A Geometrical Proof of Professor Morley's Extension of Feuerbach's Theorem.

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1. In the *Proceedings of the National Academy of Sciences of the U.S.A.*, Vol. II. (1916), page 171, Professor F. Morley has established a theorem which both extends and simplifies the theorem of Feuerbach, viz., *All curves of class three which* (i) *touch the six lines OP, OQ, OR, QR, RP, PQ joining four orthocentric points, O, P, Q, R, and* (ii) *pass through the circular points, also touch the common nine-points-circle of the triangles PQR, OQR, ORP, OPQ.* Sixteen of these curves of class three break up into one of the four points and a circle touching the sides of the triangle formed by the other three. Thus the sixteen instances of Feuerbach's theorem derivable from the four triangles are included as special cases in Morley's theorem. A purely geometrical proof of the theorem may be worth consideration.

2. In what follows C_n and Γ_n will be used to denote "curve of order n, and class n" respectively.

Let there be given a group of four points O, P, Q, R having the diagonal triangle ABC; then any two Γ_3's which touch OP, OQ, OR, QR, RP, PQ have three other common tangents. Two of the

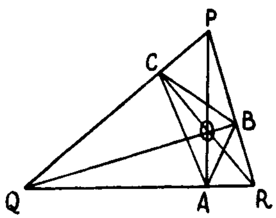

Fig. 1.

three may be taken arbitrarily, and the third is then easily determined from the property that *all* Γ_3*'s* which touch eight given lines also touch a certain ninth line.

Suppose the three unknown common tangents of the two Γ_3*'s* form a triangle XYZ. All Γ_3*'s* which touch ZX and ZY and the six lines named above also touch XY. One of these Γ_3's breaks into the point P and a Γ_2 touching OQ, OR, QR: thus the sides of the triangles OQR, XYZ touch a Γ_2, and therefore their vertices X, Y, Z, O, Q, R lie on a C_2. Similarly X, Y, Z, O, P, R lie on a C_2, which must be the same as the former. Hence

If the three unknown common tangents of two Γ_3*'s which touch* OP, OQ, OR, QR, RP, PQ *form a triangle* XYZ, *the seven points* X, Y, Z, O, P, Q, R *lie on a conic.*

Conversely it is true that

If the vertices of a triangle XYZ lie on a conic with the points O, P, Q, R, then all Γ_3*'s* which touch OP, OQ, OR, QR, RP, PQ and two sides of XYZ also touch the third side.

3. X, Y, Z being three points on a conic through O, P, Q, R, suppose that L, M, N are the points of contact of YZ, ZX, XY with a Γ_3 which also touches the six lines named above. Take a point Z' on the conic adjacent to Z and draw tangents to Γ_3, $Z'X'$, $Z'Y'$, adjacent to ZX, ZY, cutting the conic in X', Y', adjacent to X, Y; then $X'Y'$ is a tangent to Γ_3. Since the six vertices of XYZ, $X'Y'Z'$ lie on C_2, the six sides touch a Γ_2.

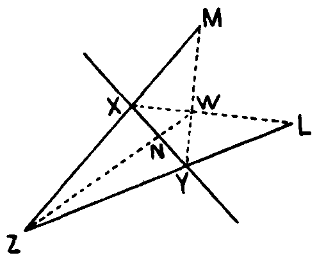

Fig. 2.

Hence, in the limit, when Z' approaches indefinitely near to Z, we see that a conic must touch YZ, ZX, XY at L, M, N respectively, *i.e.* that XL, YM, ZN are concurrent. Thus, if the positions of ZX, ZY, L and M are known, those of XY and N are easily found. But it must be noted that, when ZX and ZY are given, the positions of L and M are not arbitrary; there is a [1, 1] correspondence between them.

4. Again, when we consider the system of Γ_3's which (i) touch the six lines OP, OQ, OR, QR, RP, PQ, and (ii) pass through two given points L and M, there is a relation between the tangents LZ, MZ, which implies that Z must lie on a certain locus. In a special case, however, it is possible to find the locus of N (which locus is also the envelope of XY and part of the envelope of the Γ_3's) for such a system of Γ_3's without further investigating the locus of Z.

5. Suppose that L and M are conjugate with respect to *all* conics through O, P, Q, R. The polar of M with respect to the conic O, P, Q, R, Z, then passes through L and also divides ZX harmonically, *i.e.* it is the line LN. Thus LMN is a self-polar triangle of the C_2. N is the pole of LM; and, since for different conics through O, P, Q, R, there is a (1, 1) correspondence between LN and MN, the locus of N is a conic passing through L and M. Further, three conics through O, P, Q, R consist of a pair of straight lines. For these N lies at the intersection of the pair of lines, *i.e.* at A, B or C. Hence the locus of N is the conic $LMABC$.

In this case when two adjacent Γ_3's touching the six lines OP, OQ, OR, QR, RP, PQ and passing through L and M approach coincidence, their nine common tangents are ultimately

(1) the six lines OP, OQ, OR, QR, RP, PQ;

(2) two lines (LYZ, MXZ) through L and M, the points of contact being L and M;

(3) A line (XNY) touching the conic $LMABC$ at N, the point of contact with the Γ_3 also being N.

The envelope of the Γ_3's consists of the points L, M, and the conic $LMABC$.

6. So far all the reasoning and properties have been projective. In the special case when L and M are the circular points, O, P, Q, R are four orthocentric points, the inscribed and escribed centres of ABC. The conic through O, P, Q, R and Z is a rectangular

hyperbola, whose centre N lies on the nine-points-circle ABC. The line XY is a common tangent at N to the nine-points-circle and to the curve of class three which touches OP, OQ, OR, QR, RP, PQ, and also has ZL, ZM as tangents at L and M, the circular points. *Q.E.D.*

7. It will be found that the locus of Z is a curve of order 5, having double points at O, P, Q, R, passing through the centres of the sixteen inscribed and escribed circles of the triangles PQR, OQR, ORP, OPQ, and through the centre of the common nine-points-circle ABC. Each point N of the nine-points-circle is the point of contact of two distinct Γ_3's of the system, one having LYZ, MXZ as tangents at L, M, the other having LWX, MWY as tangents, as shown in Figure 2.

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