{matrix} \\ \hline y_1 \ldots y_n & \epsilon_2 \end{array}\right]$$

is a Seifert matrix for $F_2$, then

$$\bar{A} = \left[\begin{array}{c|cc} A & \begin{matrix}0\\ \cdot\\ \cdot\\ \cdot\\ 0\end{matrix} & \begin{matrix}0\\ \cdot\\ \cdot\\ \cdot\\ 0\end{matrix} \\ \hline x_1 \ldots x_n & \epsilon_1 & 0 \\ y_1 \ldots y_n & 0 & \epsilon_2 \end{array}\right]$$

is an extension of both $A_1$ and $A_2$; the fibre-surface $\bar{F}$ corresponding to $\bar{A}$ satisfies the required properties.

DEFINITION. *Let $K$ be a simple $(2k - 1)$-dimensional knot in $S^{2k+1}$. $K$ is stably obtained by Hopf plumbings if there is a sequence $F_0, F_1, \ldots, F_s$* of *$(2k)$-dimensional submanifolds of $S^{2k+1}$ such that $F_0 = D^{2k}$, $\partial F_s = K$ and: either $F_{i+1}$ is obtained by plumbing together $F_i$ and a Hopf band or $F_i$ is obtained by plumbing together $F_{i+1}$ and a Hopf band.*

By repeated applications of lemma 3, this definition is equivalent to the following one:

*There is a sequence $F_0, F_1, \ldots, F_s$ of $(2k)$-dimensional submanifolds of $S^{2k+1}$ and an integer $s_0$, $0 < s_0 \leqslant s$ such that $F_0 = D^{2k}$, $\partial F_s = K$ and $F_{i+1}$ is obtained by plumbing together $F_i$ and a Hopf band for $0 \leqslant i \leqslant s_0 - 1$, $F_i$ is obtained by plumbing together $F_{i+1}$ and a Hopf band for $s_0 \leqslant i \leqslant s - 1$.*

DEFINITION. *Let $A$ and $A'$ be two integral unimodular matrices, we say that $A$ and $A'$ are equivalent if there is a sequence of matrices $A_0, A_1, \ldots, A_s$ such that $A_0 = A$, $A_s = A'$ and either $A_{i+1}$ is an extension of $A_i$ or $A_i$ is an extension of $A_{i+1}$.*

Note that if $A_1$ is congruent to $A_2$, $\left(\begin{smallmatrix} A_1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is congruent to $\left(\begin{smallmatrix} A_2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, so that $A_1$ and $A_2$ are equivalent.

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The algebraic analogue to theorem 1 is:

PROPOSITION 4. *Every unimodular matrix is equivalent to the empty matrix.*

To prove this proposition, we shall need the following three lemmas.

LEMMA 5. *Let $B:L \times L \to \mathbb{Z}$ be a unimodular bilinear form defined on a free $\mathbb{Z}$-module of rank $n \geq 3$. There is a basis $e_1, e_2, \ldots, e_n$ of $L$ such that $B(e_2; e_1) = 1$.*

PROOF. Let $f_1, \ldots, f_n$ be a basis of $L$ and set $a_{ij} = B(f_i; f_j)$; $B$ is unimodular so that there exists an element $x$ in $L$ such that $B(x; f_1) = 1$. Let $x = \sum_{i=1}^{n} \alpha_i f_i$ be the expression of $x$ in terms of the $f_i$, we have:

$$(*) \qquad B(x; f_1) = \alpha_1 a_{11} + \alpha_2 a_{21} + \alpha_3 a_{31} + \ldots + \alpha_n a_{n1} = 1.$$

By [1] (chapter 7, theorem 3.1), $f_1$ and $x$ are part of a basis of $L$ if and only if the ideal $(\alpha_2; \alpha_3; \ldots; \alpha_n)$ is equal to $\mathbb{Z}$.

If $a_{11} = 0$, (*) shows that this is indeed the case. Suppose $a_{11} \neq 0$, we shall modify $x$ to satisfy the condition above. Set $y = a_{31} f_1 - a_{11} f_3$; $B(y; f_1) = 0$ so that $x' = x + ky$ satisfies $B(x'; f_1) = 1$ for any integer $k$. The vector $x'$ has coordinates:

$$\alpha_1 + k a_{31}, \alpha_2, \alpha_3 - k a_{11}, \ldots, \alpha_n.$$

If $\alpha_3 = 0$, set $k = 1$, then (*) shows that $(\alpha_2; -a_{11}; \ldots; \alpha_n) = \mathbb{Z}$. If $\alpha_3 \neq 0$, let $\gamma$ be a greatest common divisor of $\alpha_3$ and $a_{11}$; set $\alpha_3 = \gamma \alpha_3'$, $a_{11} = \gamma a_{11}'$ where $\alpha_3'$ and $a_{11}'$ are relatively prime. Choose $k$ so that $\alpha_3' - k a_{11}'$ is relatively prime to $\alpha_2$ (for instance let $k$ be the product of prime divisors of $\alpha_2$ that do not divide $\alpha_3'$ or apply Dirichlet's theorem on primes in arithmetic progression). For such an integer $k$, we have $(\alpha_2; \alpha_3 - k a_{11}; \ldots; \alpha_n) = \mathbb{Z}$. To show this, suppose there is an integer $d$, $d > 1$, such that $d | \alpha_2$, $d | \alpha_3 - k a_{11}$, $d | \alpha_i$, $i = 4, \ldots, n$. Then $d | \gamma$ and hence $d | \alpha_2$, $d | \alpha_3$, $d | a_{11}$, $d | \alpha_i$, $i = 4, \ldots, n$, thus contradicting (*).

REMARK. Lemma 5 is false if the rank of $L$ is 2: The form whose matrix with respect to the standard basis of $\mathbb{Z}^2$ is given by $\left(\begin{smallmatrix} 8 & 21 \\ 3 & 8 \end{smallmatrix}\right)$ does not have two basis vectors $e_1$, $e_2$ such that $B(e_2; e_1) = 1$. The existence of two such vectors is equivalent to an integral solution of the following system of two equations:

$$\begin{cases} 8x_1x_3 + 3x_2x_3 + 21x_1x_4 + 8x_2x_4 = 1 \\ \\ x_1x_4 - x_2x_3 = \epsilon = \pm 1 \end{cases}$$

Introducing the second equation into the first, one sees that such a solution would imply that $1 - 21\epsilon \equiv 0 \pmod 8$.

LEMMA 6. *Every integral unimodular matrix $A$ of rank $n \geq 3$ is equivalent to a matrix $A'$ of rank $n - 1$.*

PROOF. Lemma 5 shows that $A$ is congruent to a matrix of the form:

$$A_1 = \left[\begin{array}{cc|c} a_{11} & a_{12} & \\ & & Y_1 \\ 1 & a_{22} & \\ \hline & Y_2 & Y_3 \end{array}\right]$$

where the $Y_i$ denote block matrices.

Extend this matrix to

$$A_2 = \left[\begin{array}{cc|c|c} a_{11} & a_{12} & & 0 \\ & & Y_1 & \\ 1 & a_{22} & & 0 \\ \hline & & & 0 \\ & & & \cdot \\ & Y_2 & Y_3 & \cdot \\ & & & \cdot \\ & & & 0 \\ \hline -a_{11} & x & 0\cdots 0 & 1 \end{array}\right]$$

where the integer $x$ will be determined below. Adding the last column to the first and then the last row to the first, we see that $A_2$ is congruent to

$$A_3 = \left[\begin{array}{cc|c} 1 & a_{12}+x & \\ & & Z_1 \\ 1 & a_{22} & \\ \hline & Z_2 & Z_3 \end{array}\right]$$

By substracting a suitable multiple of column 1 from column $j$, $j = 2, \ldots, n+1$, one can change the first row to $(1; 0; \ldots; 0)$. After the corresponding row operations are performed, we see that $A_3$ is congruent to

$$A_4 = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0\cdots 0 \\ & & & \\ 1 - a_{12} - x & a_{22} - a_{12} - x & z_3 & z_4\cdots z_{n+1} \\ \hline & Z_2' & & Z_3' \end{array}\right]$$

Choose $x$ so that $a_{22} - a_{12} - x = 1$ and perform the same operations as above to annihilate $z_i$, $i = 3, \ldots, n+1$. This shows that $A_4$ is congruent to a matrix of the form:

$$\left[\begin{array}{cc|c} 1 & 0 & \\ & & 0 \\ 2 - a_{22} & 1 & \\ \hline Z_2'' & & Z_3'' \end{array}\right]$$

The matrix $A' = Z_3''$ is unimodular of rank $n-1$ and equivalent to $A$.

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LEMMA 7. *Let* $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ *be an integral unimodular matrix and set:* $\mu(A) = \min\{|p|; |q|; |r|; |s|\}$. *If* $\mu(A) \geq 1$ *there is a rank 2 matrix* $A'$ *equivalent to* $A$ *such that* $\mu(A') < \mu(A)$.

PROOF. After a permutation of the rows and columns if necessary, one may suppose that $\mu(A) = |p|$ or $\mu(A) = |r|$. If $\mu(A) = |p|$ there is an integer $k$ such that $|r - kp| < |p|$. By substracting $k$ times row 1 from row 2 and then $k$ times column 1 from column 2, we see that $A$ is congruent to

$$A' = \begin{pmatrix} p & q - kp \\ r - kp & s - kq - kr + k^2 p \end{pmatrix}.$$

This shows that $\mu(A') \leqslant |r - kp| < \mu(A)$.

If $\mu(A) = |r|$, extend $A$ to

$$\begin{pmatrix} p & q & 0 \\ r & s & 0 \\ -p & y & 1 \end{pmatrix}.$$

As in the proof of lemma 6, one sees that this matrix is congruent to

$$\begin{pmatrix} 1 & 0 & 0 \\ r - q - y & s - (q + y)r & -r \\ -p & y - (1 - p)(q + y) & p \end{pmatrix}.$$

This shows that $A$ is equivalent to

$$A' = \begin{pmatrix} s - (q + y)r & -r \\ y - (1 - p)(q + y) & p \end{pmatrix}.$$

Choose $y$ so that $|s - (q + y)r| < |r|$. This implies that $\mu(A') \leqslant |s - (q + y)r| < \mu(A)$.

PROOF OF PROPOSITION 4. Let $A$ be an integral unimodular matrix of rank $n$. At most $n - 2$ applications of lemma 6 show that $A$ is equivalent to a rank 2 matrix $A'$. Using lemma 7 repeatedly, we see that $A'$ is equivalent to a unimodular matrix $A''$ of rank 2 such that $\mu(A'') = 0$. After a permutation of the rows and columns if necessary, $A''$ is either of the form $\begin{pmatrix} \epsilon_1 & 0 \\ a & \epsilon_2 \end{pmatrix}$ or $\begin{pmatrix} 0 & \epsilon_1 \\ \epsilon_2 & a \end{pmatrix}$ where $\epsilon_1 = \pm 1$, $\epsilon_2 = \pm 1$, $a \in \mathbb{Z}$. In the first case $A''$ is equivalent to the empty matrix. In the second case, extend $A''$ to

$$\begin{pmatrix} 0 & \epsilon_1 & 0 \\ \epsilon_2 & a & 0 \\ 0 & \epsilon_2 a & 1 \end{pmatrix}$$

which by the same procedure as above is equivalent to $\begin{pmatrix} -\epsilon_1\epsilon_2 & -\epsilon_2 \\ -\epsilon_1 & 0 \end{pmatrix}$. The latter is congruent to $\begin{pmatrix} -\epsilon_1\epsilon_2 & 0 \\ \epsilon_2 - \epsilon_1 & 1 \end{pmatrix}$ and hence equivalent to the empty matrix.

REMARKS. i) The proof of theorem 1 is straightforward from proposition 4 and lemma 2.

ii) Each application of lemma 6 corresponds geometrically to one plumbing and two deplumbing operations. A simple fibred high odd-dimensional knot of rank $n$ is therefore connected by at most $3(n-2)$ plumbing operations to a rank 2 knot but the number of operations needed to connect it to the trivial knot cannot be predicted.

iii) Only positive extensions are used in the proofs of lemma 6 and lemma 7. As $\begin{pmatrix} -1 & 0 \\ a & -1 \end{pmatrix}$ can also be shown to be connected to some $\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$ by positive extensions, every high odd-dimensional simple fibred knot with associated unimodular matrix $A$ can be connected to the trivial knot by a sequence of plumbings of positive Hopf bands if $\det(A) = +1$, positive Hopf bands and one negative Hopf band if $\det(A) = -1$.

iv) As noted in [4] (final remark), proposition 4 is equivalent to the following property: If $A$ is any $m \times m$ integral unimodular matrix, there exists an integral unimodular triangular $n \times n$ matrix $T$ and an integral $n \times m$ matrix $B$ such that $\begin{pmatrix} A & 0 \\ B & T \end{pmatrix}$ is congruent to a triangular matrix.

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