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DEDEKIND SUMS FOR A FUCHSIAN GROUP, I

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§ 1. Introduction

The well-known first limit formula of Kronecker asserts that

$$\lim_{s\to 1}\left[y^s \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} |m+nz|^{-2s} - \frac{\pi}{s-1}\right]$$
$$= 2\pi(C - \log 2 - \log(\sqrt{y}\,|\eta(z)|^2))$$

where $z = x + iy$ is contained in the complex upper halfplane $\boldsymbol{H}$, $C =$ the Euler-Mascheroni constant, and $\eta(z)$ is the Dedekind eta-function defined by

$$\eta(z) = e^{\pi iz/12} \prod_{n=1}^{\infty} (1 - e^{2\pi inz}) \qquad (z \in \boldsymbol{H}) .$$

It is a simple matter to deduce from the first limit formula that

$$\eta\left(\frac{az+b}{cz+d}\right) = \varepsilon\sqrt{cz+d}\,\eta(z) , \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \boldsymbol{Z}) ,$$

where $|\varepsilon| = 1$, $\varepsilon = \varepsilon(a, b, c, d)$. It is possible to calculate ε explicitly, as was first accomplished by Dedekind [1], who proved that if all branches of the logarithm are taken with respect to the principal branch,[1] then

$$\log \eta\left(\frac{az+b}{cz+d}\right) = \log \eta(z) + \frac{1}{2}\log(cz+d) + \pi i S(a, b, c, d) ,$$

where

$$S(a, b, c, d) = \frac{a+d}{12c} - \frac{1}{4}\frac{c}{|c|} - s(c, d) ,$$

and where $s(c, d)$ is a so-called *Dedekind sum* which has the following elementary expression:

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[1] That is, the branch for which $-\pi > \arg \log z \leq \pi$.

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$$s(c, d) = \sum_{\mu=0}^{|c|-1} \left(\left(\frac{d\mu}{|c|}\right)\right) \left(\left(\frac{\mu}{|c|}\right)\right)$$

Here $((x)) = x - [x] - \frac{1}{2}$, $[x] =$ the largest integer $\leq x$.[1]

It is our purpose, in the following paper, to construct a generalization of the Kronecker first limit formula which leads to generalizations of all the above classical facts. The main observation is that the sum on the left hand side of the Kronecker first limit formula is, apart from a simple factor, the Eisenstein series (in the sense of Selberg) associated to the cusp at infinity for the classical modular group. In this paper, we will exhibit a corresponding limit formula for any Eisentein series and any cusp of a Fuchsian group of the first kind Γ—that is, Γ is a discrete subgroup of $SL(2, \boldsymbol{R})$ with finite invariant volume. If κ is a cusp of Γ, then the limit formula for the pair (Γ, κ) will lead to an everywhere non-zero automorphic from of weight 1/2 which is the analogue of the classical function $\eta(z)$. In analogy with the classical theory, we will derive a transformation law for our generalized η-function. And this law of transformation will lead to generalized Dedekind sums. We work out the theory explicitly for the principal congruence groups $\Gamma(N)$ $(N \geq 1)$ defined by

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a \equiv d \equiv 1 \,(\mathrm{mod}\, N),\, b \equiv c \equiv 0 \,(\mathrm{mod}\, N) \right\}$$

and we find explicit formulas for the Dedekind sums in this case. It appears that these generalized Dedekind sums have interesting arithmetic properties connected with reciprocity laws in certain abelian extensions of the rationals, but this topic will not be taken up in this paper.

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§ 2. The Fourier expansion of the Eisenstein series at a cusp

Throughout this paper, let Γ denote a discrete subgroup of $SL(2, \boldsymbol{R})/\{\pm 1\}$ having a fundamental domain of finite invariant measure. Then Γ acts discontinuously on the complex upper half-plane $\boldsymbol{H}$. Let $\overline{\boldsymbol{H}} = \boldsymbol{H} \cup \{\infty\}$

[1] For x an integer, set $((x)) = 0$.

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and let the action of Γ be extended to $\overline{\boldsymbol{H}}$ in the usual manner. Further, let $\kappa_1, \cdots, \kappa_h$ be a complete set of inequivalent cusps of Γ. Choose $\sigma_i \in SL(2, \boldsymbol{R})$ so that

$$\sigma_i(\infty) = \kappa_i ,$$
$$\sigma_i^{-1}\Gamma_i\sigma_i = \left\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} \middle| x \in \boldsymbol{Z}\right\} .$$

where $\Gamma_i = \{\sigma \in \Gamma \,|\, \sigma(\kappa_i) = \kappa_i\} =$ the Γ-stabilizer of κ_i. For $z = x + iy \in \boldsymbol{H}$, let $y(z) = y$. Then the Eisenstein series[1] for the cusp κ_i is defined by

$$E_i(z, s) = \sum_{\sigma \in \Gamma_i \backslash \Gamma} y(\sigma_i^{-1}\sigma z)^s \qquad (\mathrm{Re}\,(s) > 1,\ z \in \boldsymbol{H}) . \tag{1}$$

This series converges absolutely and uniformly for s in a compact subset of $\mathrm{Re}\,(s) > 1$. Moreover, one of the fundamental theorems in the theory of Eisenstein series asserts that $E_i(z, s)$ has an analytic continuation in s, namely:

THEOREM A. *$E_i(z, s)$ can be analytically continued to a meromorphic function in the entire s-plane and the only poles of the continued function are simple and lie in the interval* $[0, 1]$. *Moreover, $E_i(z, s)$ always has a pole at $s = 1$.*

As a function of $z, E_i(z, s)$ is an automorphic function for Γ—that is,

$$E_i(\sigma z, s) = E_i(z, s) \qquad (\sigma \in \Gamma) .$$

Moreover, $E_i(z, s)$ is an eigenfunction of the Laplace-Beltrami operator

$$D = y^{-2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{2}$$

for the symmetric space $\boldsymbol{H}$:

$$DE_i(z, s) = s(s - 1)E_i(z, s) . \tag{3}$$

Because of the property (2), $E_i(z, s)$ can be expanded in a Fourier series in a neighbourhood of the cusp κ_j. Elementary computations suffice to show that this Fourier series is of the form [3, p. 28].

$$E_i(\sigma_j z, s) = \sum_{m=-\infty}^{\infty} a_{ij,m}(s, y)e(mx) , \qquad (z = x + iy) , \tag{4}$$

where $e(x) = \exp(2\pi i x)$, and where

[1] All facts about Eisenstein series which are cited here can be found in the excellent monograph [3].

$$
\begin{aligned}
a_{ij,m}(y,s) &= 2\pi^{s}|m|^{s-1/2}\Gamma(s)^{-1}y^{1/2}K_{s-1/2}(2\pi|m|y)\phi_{ij,m}(s) \qquad (m \neq 0) \\
&= \delta_{ij}y^{s} + \phi_{ij}(s)y^{1-s} \qquad (m = 0)\ ,
\end{aligned} \tag{5}
$$

where

$$
\begin{gathered}
\phi_{ij,m}(s) = \sum_{c} \frac{1}{|c|^{2s}}\left(\sum_{d} e\left(\frac{md}{c}\right)\right)\ , \\
\left(c > 0,\ d \bmod c,\ \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \sigma_i^{-1}\Gamma\sigma_j\right)
\end{gathered} \tag{6}
$$

$$
\phi_{ij}(s) = \pi^{1/2}\frac{\Gamma(s-1/2)}{\Gamma(s)}\phi_{ij,0}(s)\ , \tag{7}
$$

$$
K_{s-1/2}(u) = 2^{-1}\pi^{-s}|u|^{1/2-s}\Gamma(s)\int_{-\infty}^{\infty}\frac{e(-ut)}{(1+t^2)^{s}}dt \qquad (u > 0) \tag{8}
$$

Moreover, it is fairly easy to show that $a_{ij,0}(s,y)$ has a pole at $s=1$, whereas $a_{ij,m}(s,y)$ is continuous at $s=1$ for $m \neq 0$. Thus, it is fairly easy to see that

$$
\lim_{s\to 1}\left[E_i(\sigma_j z, s) - a_{ij,0}(y,s)\right] = \sum_{m=-\infty}^{\infty}{}' a_{ij,m}(y,1)e(mx)\ , \tag{9}
$$

where $\sum'$ denotes a sum which excludes the term for $m=0$. Let us first make the formula (9) somewhat more explicit. It is immediate from definitions that

$$
a_{ij,m}(y,1) = 2\pi|m|^{1/2}y^{1/2}K_{1/2}(2\pi|m|y)\phi_{ij,m}(1) \qquad (m \neq 0)\ , \tag{10}
$$

$$
\begin{aligned}
K_{1/2}(2\pi|m|y) &= (2\pi)^{-1}(|m|y)^{-1/2}\int_{-\infty}^{\infty}\frac{e(-2\pi|m|ty)}{(t^2+1)}dt \\
&= \frac{1}{2}(|m|y)^{-1/2}\exp\left(-2\pi|m|y\right) \qquad (m \neq 0)\ .
\end{aligned} \tag{11}
$$

Therefore, by (6), (10) and (11),

$$
a_{ij,m}(y,1) = \pi\exp\left(-2\pi|m|y\right)\phi_{ij,m}(1) \qquad (m \neq 0) \tag{12}
$$

where

$$
\begin{gathered}
\phi_{ij,m}(1) = \lim_{s\to 1}\sum_{c}\frac{1}{|c|^{2s}}\sum_{d} e\left(\frac{md}{e}\right) \\
\left(c > 0,\ d\,(\bmod c),\ \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \sigma_i^{-1}\Gamma\sigma_j\right)\ .
\end{gathered} \tag{13}
$$

§ 3. The formal limit formula

In this section, we will prove a formal Kronecker limit formula for the Eisenstein series $E_i(z, s)$.

From equations (9)–(13), we see that

$$\begin{aligned}&\lim_{s\to 1}[E_i(\sigma_j(z), s) - a_{ij,0}(y, s)]\\ &\qquad = \pi \sum_{m=-\infty}^{\infty}{}' \exp\{-2\pi|m|x + 2\pi imx\}\phi_{ij,m}(1)\,.\end{aligned} \tag{14}$$

Let $z = x + iy \in \boldsymbol{H}$, and let $q = e^{\pi iz}$. Then, from (14), we see that

$$\begin{aligned}&\lim_{s\to 1}[E_i(\sigma_j(z), s) - a_{ij,0}(y, s)]\\ &\qquad = \pi \sum_{m=1}^{\infty} \phi_{ij,m}(1)q^m + \pi \sum_{m=1}^{\infty} \phi_{ij,-m}(1)q'^m\,,\end{aligned} \tag{15}$$

where $q' = e^{-2\pi i\bar{z}}$. However, an easy computation implies that

$$\phi_{ij,-m}(s) = \overline{\phi_{ij,m}(\bar{s})}$$

for any s for which the Dirichlet series $\phi_{ij,m}(s)$ converges. Therefore, by analytic continuation,

$$\phi_{ij,-m}(1) = \overline{\phi_{ij,m}(1)}\,.$$

Therefore, from (15), we have

$$\begin{aligned}&\lim_{s\to 1}[E_i(\sigma_j(z, s) - a_{ij,0}(y, s)]\\ &\qquad = \pi \sum_{m=1}^{\infty} \phi_{ij,m}(1)q^m + \pi \sum_{m=1}^{\infty} \overline{\phi_{ij,m}(1)}q'^m\,.\end{aligned} \tag{16}$$

Let us investigate the structure of $a_{ij,0}(y, s)$ somewhat more closely. Equation (5) implies that in a neighborhood of $s = 1$, we have

$$a_{ij,0}(y, s) = \frac{c_{ij}}{s-1} + d_{ij} + \text{higher order terms}\,,$$

where c_{ij} and d_{ij} do not depend on s. Let us find formulas for c_{ij} and d_{ij}. We know that in a neighborhood of $s = 1$, we have

$$\phi_{ij,0}(s) = \frac{\alpha_{ij}}{s-1} + \beta_{ij} + \text{higher order terms}\,.$$

However,

$$y^{1-s} = 1 + (s-1)\log(y^{-1}) + \text{higher order terms}$$

$$\frac{\Gamma(s-1/2)}{\Gamma(s)} = \sqrt{\pi}(1-(2\log 2)(s-1)+\cdots)$$

(For the latter expansion, set [5, p. 15]). Therefore, by equation (7), we see that

$$\begin{aligned} a_{ij,0}(y,s) &= \pi^{1/2}y^{1-s}\frac{\Gamma(s-1/2)}{\Gamma(s)}\phi_{ij,0}(s) + \delta_{ij}y^s \\ &= \pi^{1/2}(1+(s-1)\log(y^{-1})+\cdots)(\sqrt{\pi}-2\sqrt{\pi}\log 2(s-1)+\cdots) \\ &\quad \cdot\left(\frac{\alpha_{ij}}{s-1}+\beta_{ij}+\cdots\right)+(\delta_{ij}y+\cdots)\,. \end{aligned}$$

Therefore, by Cauchy multiplication of power series, we see that

$$a_{ij,0}(s,y) = \frac{c_{ij}}{s-1} + d_{ij} + \cdots \tag{17}$$

where

$$c_{ij} = \pi\alpha_{ij} \tag{18}$$

$$d_{ij} = \pi\{\beta_{ij}+\alpha_{ij}\log(y^{-1})-2\alpha_{ij}\log 2\}+\delta_{ij}y\,. \tag{19}$$

Combining (17)–(19) with (16), we derive that

$$\begin{aligned} &\lim_{s\to 1}\left[E_i(\sigma_j(z),s)-\frac{\pi\alpha_{ij}}{s-1}\right] \\ &\quad = \pi\{\beta_{ij}+\alpha_{ij}\log(y^{-1})-2\alpha_{ij}\log 2\}+\delta_{ij}y \\ &\quad\quad + \pi\sum_{m=1}^{\infty}\phi_{ij,m}(1)q^m + \pi\sum_{m=1}^{\infty}\phi_{ij,m}(1)q'^m\,. \end{aligned} \tag{20}$$

Let us rewrite (20) as follows:

$$\begin{aligned} &\lim_{s\to 1}\left[\frac{1}{2\pi}E_i(\sigma_j(z),s)-\frac{\alpha_{ij}}{2(s-1)}\right] \\ &\quad = \frac{1}{2}\beta_{ij}-\alpha_{ij}\log 2+\alpha_{ij}\left\{\frac{1}{2}\log(y^{-1})+\frac{y}{2\pi\alpha_{ij}}\delta_{ij}\right. \\ &\quad\quad \left. +\alpha_{ij}^{-1}\sum_{m=1}^{\infty}\phi_{ij,m}(1)q^m+\alpha_{ij}^{-1}\sum_{m=1}^{\infty}\overline{\phi_{ij,m}(1)q'^m}\right\}\,. \end{aligned} \tag{21}$$

We will prove below that α_{ij} is real, so that (21) can be rewritten as

$$\lim_{s\to 1}\left[\frac{1}{2\pi}E_i(\sigma_j z,s)-\frac{\alpha_{ij}}{2(s-1)}\right]$$

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$$= \frac{1}{2}\beta_{ij} - \alpha_{ij}\log 2 + \frac{\alpha_{ij}}{2}\log(y^{-1})$$
$$+ \alpha_{ij}\left\{\frac{z}{4\pi\sqrt{-1}\alpha_{ij}}\delta_{ij} + \alpha_{ij}^{-1}\sum_{m=1}^{\infty}\phi_{ij,m}(1)q^m\right\}$$
$$+ \alpha_{ij}\left\{\frac{-\bar{z}}{4\pi\sqrt{-1}\alpha_{ij}}\delta_{ij} + \alpha_{ij}^{-1}\sum_{m=1}^{\infty}\overline{\phi_{ij,m}(1)}q'^m\right\}.$$

Let us denote α_{ii} by α_i and β_{ii} by β_i, and let us define $\log\eta_{\Gamma,i}(z)$ by

$$\log\eta_{\Gamma,i}(z) = -\frac{z}{4\pi\sqrt{-1}\alpha_i} - \alpha_i^{-1}\sum_{m=1}^{\infty}\phi_{ii,m}(1)q^m, \qquad q = e^{2\pi iz},$$

where the logarithm is taken with respect to the principal branch. It is clear that $\log\eta_{\Gamma,i}(z)$ is analytic for $z \in \boldsymbol{H}$, so that $\eta_{\Gamma,i}(z)$ is analytic and non-zero throughout the upper half-plane $\boldsymbol{H}$. Moreover, from (22) we have

$$\lim_{s\to 1}\left[\frac{1}{2\pi}E_i(\sigma_i z, s) - \frac{\alpha_i}{2(s-1)}\right]$$
$$= \frac{1}{2}\beta_i - \alpha_i\log 2 + \frac{\alpha_i}{2}\log(y^{-1}) - \log\eta_{\Gamma,i}(z) - \overline{\log\eta_{\Gamma,i}(z)}$$
$$= \frac{1}{2}\beta_i - \alpha_i\log 2 - \alpha_i\log|y^{1/2}\eta_{\Gamma,i}(z)^2|.$$

Thus, summarizing our results thus far, we have

THEOREM 3–1. *Let $E_i(z, s)$ be the Eisenstein series at the cusp κ_i for the Fuchsian group Γ, and let $\phi_{ii,0}(s)$ be the Dirichlet series appearing in the constant term of the Fourier expansion of $E_i(z, s)$ about κ_i. Assume that*

$$\phi_{ii,0}(s) = \frac{\alpha_i}{s-1} + \beta_i + \cdots$$

in a neighborhood of $s = 1$. Then

$$\lim_{s\to 1}\left[\frac{1}{2\pi}E_i(z, s) - \frac{\alpha_i}{2(s-1)}\right] = \frac{1}{2}\beta_i - \alpha_i\log 2 - \alpha_i\log|y(z)^{1/2}\eta_{\Gamma,i}(z)^2|$$

where

$$\log\eta_{\Gamma,i}(z) = -\frac{z}{4\pi\sqrt{-1}\alpha_i} - \alpha_i^{-1}\sum_{m=1}^{\infty}\phi_{ii,m}(1)q^m, \qquad q = e^{2\pi iz}.$$

Next, let us explore the analytical properties of the function $\eta_{\Gamma,i}(z)$. Our main result will be

THEOREM 3–2. *Let* $\sigma \in \sigma_i^{-1}\Gamma\sigma_i$. *Then*

$$\log \eta_{\Gamma,i}(\sigma(z)) = \log \eta_{\Gamma,i}(z) + \frac{1}{2}\log(cz+d) + \pi\sqrt{-1}S_{\Gamma,i}(\sigma)\,, \qquad \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $S_{\Gamma,i}(\sigma)$ *is real and depends only on* Γ, i *and* σ *and not on* s.

Theorem 3–2 is a generalization of the classical transformation formula for $\log \eta(z)$ and the quantities $S_{\Gamma,i}(\sigma)$ are generalizations of the classical Dedekind sums. We will call the quantities $S_{\Gamma,i}(\sigma)$ the *Dedekind sum attached to* Γ *and* i.

Proof. By Theorem 3–1 and the facts that (i) $E_i(z,s)$ is automorphic in z, (ii) α_i and β_i do not depend on z, we see that

$$\log|y(\sigma(z))^{1/2}\eta_{\Gamma,i}(\sigma(z))^2| = \log|y^{1/2}\eta_{\Gamma,i}(z)^2|\,. \tag{$*$}$$

However, $y(\sigma(z)) = y/(|cz+d|^2)$, so that

$$\log|y(\sigma(z))^{1/2}\eta_{\Gamma,i}(\sigma(z))^2| = \log y^{1/2} - \log|cz+d| + \log|\eta_{\Gamma,i}(\sigma(z))|^2\,. \tag{$**$}$$

From equations $(*)$ and $(**)$, we have

$$\log|\eta_{\Gamma,i}(\sigma(z))| = \log|\eta_{\Gamma,i}(z)| + \tfrac{1}{2}\log|cz+d|\,.$$

Therefore, the function $F(z)$ defined by

$$F(z) = \log \eta_{\Gamma,i}(\sigma(z)) - \log \eta_{\Gamma,i}(z) - \tfrac{1}{2}\log(cz+d)\,,$$

where all logarithms are taken with respect to the principal branch, has a real part identically zero. Thus, $F(z)$ is identically constant and this constant must be purely imaginary, depending only on Γ, i and σ.

Next, let us give a complement to the Kronecker limit formula of Theorem 3–1 by providing a geometric interpretation of the constant α_{ij}. It is well-known that $dxdy/y^2$ is an invariant valume element on the upper half-plane $\boldsymbol{H}$. If D is a fundamental domain for Γ, then by hypothesis, D has finite invariant volume which does not depend on the choice of D. Let us denote the volume of such a fundamental domain by $\mathrm{vol}(\boldsymbol{H}/\Gamma)$. Then we will prove

THEOREM 3–3. *Suppose that* $\phi_{ij}(s) = \alpha_{ij}/(s-1) + \beta_{ij} + \cdots$. *Then*

$$\alpha_{ij} = \frac{1}{\pi \operatorname{vol}(\boldsymbol{H}/\Gamma)}\ .$$

To prove Theorem 3–2, we require some preliminaries. Let Y be a positive real number, which will be chosen large. Let us define the *compact part* $E_i^Y(z, s)$ of $E_i(z, s)$ by

$$\begin{aligned} E_i^Y(z, z) &= E_i(z, s) - a_{ij,0}(s, y(\sigma_i^{-1}(z))) \qquad \text{if } y(\sigma_i^{-1}(z)) > Y \\ &= E_i(z, s) \qquad \text{otherwise}\ . \end{aligned}$$

Then it is known that $E_i^Y(z, s)$ belongs to $L^2(\boldsymbol{H}/\Gamma)$ where integration is taken with respect to the invariant volume element $dxdy/y^2$. Moreover, the following inner product formula is a consequence of the so-called *Maass-Selberg formula* [3, Theorem 2.3.2]:

$$\begin{aligned} \int_{\boldsymbol{H}/\Gamma} & E_i^Y(z, s) E_i^Y(z, s') \frac{dxdy}{y^2} \\ &= \frac{Y^{s+s'-1} - \sum_{k=1}^{k} \phi_{ik}(s)\phi_{ik}(s') Y^{-s-s'+1}}{s + s' - 1} \\ &\quad - \phi_{ii}(s) \frac{Y^{-s+s'}}{s - s'} + \phi_{ii}(s') \frac{Y^{s-s'}}{s - s'}\ , \end{aligned} \tag{23}$$

where the formula is valid for $\operatorname{Re}(s) > 1$, $\operatorname{Re}(s') > 1$, $s \neq s'$.

Let us fix a fundamental domain D for Γ and let us define D^Y by

$$D^Y = D - \bigcup_{k=1}^{k} D_k^Y\ ,$$

where

$$D_k^Y = \{z \in \boldsymbol{H} \mid y(\sigma_k^{-1}(z)) > Y\}\ .$$

From the definition of $E_i^Y(z, s)$ and the Fourier expansion of $E_i(z, s)$ about the cusp κ_j, it is easy to verify the following facts:

I. $(s - 1)E_i(z, s) - \pi\alpha_{ii}$ is bounded uniformly for $0 \leq s \leq 1$, $z \in D^Y$ and as s tends to 1, this quantity tends to zero uniformly for $z \in D^Y$.

II. $(s - 1)E_i^Y(z, s)$ is uniformly bounded for $0 \leq s \leq 1$, $z \in D_k^Y$ $(k = 1, \cdots, h)$, $Y \geq 1$.

It is clear that

$$\begin{aligned} \int_{\boldsymbol{H}/\Gamma} & (s - 1)(s' - 1) E_i^Y(z, s) E_i^Y(z, s') \frac{dxdy}{y^2} \\ &= \int_{D^Y} (s - 1)(s' - 1) E_i(z, s) E_i(z, s') \frac{dxdy}{y^2} \end{aligned}$$

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$$+\sum_{k=1}^{k}\int_{D_k^Y}(s-1)(s'-1)E_i^Y(z,s)E_i^Y(z,s)\frac{dxdy}{y^2}$$
$$=\pi^2\alpha_{ii}^2\int_{D^Y}\frac{dxdy}{y^2}+O(s-1)+O(s'-1)\qquad(s,s'\to 1)\ ,$$

by I and II. Therefore,

$$\lim_{\substack{s\to 1\\ s'\to 1}}\int_{H/\Gamma}(s-1)(s'-1)E_i^Y(z,s)E_i^Y(z,s')\frac{dxdy}{y^2}=\pi^2\alpha_{ii}^2\int_{D^Y}\frac{dxdy}{y^2}\ . \tag{24}$$

However, by equation (23),

$$(s-1)(s'-1)\int_{H/\Gamma}E_i^Y(z,s)E_i^Y(z,s')\frac{dxdy}{y^2}$$
$$=\frac{(s-1)(s'-1)}{s-s'}\left[Y^{s+s'-1}-\sum_{k=1}^{k}\phi_{ik}(s)\phi_{ik}(s')Y^{-s-s'+1}\right]$$
$$-\frac{(s-1)(s'-1)}{s-s'}\phi_{ii}(s)Y^{-s+s'}+\frac{(s-1)(s'-1)}{s-s'}\phi_{ii}(s')Y^{s-s'}$$
$$=\frac{\pi\alpha_{ii}}{s+s'-1}Y^{-s-s'+1}-\pi\alpha_{ii}\frac{s'-1}{s-s'}Y^{-s+s'}+\pi\alpha_{ii}\frac{s-1}{s-s'}Y^{s-s'}$$
$$+O(s-1)+O(s'-1)$$
$$=-\frac{\pi\alpha_{ii}}{s+s'-1}Y^{-s-s'+1}+\pi\alpha_{ii}+O(s-1)+O(s'-1)\ .$$

Therefore,

$$\lim_{\substack{s\to 1\\ s'\to 1}}\int_{H/\Gamma}(s-1)(s'-1)E_i^Y(z,s)E_i^Y(z,s')\frac{dxdy}{y^2}=\pi\alpha_{ii}-\pi\alpha_{ii}Y^{-1}\ ,$$

so that

$$\pi^2\alpha_{ii}^2\int_{D^Y}\frac{dxdy}{y^2}=\pi\alpha_{ii}-\pi\alpha_{ii}Y^{-1}\ .$$

But as $Y\to\infty$,

$$\int_{D^Y}\frac{dxdy}{y^2}\longrightarrow\int_{D}\frac{dxdy}{y^2}\ ,$$

so that

$$\pi^2\alpha_{ii}^2\,\mathrm{vol}\,(\boldsymbol{H}/\Gamma)=\pi\alpha_{ii}\ ,$$

which is just Theorem 3–3 for $i=j$. But since $\pi\alpha_{ij}$ is the residue of $E_i(z,s)$ at $s=1$, we see that $\pi\alpha_{ii}=\pi\alpha_{ij}$ $(1\leq i,j\leq h)$.

There is a classical formula for $\mathrm{vol}(\boldsymbol{H}/\Gamma)$ which gives us arithmetic information about α_{ij}. Let g denote the genus of $\overline{\boldsymbol{H}/\Gamma}$ and let $e_1, \cdots, e_r$ denote the orders of the Γ-inequivalent fixed points of Γ on $\boldsymbol{H}$, h the number of cusps of Γ. Then [3, p. 42], we have

$$\mathrm{vol}\,(\boldsymbol{H}/\Gamma) = 2\pi\left\{2g - 2 + h + \sum_{\nu=1}^{r}\left(1 - \frac{1}{e_\nu}\right)\right\} .$$

Therefore, we have

COROLLARY 3–4. $\alpha_{ij} = \dfrac{1}{2}\pi^{-2}\left\{2g - 2 + h + \sum_{\nu=1}^{r}\left(1 - \dfrac{1}{e_\nu}\right)\right\}^{-1}$.

In particular, we have

COROLLARY 3–5. *The quantity* $-(4\pi\sqrt{-1}\alpha_{ii})^{-1}$ *is of the form* $2\pi\sqrt{-1}r_i$ *where* $r_i = r_i(\Gamma)$ *is the positive rational number given by*

$$r_i = \frac{1}{4}\left\{2g - 2 + h + \sum_{\nu=1}^{r}\left(1 - \frac{1}{e_\nu}\right)\right\} .$$

Our next task is to prove that $\eta_{\Gamma,i}(z)$ is an automorphic form for Γ corresponding to a certain multiplier system. Namely we will prove

THEOREM 3–6. *The function* $\eta_{\Gamma,i}(z)$ *is an automorphic form for* $\sigma_i^{-1}\Gamma\sigma_i$ *of weight* $1/2$ *corresponding to the multiplier system*

$$v(\sigma) = \exp\,\{\pi\sqrt{-1}S_{\Gamma,i}(\sigma)\} .$$

Proof. It is clear from Theorem 3–2 that

$$\eta_{\Gamma,i}(\sigma z) = v(\sigma)\sqrt{cz + d}\,\eta_{\Gamma,i}(z) ,$$

for $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \sigma_i^{-1}\Gamma\sigma_i$, $z \in \boldsymbol{H}$, where the branch of the square root is the one which is positive on the positive real axis. Therefore, it suffices to check that $\eta_{\Gamma,i}(z)$ has the appropriate Fourier expansion about the cusps of $\sigma_i^{-1}\Gamma\sigma_i$. It is easy to check that the cusps of $\sigma_i^{-1}\Gamma\sigma_i$ are given by $\sigma_i^{-1}(\kappa_j)$ $(j = 1, \cdots, h)$. Moreover, the stability subgroup of $\sigma_i^{-1}\kappa_j$ in $\sigma_i^{-1}\Gamma\sigma_i$ is just $\sigma_i^{-1}\Gamma_j\sigma_i$, where Γ_j is the stability subgroup of κ_j in Γ. Also, note that

$$(\sigma_i^{-1}\sigma_j)^{-1}(\sigma_i^{-1}\Gamma_j\sigma_i)(\sigma_i^{-1}\sigma_j) = \left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\middle|\, x \in \boldsymbol{Z}\right\} .$$

Therefore, if $\sigma_i^{-1}\sigma_j = \begin{pmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{pmatrix}$, then let us set

$$f_{ij}(z) = \sqrt{c_{ij}z + d_{ij}}^{-1}\eta_{\Gamma,i}(\sigma_i^{-1}\sigma_j z) .$$

Choose $\rho \in \sigma_i^{-1}\Gamma\sigma_i$ so that $(\sigma_i^{-1}\sigma_i)^{-1}\rho(\sigma_i^{-1}\sigma) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then by the fundamental transformation property of $\eta_{\Gamma,i}(z)$, we derive that

$$\begin{aligned} f_{ij}(z+1) &= \sqrt{c_{ij}(z+1) + d_{ij}}^{-1}\eta_{\Gamma,i}(\sigma_i^{-1}\sigma_j(z+1)) \\ &= \sqrt{c_{ij}(z+1) + d_{ij}}^{-1}\eta_{\Gamma,i}(\rho\sigma_i^{-1}\sigma_j(z)) \\ &= \sqrt{c_{ij}(z+1) + d_{ij}}^{-1}\sqrt{c\sigma_i^{-1}\sigma_j z + d}\,v(\rho)\eta_{\Gamma,i}(\sigma_i^{-1}\sigma_j(z)) \end{aligned}$$

where $\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $\theta = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \boldsymbol{R})$, let us set

$$J(\theta, z) = \sqrt{\gamma z + \delta} ,$$

where the branch of the square root is the one which is positive on the positive real axis. Then $J(\theta, z)$ is well-defined and analytic for $z \in \boldsymbol{H}$ and a trivial computation involving $J(\theta, z)^2$ shows that

$$J(\theta\zeta, z) = \pm J(\theta, \zeta z)J(\zeta, z) , \qquad (\zeta, \theta \in SL(2, \boldsymbol{R}), z \in \boldsymbol{H}) .$$

But it is clear that for large z very close to the positive real axis, the plus sign must prevail, so that

$$J(\theta\zeta, z) = J(\theta, \zeta z)J(\zeta, z) \qquad (\zeta, \theta \in SL(2, \boldsymbol{R}), z \in \boldsymbol{H}) .$$

However, since

$$\begin{aligned} \sqrt{c_{ij}(z+1) + d_{ij}} &= J(\sigma_i^{-1}\sigma_j, (\sigma_i^{-1}\sigma_j)^{-1}\rho(\sigma_i^{-1}\sigma_j)z) , \\ \sqrt{c\sigma_i^{-1}\sigma_j z + d} &= J(\rho, \sigma_i^{-1}\sigma_j z) , \end{aligned}$$

we see that

$$\begin{aligned} &\sqrt{c_{ij}(z+1) + d_{ij}}^{-1}\sqrt{c\sigma_i^{-1}\sigma_j z + d} \\ &\quad = J(\rho(\sigma_i^{-1}\sigma_j), z)^{-1}J((\sigma_i^{-1}\sigma_j)^{-1}\rho(\sigma_i^{-1}\sigma_j), z)J(\rho(\sigma_i^{-1}\sigma_j), z)J(\sigma_i^{-1}\sigma_j, z)^{-1} \\ &\quad = J(\sigma_i^{-1}\sigma_j, z)^{-1} . \end{aligned}$$

Thus, we see that

$$f_{ij}(z+1) = v(\rho)J(\sigma_i^{-1}\sigma_j, z)^{-1}\eta_{\Gamma,i}(\sigma_i^{-1}\sigma_j z) = v(\rho)f_{ij}(z) . \tag{25}$$

Let us define θ_{ij} to be the unique real number satisfying

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$$v(\rho) = e^{2\pi\sqrt{-1}\theta_{ij}}, \qquad 0 \leq \theta_{ij} < 1 .$$

Then (25) implies that $f_{ij}(z)$ has a Fourier expansion of the form

$$f_{ij}(z) = \sum_{m=-\infty}^{\infty} a_m e^{2\pi\sqrt{-1}(m+\theta_{ij})z} \tag{26}$$

We assert that all terms in the Fourier expansion for which $m + \alpha_{ij} < 0$ are zero. If not, the function $f_{ij}(z)$ will assume values of arbitrarily large absolute value in every neighborhood of infinity. However, it follows immediately from the Fourier expansion (22) that $f_{ij}(z)$ remains bounded in a neighborhood of infinity. Therefore, the Fourier expansion (26) has the form

$$\frac{\eta_{\Gamma,i}(\sigma_i^{-1}\sigma_j z)}{\sqrt{c_{ij}z + d_{ij}}} = e^{2\pi\sqrt{-1}\theta_{ij}z} \sum_{m=0}^{\infty} a_m e^{2\pi\sqrt{-1}mz} ,$$

which is the desired Fourier expansion of $\eta_{\Gamma,i}(z)$ about the cusp $\sigma_i^{-1}\kappa_j$. Thus we have established the character of $\eta_{\Gamma,i}(z)$ as an automorphic form.

§ 4. Properties of the Dedekind sums

In this section, we will prove certain general facts about the Dedekind sums $S_{\Gamma,i}(\sigma)$.

THEOREM 4–1. *The mapping $\theta_{\Gamma,i}: \sigma_i^{-1}\sigma_i \rightarrow \boldsymbol{R}$ defined by*

$$\theta_{\Gamma,i}(\sigma) = S_{\Gamma,i}(\sigma)$$

is a homomorphism.

Proof. Let $\sigma, \tau \in \sigma_i^{-1}\Gamma\sigma_i$ Then by Theorem 3–2 we have

$$\log \eta_{\Gamma,i}(\sigma\tau(z)) = \log \eta_{\Gamma,i}(z) + \tfrac{1}{2}\log(c''z + d'') + \pi\sqrt{-1}S_{\Gamma,i}(\sigma\tau) , \tag{27}$$

where we set

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \tau = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}, \quad \sigma\tau = \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}.$$

On the other hand, by applying Theorem 3–2 twice, we see that

$$\begin{aligned} \log \eta_{\Gamma,i}(\sigma\tau(z)) &= \log \eta_{\Gamma,i}(\tau(z)) + \tfrac{1}{2}\log(c\tau(z) + d) + \pi\sqrt{-1}S_{\Gamma,i}(\sigma) \\ &= \log \eta_{\Gamma,i}(z) + \tfrac{1}{2}\log(c\tau(z) + d) + \tfrac{1}{2}\log(c'z + d') \\ &\quad + \pi\sqrt{-1}S_{\Gamma,i}(\sigma) + \pi\sqrt{-1}S_{\Gamma,i}(\tau) . \end{aligned} \tag{28}$$

However,

$$\log(c\tau(z)+d)+\log(c'z+d') = \log\left(c\frac{a'z+b'}{c'z+d'}+d\right)+\log(c'z+d')$$
$$= \log((ca'+dc')z+(cb'+dd'))+2k\pi\sqrt{-1}\,, \tag{29}$$

where $k \in \boldsymbol{Z}$ does not depend on $\boldsymbol{Z}$. However, by taking $z \in \boldsymbol{H}$ large and close to the positive real axis, and recalling that all logarithms are taken with respect to the principal branch, we see that $k=0$. Therefore, combining (28) and (29) noting that $c''=ca'+dc'$, $d''=cb'+dd'$, we see that

$$S_{\Gamma,i}(\sigma\tau) = S_{\Gamma,i}(\sigma)+S_{\Gamma,i}(\tau)\ .$$

In the remainder of this section, we would like to make some comments on the rationality of the quantities $S_{\Gamma,i}(\sigma)$, since many of the most significant and interesting properties of the classical Dedekind sums come from the fact that they are rational numbers. As trivial consequences of Theorem 4–1, we have

COROLLARY 4–2. *If $\sigma \in \sigma_i^{-1}\Gamma\sigma_i$ is elliptic, then $S_{\Gamma,i}(\sigma)$ is rational with denominator dividing the order of σ.*

COROLLARY 4–3. (1) $S_{\Gamma,i}(\sigma) = -S_{\Gamma,i}(\sigma^{-1})$.
(2) *If $\sigma,\tau \in \Gamma$ are conjugate, then $S_{\Gamma,i}(\sigma)=S_{\Gamma,i}(\tau)$.*

It is known that Γ is finitely generated. Then we have:

COROLLARY 4–4. *Let $\tau_1,\cdots,\tau_r$ be generators for $\sigma_i^{-1}\Gamma\sigma_i$. Then $S_{\Gamma,i}(\sigma)$ is rational for all $\sigma\in\sigma_i^{-1}\Gamma\sigma_i$ if and only if $S_{\Gamma,i}(\tau_j)$ is rational for $1\le j\le r$. Moreover, in case the latter is true, the denominator of $S_{\Gamma,i}(\sigma)(\sigma\in\sigma_i^{-1}\Gamma\sigma_i)$ always divides the least common multiple of the denominators of $S_{\Gamma,i}(\tau_j)$ $(1\le j\le r)$. In particular, the Dedekind sums have bounded denominators.*

There is one obvious case (other than Corollary 4–2) where we can conclude the rationality of the Dedekind sum. Namely:

THEOREM 4–5. $S_{\Gamma,i}\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)$ *is rational.*

Proof. By Theorem 4–1, we have

$$S_{\Gamma,i}\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) = xS_{\Gamma,i}\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right)\,,$$

so that it suffices to prove that

$$S_{\Gamma,i}\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right)$$

is rational. But from the definition of the Dedekind sum, we see that

$$\pi\sqrt{-1}S_{\Gamma,i}\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right) = \log\eta_{\Gamma,i}(z+1) - \log\eta_{\Gamma,i}(z)$$

However, by the definition of $\log\eta_{\Gamma,i}(z)$, we see that

$$\pi\sqrt{-1}S_{\Gamma,i}\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right) = -\frac{1}{4\pi\sqrt{-1}\alpha_i}\,.$$

Therefore, by Corollary 3–5,

$$S_{\Gamma,i}\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right) = 2r_i\,, \tag{30}$$

where

$$r_i = \frac{1}{4}\left\{2g - 2 + h + \sum_{\nu=1}^{r}\left(1 - \frac{1}{e_\nu}\right)\right\}\,.$$

COROLLARY 4–6. *Suppose that Γ has a single cusp and is generated by parabolic and elliptic elements· Then $S_{\Gamma,1}(\sigma)$ is rational for all $\sigma\in\sigma_1^{-1}\Gamma\sigma_1$.*

Proof. By Corollary 4–2 and Corollary 4–4, it suffices to show that $S_{\Gamma,1}(\sigma)$ whenever σ is one of the parabolic generators $\sigma = \sigma_1^{-1}\rho\sigma_1$, $\rho\in\Gamma$ is parabolic. Then the fixed point λ of ρ is a cusp of Γ and is therefore equivalent to κ_1, so that $\lambda = \theta(\kappa_1)$ for some $\theta\in\Gamma$. Therefore, since λ is a fixed point of ρ, κ_i is a fixed point of $\theta^{-1}\rho\theta$ and $\sigma_1^{-1}\kappa_1 = \infty$ is a fixed point of $\sigma_1^{-1}\theta^{-1}\rho\theta\sigma_1$. However, by the way in which σ was chosen, this implies that

$$\sigma_1^{-1}\theta^{-1}\rho\theta\sigma_1 = \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$$

for some $x\in\boldsymbol{Z}$. Moreover, by Corollary 4–3, (2), we have

$$S_{\Gamma,1}(\sigma) = S_{\Gamma,1}(\sigma_1^{-1}\rho\sigma_1) = S_{\Gamma,1}(\sigma_1^{-1}\theta^{-1}\rho\theta\sigma_1) = S_{\Gamma,1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\,.$$

Therefore, $S_{\Gamma,1}(\sigma)$ is rational by Corollary 4–5.

For example, Corollary 4–6 implies to $\Gamma = SL(2,\boldsymbol{Z})$ which is generated by $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}0 & -1\\ 0 & 1\end{pmatrix}$. Here, by (30), we have

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$$S_{\Gamma,1}\left(\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\right) = 2\cdot\frac{1}{4}\left[(2-0-2)+1+\left(\frac{1}{2}+\frac{1}{3}\right)\right] = -\frac{1}{12}\,,$$

and the denominator of $S_{\Gamma,1}\left(\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\right)$ divides 2. Therefore, by Corollary 4–4, $S_{\Gamma,1}(\sigma)$ is rational for all $\sigma \in SL(2, \boldsymbol{Z})$ and has denominator always divisible by 12. Therefore, in this case, the multiplier

$$v(\sigma) = \exp\{\pi\sqrt{-1}S_{\Gamma,1}(\sigma)\}$$

is a 24^{th} root of 1.

It would be interesting to determine precisely for which groups Γ it is true that $S_{\Gamma,1}(\sigma)$ is rational for all $\sigma \in \sigma_i^{-1}\Gamma\sigma_i$. There is some reason to believe that all arithmetic subgroups of $SL(2, \boldsymbol{R})$ have the property that the Dedekind sums are algebraic, but we have no way of proving this. In the remainder of this paper, we will explicitly compute the Dedekind sums for the groups $\Gamma(N)$.

§ 5. Calculation of the Dedekind sums for $\Gamma(N)$

Throughout this section, let N be a positive integer and let $\Gamma(N)$ denote the principal congruence subgroup of $SL(2, \boldsymbol{Z})$ of level N-that is,

$$\Gamma(N) = \left\{\begin{pmatrix}a & b\\c & d\end{pmatrix}\middle| a, b, c, d \in \boldsymbol{Z}, a \equiv d \equiv 1 \,(\mathrm{mod}\, N), c \equiv b \equiv 0 \,(\mathrm{mod}\, N)\right\}.$$

Set

$$\mu_N = \begin{cases}\dfrac{N^3}{2}\prod_{p\mid N}\left(1-\dfrac{1}{p^2}\right) & (N > 2)\\ 6 & (N = 2)\\ 1 & (N = 1)\end{cases}$$

Moreover, let $x \to \bar{x}$ denote the canonical homomorphism of $SL(2, \boldsymbol{R})$ into $SL(2, \boldsymbol{R})/\{\pm 1\}$. Then it is well-known that $\overline{\Gamma(N)}$ is a Fuchsian group of the first kind and that $\Gamma(N)$ has μ_N/N inequivalent cusps. Moreover, it is possible to get a set of representatives for these cusps as follows: Every cusp κ is of the form $\kappa = a/b$, $a, b \in \boldsymbol{Z}$, $(a, b) = 1$. By convention, we include $\kappa = 1/0 = \infty$.) Moreover, $\kappa_1 = a/b$, $\kappa_2 = c/d$ are $\overline{\Gamma(N)}$ equivalent if and only if $a \equiv c \,(\mathrm{mod}\, N)$, $b \equiv d \,(\mathrm{mod}\, N)$ or $a \equiv -c \,(\mathrm{mod}\, N)$, $b \equiv -d \,(\mathrm{mod}\, N)$.

Let $\kappa = \alpha/\beta$ be a cusp of $\overline{\Gamma(N)}$ and let $\sigma_\kappa \in SL(2, \boldsymbol{R})/\{\pm 1\}$ be such that $\sigma_\kappa\infty = \kappa$ and

$$\sigma_\kappa^{-1}\Gamma_\kappa\sigma_\kappa = \left\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\middle| x\in \boldsymbol{Z}\right\},$$

where Γ_κ is the subgroup of $\overline{\Gamma(N)}$ leaving κ invariant. Then it is a reasonably elementary computation to show that

$$E_\kappa(z,s) = \frac{\delta(N)}{2}\sum_{\substack{c,d\in \boldsymbol{Z}\\ (c,d)=1\\ c\equiv\beta\,(\mathrm{mod}\,N)\\ d\equiv\alpha\,(\mathrm{mod}\,N)}} \frac{y^s}{|cz+d|^{2s}},$$

where $\delta(N)=1$ if $N=1$ or 2 and $\delta(N)=2$ otherwise.

Let us compute the Dirichlet series $\phi_{ii,m}(s)$ which appear in the Fourier expansion of $E_\kappa(\sigma_\kappa z, s)$. Let κ_∞ denote the cusp at ∞ and let us write σ_∞ instead of σ_{κ_∞}. Then, it is easy to see that

$$\Gamma_{\kappa_\infty} = \left\{\begin{pmatrix}1 & Nx\\ 0 & 1\end{pmatrix}\middle| x\in \boldsymbol{Z}\right\},$$

so that

$$\sigma_\infty = \begin{pmatrix}N^{1/2} & 0\\ 0 & N^{-1/2}\end{pmatrix}.$$

Since $\overline{\Gamma(1)}$ has only one cusp, all cusps of $\overline{\Gamma(N)}$ are $\overline{\Gamma(1)}$-equivalent. Therefore, there exists $\theta_\kappa\in\overline{\Gamma(1)}$ such that $\theta_\kappa(\infty)=\kappa$. But then an elementary argument shows that

$$\theta_\kappa\Gamma_{\kappa_\infty}\theta_\kappa^{-1} = \Gamma_\kappa^*$$

is the stabilizer of κ in $\theta_\kappa\overline{\Gamma(N)}\theta_\kappa^{-1}$. However, since $\overline{\Gamma(N)}$ is a normal subgroup of $\overline{\Gamma(1)}$, this shows that Γ_κ^* is the stabilizer of κ in $\overline{\Gamma(N)}$, and therefore we have

$$\theta_\kappa\Gamma_{\kappa_\infty}\theta_\kappa^{-1} = \Gamma_\kappa .$$

Now on the one hand, we have

$$(\theta_\kappa\sigma_\infty)^{-1}(\theta_\kappa\Gamma_\infty\theta_\kappa^{-1})(\theta_\kappa\sigma_\infty) = \sigma_\infty^{-1}\Gamma_\infty\sigma_\infty = \left\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\middle| x\in \boldsymbol{Z}\right\}.$$

But on the other hand,

$$(\theta_\kappa\sigma_\infty)^{-1}(\theta_\kappa\Gamma_\infty\theta_\kappa^{-1})(\theta_\kappa\sigma_\infty) = (\theta_\kappa\sigma_\infty)^{-1}\Gamma_\kappa(\theta_\kappa\sigma_\infty).$$

Therefore, by comparing the last two equations, we have

$$\sigma_\kappa = \theta_\kappa \sigma_\infty \,. \tag{31}$$

We can explicitly construct θ_κ as follows: If $\kappa = \kappa_\infty$, set $\theta_\kappa = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. If $\kappa \neq \infty$, suppose that $\kappa = \alpha/\gamma$, $(\alpha, \gamma) = 1$, $\gamma \neq 0$. Choose $\beta, \delta \in \boldsymbol{Z}$ such that $\alpha\delta - \beta\gamma = 1$. Then we may set

$$\theta_\kappa = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} . \tag{31$'$}$$

Next, let us compute $\sigma_\kappa^{-1}\overline{\Gamma(N)}\sigma_\kappa$. By (31) and (31′),

$$\sigma_\kappa^{-1}\overline{\Gamma(N)}\sigma_\kappa = (\theta_\kappa\sigma_\infty)^{-1}\overline{\Gamma(N)}(\theta_\kappa\sigma_\infty) = \sigma_\infty^{-1}(\theta_\kappa^{-1}\overline{\Gamma(N)}\theta_\kappa)_\infty\sigma = \sigma_\infty^{-1}\overline{\Gamma(N)}\sigma_\infty$$

since $\overline{\Gamma(N)}$ is a normal subgroup of $\overline{\Gamma(1)}$. However, since

$$\begin{pmatrix} N^{-1/2} & 0 \\ 0 & N^{1/2} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} N^{1/2} & 0 \\ 0 & N^{-1/2} \end{pmatrix} = \begin{pmatrix} a & N^{-1}b \\ Nc & d \end{pmatrix}$$

we see that

$$\sigma_\kappa^{-1}\overline{\Gamma(N)}\sigma_\kappa = \left\{\begin{pmatrix} a & N^{-1}b \\ Nc & d \end{pmatrix}\middle| a, b, c, d \in \boldsymbol{Z}, ad - bc = 1, a \equiv d \equiv 1(N), b \equiv c \equiv 0(N)\right\} \tag{31$''$}$$

It is possible to use (31″) to directly calculate $\phi_{ii,m}(s)$ from the formulas of Section 2. However, the expression for $\phi_{ii,m}(s)$ thus derived is rather complicated. Thus, the following approach to calculating the η-function seems preferable:

Let g and h be arbitrary integers such that $(g, h, N) = 1$, and let us define the following series:

$$E_{g,h}(z, s; N) = \frac{\delta(N)}{2} \sum_{\substack{c,d=-\infty \\ c\equiv g (\mathrm{mod}\, N) \\ d \equiv h (\mathrm{mod}\, N) \\ (c,d)=1}}^{\infty}{}' \frac{y^s}{|cz + d|^{2s}} \,,$$

$$E^*_{g,h}(z, s; N) = \frac{\delta(N)}{2} \sum_{\substack{c,d=-\infty \\ c\equiv g (\mathrm{mod}\, N) \\ d \equiv h (\mathrm{mod}\, N)}}^{\infty}{}' \frac{y^s}{|cz + d|^{2s}} \,,$$

where the prime on the summation indicates that $(c, d) = (0, 0)$ is omitted. Note that if $\kappa = h/g$ is a cusp of $\Gamma(N)$, then $E_{g,h}(z, s; N) = E_\kappa(z, s)$.

By using the formal reasoning found in [2, pp. 44–48], we find that

$$E_{g,h}(z, s; N) = \sum_{\substack{n=1 \\ (a,N)=1}}^{N} \left(\sum_{\substack{n=1 \\ na\equiv 1 (\mathrm{mod}\, N)}}^{\infty} \frac{\mu(n)}{n^{2s}} \right) E^*_{ag,ah}(z, s) \,. \tag{32}$$

Moreover,

$$E^*_{g,h}(z, s; N) = \frac{\delta(N)y^s}{2}\{\theta_N(g)T_h + S_{g,h}\},$$

where

$$T_h = \sum_{\substack{d=-\infty \\ d\equiv h \,(\mathrm{mod}\, N)}}^{\infty} d^{-2s}, \qquad S_{g,h} = \sum_{\substack{c=-\infty \\ c\equiv g\,(\mathrm{mod}\, N) \\ c\neq 0}}^{\infty} \sum_{\substack{d=-\infty \\ d\equiv h\,(\mathrm{mod}\, N)}}^{\infty} |cz+d|^{-2s},$$

$$\theta_N(g) = \begin{cases} 1 & \text{if } g \equiv 0 \,(\mathrm{mod}\, N) \\ 0 & \text{otherwise}. \end{cases}$$

By applying the Poisson summation formula to the inner sum of $S_{g,h}$, we derive that

$$E^*_{g,h}(z, s; N) = \frac{\delta(N)y^s}{2}\Big\{\theta_N(g)T_h + \frac{y^{1-2s}}{N}\sum_{\substack{c=-\infty \\ c\equiv g\,(\mathrm{mod}\, N)}}^{\infty} \frac{1}{|c|^{2s-1}} \sum_{m=-\infty}^{\infty} e^{-2\pi i m(h+|c|x)/N} I(s, 2\pi|c|y^m/N)\Big\}$$

where

$$I(s, v) = \int_{-\infty}^{\infty} \frac{e^{ivw}}{(u^2+1)^s}du,$$

$$E^*_{g,h}(z, s; N) = \frac{\delta(N)y^s}{2}\Big\{\theta(g)T_h + \frac{y^{1-2s}}{N}\left(\int_{-\infty}^{\infty}\frac{du}{(u^2+1)^s}\right)\sum_{\substack{c=-\infty \\ c\equiv g\,(\mathrm{mod}\, N) \\ c\neq 0}}^{\infty} |c|^{1-2s} + \frac{y^{1-2s}}{N}\sum_{\substack{n=-\infty \\ n\neq 0 \\ c>0}}^{\infty} e^{2\pi i n x/N} I\left(s, -\frac{2\pi n y}{N}\right) \sum_{\substack{c\mid 0 \\ c\equiv g\,(\mathrm{mod}\, N) \\ c>0}} |c|^{1-2s} e^{2\pi i n k/cN}\Big\}. \tag{32'}$$

Thus, by combining (32) and (32′), we see that

$$E_{g,h}(z, s; N) = a_{0,g,h}(y, s; N) + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} a_{m,g,h}(y, s; N)e^{2\pi i m x/N}, \tag{33}$$

where

$$a_{0,g,h}(y, s; N) = \frac{\delta(N)}{2}\sum_{\substack{a=1 \\ (a,N)=1}}^{N}\left(\sum_{\substack{n=1 \\ na\equiv 1\,(\mathrm{mod}\, N)}}^{\infty}\frac{\mu(n)}{n^{2s}}\right)\left[y^s\theta_N(ag)\sum_{\substack{d=-\infty \\ d\equiv ah\,(\mathrm{mod}\, N) \\ d\neq 0}}^{\infty}|d|^{-2s} + \frac{y^{1-s}}{N}\left(\int_{-\infty}^{\infty}\frac{du}{(u^2+1)^s}\right)\sum_{\substack{c=-\infty \\ c\equiv ag\,(\mathrm{mod}\, N)}}^{\infty}|c|^{1-2s}\right]$$

$$a_{m,g,h}(y, s; N) = \frac{\delta(N)}{2N} \sum_{\substack{a=1 \\ (a,N)=1}}^{N} \left(\sum_{\substack{n=1 \\ na\equiv 1 (\text{mod } N)}}^{\infty} \frac{\mu(n)}{n^{2s}} \right) \cdot \left[I\left(s, -\frac{2\pi m y}{N} \sum_{\substack{c|m \\ c\equiv ag (\text{mod } N) \\ c>0}} c^{-2s+1} e^{2\pi i h m/cN}\right] (m \neq 0) .$$

LEMMA 5–1. *Let Γ be a Fuchsian group of the first kind and let κ be a cusp of $\Gamma, \tau \in SL(2, \boldsymbol{R})/\{\pm 1\}$. Then*

$$E_{\kappa,\Gamma}(z, s) = E_{\tau^{-1}(\kappa),\tau^{-1}\Gamma\tau}(z, s) .$$

In particular, if τ normalizes Γ, then

$$E_{\kappa,\Gamma}(\tau z, s) = E_{\tau^{-1}(\tau),\Gamma}(z, s) .$$

Proof. Note first that $\tau^{-1}(\kappa)$ is a cusp of $\tau^{-1}\Gamma\tau$ and that the stabilizer of $\tau^{-1}(\kappa)$ in $\tau^{-1}\Gamma\tau$ is just $\tau^{-1}\Gamma_\kappa\tau$, where $\Gamma_\kappa =$ the stabilizer of κ in Γ. Moreover, by the definition of σ_κ, we see that

$$(\tau^{-1}\sigma_\kappa)^{-1}(\tau^{-1}\Gamma_\kappa\tau)(\tau^{-1}\sigma_\kappa) = \left\{ \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} x \in \boldsymbol{Z} \right\} ,$$

so that $\sigma_{\tau^{-1}(\tau)} = \tau^{-1}\sigma_\kappa$. Therefore,

$$\begin{aligned} E_{\kappa,\Gamma}(\tau z, s) &= \sum_{\sigma \in \Gamma_\kappa\backslash\Gamma} y(\sigma_\kappa^{-1}\sigma\tau(z))^s = \sum_{\sigma \in \Gamma_\kappa\backslash\Gamma} y(\sigma_\kappa^{-1}\tau \cdot \tau^{-1}\sigma\tau(z))^s \\ &= \sum_{\eta \in \tau^{-1}\Gamma_\kappa\tau\backslash\tau^{-1}\Gamma\tau} y(\sigma_{\tau^{-1}(\kappa)}^{-1}\eta(z))^s = \sum_{\eta \in \Gamma_{\tau^{-1}(\kappa)}\backslash\tau^{-1}\Gamma\tau} y(\sigma_{\tau^{-1}(\kappa)}^{-1}\eta(z))^s \\ &= E_{\tau^{-1}(\kappa),\tau^{-1}\Gamma\tau}(z, s) . \end{aligned}$$

COROLLARY 5–2. *Let $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \boldsymbol{Z})$. Then*

$$E_{g,h}(\sigma z, s) = E_{dg-bh,-cg+ah}(z, s) .$$

Let us now use Lemma 5–1 to compute $E_\kappa(\sigma_\kappa z, s)$. By (28), we have

$$E_\kappa(\sigma_\kappa z, s) = E_\kappa(\theta_\kappa\sigma_\infty z, s) = E_{\theta_\kappa^{-1}(\kappa)}(\sigma_\infty z, s) = E_\infty(Nz, s) . \tag{34}$$

Thus, as a consequence of (34), we see that there is only one η-function for the group $\Gamma(N)$ and this one is an automorphic form for $\overline{\Gamma(N)}$ itself. The situation is typical for a normal subgroup of $SL(2, \boldsymbol{Z})/\{\pm 1\}$. Let us denote this unique automorphic from by $\eta_N(z)$. By (33), we see that

$$E_\kappa(\sigma_r z, s) = b_0(y, s; N) + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} b_m(y, s; N) e^{2\pi i m x} , \tag{35}$$

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where

$$b_m(y, s; N) = a_{m,0,1}(Ny, s; N) \qquad (m \in \boldsymbol{Z}) .$$

Then it is easy to see that

$$b_0(y, s; N) = y^s + y^{1-s}\phi_0(s; N)\int_{-\infty}^{\infty} \frac{du}{(u^2+1)^s} \tag{36}$$

where

$$\phi_0(s; N) = \delta(N)N^{1-3s}\frac{\zeta(2s-1)}{\zeta(2s)} \prod_{p \mid N} (1-p^{-2s})^{-1} .$$

Moreover,

$$\begin{aligned} b_m(y, s; N) = \frac{\delta(N)}{2N}\zeta(2s)^{-1} \prod_{p \mid N} (1-p^{-2s})^{-1} I(s, -2\pi m y) \\ \cdot \sum_{\substack{c \mid m \\ c \equiv 0 (\mathrm{mod}\, N) \\ c > 0}} c^{-2s-1}e^{2\pi i m/cN} \qquad (m \neq 0) \end{aligned} \tag{37}$$

Now an easy computation shows that

$$I(s, -2\pi m y) = 2\pi(|m| y)^{1/2}K_{1/2}(2\pi |m| y) .$$

Therefore, combining (37) and (10), we see that

$$\phi_{N,m}(s) = \frac{\delta(N)}{2N}\zeta(2s)^{-1} \prod_{p \mid N} (1-p^{-2s})^{-1} \cdot \sum_{\substack{c \mid m \\ c > 0 \\ c \equiv 0 (\mathrm{mod}\, N)}} \frac{e^{2\pi i m/cN}}{c^{2s-1}} \qquad (m \neq 0) . \tag{38}$$

(Here we write $\phi_{N,m}(s)$ instead of $\phi_{ii,m}(s)$.) Moreover, another easy computation shows that

$$\int_{-\infty}^{\infty} \frac{du}{(u^2+1)^s} = \pi^{1/2}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} ,$$

so that from (36), we see that

$$\phi_{N,0}(s) = \delta(N)N^{1-3s}\frac{\zeta(2s-1)}{\zeta(2s)} \prod_{p \mid N} (1-p^{-2s})^{-1} \tag{39}$$

(where we write $\phi_{N,0}(s)$ instead of $\phi_{ii,0}(s)$).

It is easy to check that the Laurent expansion of $\phi_{N,0}(s)$ about $s = 1$ begins

$$\phi_{N,0}(s) = \frac{\alpha_N}{s-1} + \beta_N + \cdots ,$$

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where

$$\alpha_N = \frac{3}{\pi^2}\delta(N)\left\{N^2 \prod_{p|N} (1-p^{-2})\right\}^{-1}, \tag{40}$$

$$\beta_N = \frac{6}{\pi^2}\delta(N)\left\{N^2 \prod_{p|N} (1-p^{-2})\right\}^{-1}\left\{C - \log N - \prod_{p|N} \frac{\log p}{p^2-1} - \frac{\zeta'(2)}{\zeta(2)}\right\}, \tag{41}$$

where C is the Euler-Mascheroni constant. Combining the above formulas with the definition of $\log \eta_N(z)$, we see that

$$\begin{aligned}
\log \eta_N(z) &= -\frac{z}{4\pi i \alpha_N} - N\sum_{m=1}^{\infty} \frac{e^{2\pi i m z}}{m} \sum_{\substack{d|m \\ d>0 \\ m/d \equiv 0 (\mathrm{mod}\ N)}} d e^{2\pi i d/N} \\
&= -\frac{z}{4\pi i \alpha_N} - N\sum_{k=1}^{\infty}\sum_{d=1}^{\infty} \frac{e^{2\pi i d k N z}}{kn} \cdot e^{2\pi i d/N} \\
&= -\frac{z}{4\pi i \alpha_N} - \sum_{d=1}^{\infty} e^{2\pi i d/N} \sum_{k=1}^{\infty} \frac{e^{2\pi i d k N z}}{k} \\
&= -\frac{z}{4\pi i \alpha_N} + \sum_{d=1}^{\infty} e^{2\pi i d/N} \log(1 - e^{2\pi i d N z}) \\
&= -\frac{z}{4\pi i \alpha_N} + \sum_{n=1}^{N-1} e^{2\pi i a/N} \sum_{n=0}^{\infty} \log(1 - e^{2\pi i (a+Nn) N z}) \\
&= -\frac{z}{4\pi i \alpha_N} + \sum_{a=0}^{N-1} e^{2\pi i a/N} \log \prod_{\substack{m=0 \\ m \equiv a (\mathrm{mod}\ N)}}^{\infty} (1 - e^{2\pi i m N z}) .
\end{aligned} \tag{42}$$

Thus, we may finally state

THEOREM 5–3. *Let $\overline{\Gamma(N)}$ denote the principal congruence subgroup of $SL(2, \boldsymbol{Z})$ of level N. Then*

$$\lim_{s\to 1}\left[\frac{1}{2\pi}E_\epsilon(z,s) - \frac{\alpha_N}{2(s-1)}\right] = \frac{1}{2}\beta_N - \alpha_N \log 2 - \alpha_N \log |y(z)^{1/2}\eta_N(z)^2| ,$$

where α_N and β_N are given by (40) *and* (41), *respectively, and where*

$$\log \eta_N(z) = -\frac{z}{4\pi i \alpha_N} + \sum_{a=0}^{N-1} e^{2\pi i a/N} \log \prod_{\substack{m=0 \\ m \equiv a (\mathrm{mod}\ N)}}^{\infty} (1 - e^{2\pi i m N z}) .$$

Let us now explicitly determine the Dedekind sums for the groups $\overline{\Gamma(N)}$. Our method is an adaptation of Dedekind's original proof [1] of the transformation formula for $\log \eta(\tau)$. Throughout this discussion, let $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \overline{\Gamma(N)}$, and let $s_N(\sigma)$ denote the Dedekind sum for $\overline{\Gamma(N)}$, and let ζ_N denote $e^{2\pi i/N}$.

From equation (42), we have

$$\begin{aligned}\log \eta_N(z) &= -\frac{z}{4\pi i \alpha_N} + \sum_{d=1}^{\infty} e^{2\pi i d/N} \log(1 - e^{2\pi i d N z}) \\ &= -\frac{z}{4\pi i \alpha_N} - \sum_{d=1}^{\infty}\sum_{r=1}^{\infty} e^{2\pi i d/N} \frac{e^{2\pi i d r N z}}{r} \\ &= -\frac{z}{4\pi i \alpha_N} - \sum_{r=1}^{\infty} \frac{1}{r} \frac{\zeta_N Q^r}{1 - \zeta_N Q^r},\end{aligned} \tag{43}$$

where $Q = e^{2\pi i N z}$. Now by Theorem 3–2, we have

$$\log \eta_N(\sigma z) = \log \eta_N(z) + \tfrac{1}{2}\log(cz + d) + \pi i S_N(\sigma),$$

so that

$$\pi i S_N(\sigma) = i \operatorname{Im}\{\log \eta_N(\sigma z) - \log \eta_N(z) - \tfrac{1}{2}\log(cz + d)\}. \tag{44}$$

However, by (43) and the formal identity

$$\frac{\alpha}{1-\alpha} - \frac{\beta}{1-\beta} = \frac{1}{1-\alpha} - \frac{1}{1-\beta},$$

we derive

$$\begin{aligned}\operatorname{Im}\log \eta_N(z) &= -\frac{1}{4\pi}\left[\frac{z}{i} + \frac{\bar{z}}{i}\right]\frac{1}{2i} - \frac{1}{2i}\sum_{r=1}^{\infty}\frac{1}{r}\left[\frac{\zeta_N Q^r}{1 - Q^r} - \frac{\zeta_N^{-1}\bar{Q}^r}{1 - \bar{Q}^r}\right] \\ &= \frac{1}{8\pi\alpha_N}(z + \bar{z}) - \frac{1}{2i}\sum_{r=1}^{\infty}\frac{1}{r}\left[\frac{1}{1 - \zeta_N Q^r} - \frac{1}{1 - \zeta_N^{-1}\bar{Q}^r}\right].\end{aligned} \tag{45}$$

Case 1: $c \neq 0$.

Let us set $z = -d/c + ic^2 u, u$ real and positive. Then, $\sigma(z) = a/c + iu/c^2$. By (44), we see that

$$\begin{aligned}\pi i S_N = i \lim_{u\to 0} \operatorname{Im}\Big\{&\log \eta_N\left(\frac{a}{c} + \frac{iu}{c^2}\right) \\ &- \log \eta_N\left(-\frac{d}{c} + \frac{i}{c^2 u}\right) - \frac{1}{2}\log\left(\frac{i}{cu}\right)\Big\}\end{aligned} \tag{46}$$

However,

$$\log\left(\frac{i}{cu}\right) = \log\left(\frac{1}{|cu|}\right) + i\arg\left(\frac{i}{cu}\right),$$

so that

$$\operatorname{Im}\log\left(\frac{i}{cu}\right)=\frac{\pi i}{2} \qquad \text{if } c>0$$
$$-\frac{\pi i}{2} \qquad \text{if } c<0\,.$$

Thus, by (45), we have

$$S_N(\sigma)=-\frac{1}{4}\frac{c}{|c|}+\frac{1}{\pi}\lim_{u\to 0}\operatorname{Im}\left\{\log\eta_N\left(-\frac{d}{c}+\frac{iu}{c^2}\right)-\log\eta_N\left(-\frac{d}{c}+\frac{i}{c^2u}\right)\right\}. \tag{47}$$

Now by equation (45), we have

$$\operatorname{Im}\log\eta_N(z)=\frac{1}{8\pi\alpha_N}\left(-\frac{2d}{c}\right)-\frac{1}{2i}\sum_{r=1}^{\infty}\frac{1}{r}\left\{\frac{1}{1-\zeta_N V^r Q_0^r}-\frac{1}{1-\zeta_N V^{-r}Q_0^r}\right\}, \tag{48}$$

where $V=e^{-2\pi N/c^2u}$, $Q_0=e^{2\pi N/c^2u}$. However, as $u\to 0$, Q_0 tends rapidly to infinity and it is clear that the summation in (48) tends to zero as $u\to 0$. Thus,

$$\lim_{u\to 0}\operatorname{Im}\log\eta_N\left(-\frac{d}{c}+\frac{i}{c^2u}\right)=\frac{d}{4\pi\alpha_N c}\,.$$

Thus, by (47), we see that

$$S_N(\sigma)=-\frac{1}{4}\frac{c}{|c|}+\frac{d}{4\pi^2\alpha_N c}+\frac{1}{r}\lim_{u\to 0}\operatorname{Im}\log\eta_N\left(\frac{a}{c}+\frac{iu}{c^2}\right). \tag{49}$$

Let $W=e^{2\pi iNs/c}$, $Q_1=e^{-2\pi uN/c^2}$. Then (45) implies that

$$\operatorname{Im}\log\eta_N\left(\frac{a}{c}+\frac{iu}{c^2}\right)=\frac{a}{4\pi\alpha_N c}-\frac{1}{2i}\sum_{r=1}^{\infty}\frac{1}{r}\left[\frac{1}{1-\zeta_N W^r Q^r}-\frac{1}{1-\zeta_N^{-1}W^{-r}Q_1^{-r}}\right]. \tag{50}$$

A simple argument (see [1, p. 167]) can be used to justify the interchange of the summation on the right of (50) and the operation $u\to 0$. However,

$$\lim_{u\to 0}\left\{\frac{1}{1-\zeta_N W^r Q_1^r}-\frac{1}{1-\zeta_N^{-1}W^{-r}Q_1^{-r}}\right\}=\frac{1}{1-\zeta_N W^r}-\frac{1}{1-\zeta_N^{-1}W^r} \qquad \text{if } \zeta_N W^r\neq 1 \tag{51}$$
$$0 \quad \text{if } \zeta_N W^r=1\,.$$

If $\eta \neq 1$ is a k^{th} root of 1, then a simple argument shows that

$$\frac{1}{1-\eta} = -\frac{1}{k} \sum_{i=1}^{k-1} i\eta .$$

Therefore, since $\zeta_N W^r$ is a $|c|N^{\text{th}}$ root of 1, we see that, if $\zeta_N W^r \neq 1$,

$$\frac{1}{1-\zeta_N W^r} = -\frac{1}{|c|N} \sum_{i=1}^{|c|N-1} i\zeta_N^i W^{ir}$$
$$\frac{1}{1-\zeta_N^{-1} W^{-r}} = -\frac{1}{|c|N} \sum_{i=1}^{|c|N-1} i\zeta_N^{-1} W^{-ir} .$$

Thus, by equation (51), we see that

$$\lim_{u\to 0}\left\{\frac{1}{1-\zeta_N W^r Q_1^r} - \frac{1}{1-\zeta_N^{-1} W^{-1} Q_1^r}\right\} = -\frac{1}{|c|N} \sum_{j=1}^{|c|N-1} j\{\zeta_N^j W^{jr} - \zeta_N^{-j} W^{-jr}\} , \tag{52}$$

provided that $\zeta_N W^r \neq 1$. However, if $\zeta_N W^r = 1$, then (51) shows that (52) again holds, so that by (50),

$$\begin{aligned} \lim_{u\to 0} \operatorname{Im} \log \eta_N\left(\frac{a}{c} + \frac{iu}{c^2}\right) &= \frac{a}{4\pi\alpha_N c} - \frac{1}{2i} \sum_{r=1}^{\infty} \frac{1}{r|c|N} \sum_{j=1}^{|c|N-1} j\{\zeta_N^j W^{jr} - \zeta_N^{-j} W^{-jr}\} \\ &= \frac{a}{4\pi\alpha_N c} - \frac{1}{2i|c|N} \sum_{j=1}^{|c|N-1} j \sum_{r=1}^{\infty} \frac{\zeta_N^j W^{jr} - \zeta_N^{-j} W^{-jr}}{r} . \end{aligned} \tag{53}$$

Let us now show that the inner sum of (53) is just the Forier expansion of an elementary function.

Let z be a real number and let $[z]$ = the largest integer $\leq z$. Further, let $((z)) = z - [z] - 1/2$*. Then a simple computation shows that

$$2\pi i\left(\left(z - \frac{1}{2}\right)\right) = \sum_{\mu=1}^{\infty} \frac{e^{-2\pi i z\mu} - e^{2\pi i z\mu}}{\mu} .$$

Therefore,

$$\begin{aligned} 2\pi i\left(\left(z - \frac{1}{4}\right)\right) &= \sum_{\mu=1}^{\infty} \frac{e^{-\pi i/2} e^{-2\pi i z\mu} - e^{\pi i/2} e^{2\pi i z\mu}}{\mu} \\ &= -i \sum_{\mu=1}^{\infty} \frac{e^{-2\pi i z\mu} + e^{2\pi i z\mu}}{\mu} . \end{aligned}$$

* See the footnote on p. 22.

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Thus, as a consequence, we see that

$$\sum_{\mu=1}^{\infty} \frac{e^{-2\pi i z\mu}}{\mu} = -\pi\left(\left(z - \frac{1}{4}\right)\right) + \pi i\left(\left(z - \frac{1}{2}\right)\right)$$
$$\sum_{\mu=1}^{\infty} \frac{e^{2\pi i z\mu}}{\mu} = -\pi\left(\left(z - \frac{1}{4}\right)\right) - \pi i\left(\left(z - \frac{1}{2}\right)\right).$$

Thus, finally we see that for $\lambda \in \boldsymbol{C}$,

$$\sum_{\mu=1}^{\infty} \frac{\lambda e^{2\pi i z\mu} - \lambda^{-1}e^{-2\pi i z\mu}}{\mu} = -\pi\left(\left(z - \frac{1}{4}\right)\right)\{\lambda - \lambda^{-1}\} - \pi i\left(\left(z - \frac{1}{2}\right)\right)\{\lambda + \lambda^{-1}\}.$$

Let us set $\lambda = \zeta_N^j$, $z = Naj/c$. Then we see immediately that

$$\begin{aligned}\sum_{r=1}^{\infty} &\frac{\zeta_N^j W^{jr} - \zeta_N^{-j} W^{-jr}}{r} \\ &= -\pi\left(\left(\frac{Naj}{c} - \frac{1}{4}\right)\right)\{\zeta_N^j - \zeta_N^{-j}\} - \pi i\left(\left(\frac{Naj}{c} - \frac{1}{2}\right)\right)\{\zeta_N^j + \zeta_N^{-j}\} \qquad (54)\\ &= -2\pi i \sin\left(\frac{2\pi j}{N}\right)\left(\left(\frac{Naj}{c} - \frac{1}{4}\right)\right) - \cos 2\pi i\left(\frac{2\pi j}{c}\right)\left(\left(\frac{Naj}{c} - \frac{1}{2}\right)\right).\end{aligned}$$

Thus, by combining (55) with (49), we have, finally, that

$$\begin{aligned}S_N(\sigma) &= -\frac{1}{4}\frac{c}{|c|} + \frac{d}{4\pi^2\alpha_N c} + \frac{a}{4\pi^2\alpha_N c} + \frac{1}{|c|N} \\ &\qquad \times \sum_{j=1}^{|c|N-1} j \sin\left(\frac{2\pi j}{N}\right)\left(\left(\frac{Naj}{c} - \frac{1}{4}\right)\right) + \frac{1}{|c|N} \\ &\qquad \times \sum_{j=1}^{|c|N-1} j \cos\left(\frac{2\pi j}{N}\right)\left(\left(\frac{Naj}{c} - \frac{1}{2}\right)\right) \\ &= \frac{1}{4\pi^2\alpha_N}\left(\frac{a+d}{c}\right) - \frac{1}{4}\frac{c}{|c|} + \frac{1}{|c|N} \qquad (55)\\ &\qquad \times \sum_{j=1}^{|c|N-1} j\left[\sin\left(\frac{2\pi j}{N}\right)\left(\left(\frac{Naj}{c} - \frac{1}{4}\right)\right)\right. \\ &\qquad\qquad \left. + \cos\left(\frac{2\pi j}{N}\right)\left(\left(\frac{Naj}{c} - \frac{1}{2}\right)\right)\right].\end{aligned}$$

This completes the discussion of Case 1.

Case 2: $c = 0$.

In this case, $\sigma = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, $b \equiv 0 \pmod N$. Thus, by equation (42),

$$\log \eta_N(\sigma z) = \log \eta_N(z+b) = \log \eta_N(z) - \frac{b}{4\pi i \alpha_N} .$$

Thus, by the definition of $S_N(\sigma)$, we have

$$S_N(\sigma) = \frac{b}{4\pi^2 \alpha_N} .$$

Combining Cases 1 and 2, we may state the following

THEOREM 5–4. *Let* $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N)$. *Then the Dedekind sum* $S_N(\sigma)$ *is given by*

$$\begin{aligned} S_N(\sigma) &= \frac{6}{4\pi^2 \alpha_N} , \qquad \text{if } c = 0 \\ &= \frac{1}{4\pi^2 \alpha_N}\left(\frac{\alpha + d}{c}\right) - \frac{1}{4}\frac{c}{|c|} + \frac{1}{|c|N} \\ &\qquad \times \sum_{j=1}^{|c|N-1} j\left[\sin\left(\frac{2\pi j}{N}\right)\left(\left(\frac{Naj}{c} - \frac{1}{4}\right)\right)\right. \\ &\qquad \left. + \cos\left(\frac{2\pi j}{N}\right)\left(\left(\frac{Nai}{c} - \frac{1}{2}\right)\right)\right] , \qquad (c \neq 0) \end{aligned}$$

where

$$\alpha_N = \left(\frac{3}{\pi^2}\right)\delta(N)\left\{N^2 \prod_{p|N}(1 - p^{-3})\right\}^{-1} .$$

In particular, $S_N(\sigma)$ *is an algebraic number belonging to the* N^{th} *cyclotomic field.*

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