

# 1

## Introduction

It is well known that supersymmetric theories may have Bogomol'nyi–Prasad–Sommerfield (BPS) sectors in which some data can be computed at strong coupling even when the full theory is not solvable. Historically, this is how the first exact results on particle spectra were obtained [1]. Seiberg–Witten's breakthrough results [2, 3] in the mid 1990s gave an additional motivation to the studies of the BPS sectors.

BPS solitons can emerge in those supersymmetric theories in which superalgebras are centrally extended. In many instances the corresponding central charges are seen at the classical level. In some interesting models central charges appear as quantum anomalies.

First studies of BPS solitons (sometimes referred to as critical solitons) in supersymmetric theories at weak coupling date back to the 1970s. De Vega and Schaposnik were the first to point out [4] that a model in which classical equations of motion can be reduced to first-order Bogomol'nyi–Prasad–Sommerfeld (BPS) equations [5, 6] is, in fact, a bosonic reduction of a supersymmetric theory. Already in 1977 critical soliton solutions were obtained in the superfield form in some two-dimensional models [7]. In the same year miraculous cancellations occurring in calculations of quantum corrections to soliton masses were noted in [8] (see also [9]). It was observed that for BPS solitons the boson and fermion modes are degenerate and their number is balanced. It was believed (incorrectly, we hasten to add) that the soliton masses receive no quantum corrections. The modern – correct – version of this statement is as follows: if a soliton is BPS-saturated at the classical level and belongs to a shortened supermultiplet, it stays BPS-saturated after quantum corrections, and its mass exactly coincides with the central charge it saturates. The latter may or may not be renormalized. Often – but not always – central charges that do not vanish at the classical level and have quantum anomalies *are* renormalized. Those that emerge as anomalies and have no classical part typically receive no

renormalizations. In many instances holomorphy protects central charges against renormalizations.

Critical solitons play a special role in gauge field theories. Numerous parallels between such solitonic objects and basic elements of string theory have been revealed in recent years. At first, the relation between string theory and supersymmetric gauge theories was mostly a “one-way street” – from strings to field theory. Now it is becoming exceedingly more evident that field-theoretic methods and results, in their turn, provide insights in string theory.

String theory, which emerged from dual hadronic models in the late 1960s and 70s, elevated to the “theory of everything” in the 1980s and 90s when it experienced an unprecedented expansion, has seemingly entered a “return-to-roots” stage. The task of finding solutions to “down-to-earth” problems of QCD and other gauge theories by using results and techniques of string/ $D$ -brane theory is currently recognized by many as one of the most important and exciting goals of the community. In this area the internal logic of development of string theory is fertilized by insights and hints obtained from field theory. In fact, this is a very healthy process of cross-fertilization.

If supersymmetric gauge theories are, in a sense, dual to string/ $D$ -brane theory – as is generally believed to be the case – they must support domain walls (of the  $D$ -brane type) [10], and we know, they do [11, 12]. A  $D$ -brane is defined as a hypersurface on which a string may end. In field theory both the brane and the string arise as BPS solitons, the brane as a domain wall and the string as a flux tube. If their properties reflect those inherent to string theory, at least to an extent, the flux tube must end on the wall. Moreover, the wall must house gauge fields living on its world volume, under which the end of the string is charged.

The purpose of this review is to summarize developments in critical solitons in two, three and four dimensions, with emphasis on four dimensions and on most recent results. A large variety of BPS-saturated solitons exist in four-dimensional field theories: domain walls, flux tubes (strings), monopoles and dyons, and various junctions of the above objects. A list of recent discoveries includes localization of gauge fields on domain walls, non-Abelian strings that can end on domain walls, developed boojums, confined monopoles attached to strings, and other remarkable findings. The BPS nature of these objects allows one to obtain a number of exact results. In many instances nontrivial dynamics of the bulk theories we will consider lead to effective low-energy theories in the world volumes of domain walls and strings (they are related to zero modes) exhibiting novel dynamical features that are interesting by themselves.

We do not try to review the vast literature accumulated since the mid 1990s in its entirety. A comparison with a huge country the exploration of which is not yet completed is in order here. Instead, we suggest what may be called “travel diaries”

of the participants of the exploratory expedition. Recent publications [13, 14, 15, 16, 17] facilitate our task since they present the current developments in this field from a complementary point of view.

The “diaries” are organized in two parts. The first part (entitled “Short excursion”) is a bird’s eye view of the territory. It gives a brief and largely nontechnical introduction to basic ideas lying behind supersymmetric solitons and particular applications. It is designed in such a way as to present a general perspective that would be understandable to anyone with an elementary knowledge in classical and quantum fields, and supersymmetry.

Here we present some historic remarks, catalog relevant centrally extended superalgebras and review basic building blocks we consistently deal with – domain walls, flux tubes, and monopoles – in their classic form. The word “classic” is used here not in the meaning “before quantization” but, rather, in the meaning “recognized and cherished in the community for years.”

The second part (entitled “Long journey”) is built on other principles. It is intended for those who would like to delve in this subject thoroughly, with its specific methods and technical devices. We put special emphasis on recent developments having direct relevance to QCD and gauge theories at large, such as non-Abelian flux tubes (strings), non-Abelian monopoles confined on these strings, gauge field localization on domain walls, etc. We start from presenting our benchmark model, which has extended  $\mathcal{N} = 2$  supersymmetry. Here we go well beyond conceptual foundations, investing efforts in detailed discussions of particular problems and aspects of our choosing. Naturally, we choose those problems and aspects which are instrumental in the novel phenomena mentioned above. In addition to walls, strings and monopoles, we also dwell on the string-wall junctions which play a special role in the context of dualization.

Our subsequent logic is from  $\mathcal{N} = 2$  to  $\mathcal{N} = 1$  and further on. Indeed, in certain instances we are able to descend to non-supersymmetric gauge theories which are very close relatives of QCD. In particular, we present a fully controllable weakly coupled model of the Meissner effect which exhibits quite nontrivial (strongly coupled) dynamics on the string world sheet. One can draw direct parallels between this consideration and the issue of  $k$ -strings in QCD.

