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Homological Planes in the Grothendieck Ring of Varieties

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Abstract. In this note we identify the classes of $\mathbf{Q}$-homological planes in the Grothendieck group of complex varieties $K_0(\mathrm{Var}_{\mathbf{C}})$. Precisely, we prove that a connected, smooth, affine, complex, algebraic surface X is a $\mathbf{Q}$-homological plane if and only if $[X] = [\mathbf{A}^2_{\mathbf{C}}]$ in the ring $K_0(\mathrm{Var}_{\mathbf{C}})$ and $\mathrm{Pic}(X)_{\mathbf{Q}} := \mathrm{Pic}(X) \otimes_{\mathbf{Z}} \mathbf{Q} = 0$.

1 Introduction

1.1 Grothendieck Ring of Varieties

If k is a field, a *k-variety* is a separated k-scheme of finite type. The *Grothendieck ring of varieties* was introduced by A. Grothendieck in 1964 and can be defined as follows. Let $\mathbf{Z}[\mathrm{Var}_k]$ be the free abelian group generated by the isomorphism classes of k-varieties; let us denote by $\{X\}$ the isomorphism class of the k-variety X in $\mathbf{Z}[\mathrm{Var}_k]$. If N is the subgroup of $\mathbf{Z}[\mathrm{Var}_k]$ generated by the elements of the form $\{X\} - \{Y\} - \{X \setminus Y\}$, where X is a k-variety and Y a closed subscheme of X, then one sets

$$K_0(\mathrm{Var}_k) := \mathbf{Z}[\mathrm{Var}_k]/N.$$

The class of the variety X is denoted by $[X]$ in the group $K_0(\mathrm{Var}_k)$. By bilinearity, the formula

$$[X] \cdot [X'] := [X \times_k X']$$

for every pair of k-varieties (X, X') provides a ring structure on the group $K_0(\mathrm{Var}_k)$ whose neutral element is $[\mathrm{Spec}(k)]$. We denote by $\mathbf{L}$ the class of the affine line $\mathbf{A}^1_k$.

1.2 Piecewise Isomorphism

We say that two k-varieties X, X' are *piecewise isomorphic* if there exist a finite set I and a partition $(X_i)_{i\in I}$ (resp. $(X'_i)_{i\in I}$) of X (resp. X') into locally closed subsets such that, for every $i \in I$, there exists an isomorphism of k-schemes $\varphi_i\colon (X_i)_{\mathrm{red}} \to (X'_i)_{\mathrm{red}}$. Such a family of isomorphisms $(\varphi_i)_{i\in I}$ is called a *piecewise isomorphism* between X and X'. By definition, it is easy to check that two piecewise isomorphic k-varieties have the same class in $K_0(\mathrm{Var}_k)$. (The converse is essentially an open question; see for example [4–6, 8].)

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Definition 1.1 A $\mathbf{Q}$-*homological plane* is a connected smooth affine complex surface X, with vanishing rational homology, *i.e.,* such that for every $n \in \mathbf{N}^*$, $H_n(X(\mathbf{C}), \mathbf{Q}) = 0$.

1.3 Main Result

In this note we give a characterization of $\mathbf{Q}$-homological planes involving the expression of their class in $K_0(\mathrm{Var}_{\mathbf{C}})$. It shows in particular that every $\mathbf{Q}$-homological plane is piecewise isomorphic to the affine plane $\mathbf{A}^2_{\mathbf{C}}$. (To the best of our knowledge, this direct consequence of our main theorem is new.)

Precisely, let $(\star)$ be the following family of five conditions on the complex variety X:

$$(\star) = \left\{ \begin{matrix} H_1(X(\mathbf{C}), \mathbf{Q}) = 0, & H^1(X(\mathbf{C}), \mathbf{Q}) = 0, \\ H_2(X(\mathbf{C}), \mathbf{Q}) = 0, & H^2(X(\mathbf{C}), \mathbf{Q}) = 0, \\ \mathrm{Pic}(X)_{\mathbf{Q}} = 0 & \end{matrix} \right\}.$$

Theorem 1.2 *Let X be a connected smooth affine complex algebraic surface X that statisfies one of the conditions of the family* $(\star)$. *Then the following assertions are equivalent:*

(i) *The surface X is a* $\mathbf{Q}$*-homological plane;*
(ii) *In the ring* $K_0(\mathrm{Var}_k)$*, we have* $[X] = \mathbf{L}^2$*;*
(iii) *The surfaces Xand* $\mathbf{A}^2_{\mathbf{C}}$ *are piecewise isomorphic.*

In this case, all the conditions of the family $(\star)$ *are satisfied.*

Furthermore, Remark 2.7 emphasizes the fact that the conditions in $(\star)$ cannot be deleted.

2 Preliminary Results

2.1 Smooth Completions

If X is a connected, smooth, affine k-surface, the datum of a pair (V, D), where V is a connected smooth projective k-surface and D is a closed subscheme of V such that $V \backslash D = X$, is called a *smooth completion of* X. If (V, D) is a smooth completion of X, we denote by $(D_i)_{i \in \{1,\dots,n\}}$ the family of the irreducible components of D_{red}, and we set

$$\Gamma := \{x \in D\, ;\, \exists i, j \in \{1, \dots, n\},\ i \neq j,\ x \in D_i \cap D_j\}. \qquad (2.1)$$

In that case, the scheme D is connected, and all the D_i are of dimension one. In addition, if the closed subscheme D defines a simple normal crossings divisor in V, we call the pair (V, D) a *log-smooth completion of* X. If k is a field of characteristic 0, Nagata's theorem and Hironaka's theorem allow us to associate a log-smooth completion with any connected smooth affine k-surface.

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2.2 A Preliminary Result

If T is a complex (algebraic) variety, let us denote by T^{an} the analytic space associated with the set $T(\mathbf{C})$ of its complex points. The following statement is classical; the first assertion can be deduced from a study of the Mayer–Vietoris sequence (for cohomology). (See [2, (12.1),(17.6),(23.13)].)

Lemma 2.1 *Let V be a rational, connected, smooth, projective, complex surface, with D a simply connected strictly normal crossings divisor on V. Then we have the following formulae:*

(i) $H^1(D^{\mathrm{an}}, \mathbf{Q}) = (0)$; $H^2(D^{\mathrm{an}}, \mathbf{Q}) \cong \mathbf{Q}^n$;

(ii) $H^0(V^{\mathrm{an}}, \mathbf{Q}) \cong \mathbf{Q}$; $H^1(V^{\mathrm{an}}, \mathbf{Q}) = H^3(V^{\mathrm{an}}, \mathbf{Q}) = (0)$; $H^2(V^{\mathrm{an}}, \mathbf{Q}) \cong \mathbf{Q}^{b_2}$; $H^4(V^{\mathrm{an}}, \mathbf{Q}) \cong \mathbf{Q}$,

where $b_2 := \dim_{\mathbf{Q}}(H^2(V^{\mathrm{an}}, \mathbf{Q}))$ *and n is the number of irreducible components of D_{red}.*

Let us establish an important technical proposition related to piecewise algebraic geometry. If k is a field of characteristic 0 with a fixed algebraic closure $\overline{k}$ and if X is a k-variety, we denote by $H^*_{\text{ét}}(\overline{X}, \mathbf{Q}_\ell)$ the *ℓ-adic cohomology* of X, where we set $\overline{X} := X \otimes_k \overline{k}$.

Proposition 2.2 *With notation (2.1) as above, let k be an algebraically closed field of characteristic 0, and let X be a connected smooth affine k-surface. Then the following assertions are equivalent.*

(i) *The k-surfaces $X, \mathbf{A}^2_k$ are piecewise isomorphic;*

(ii) *In the ring $K_0(\mathrm{Var}_k)$, we have $[X] = \mathbf{L}^2$;*

(iii) *Every log-smooth completion (V, D) of X satisfies the following properties, where n is the number of irreducible components of D_{red}:*

- *the k-variety V is rational;*
- $n = \dim_{\mathbf{Q}_\ell}(H^2_{\text{ét}}(V, \mathbf{Q}_\ell))$;
- $D_i \cong \mathbf{P}^1_k$ *for every* $i \in \{1, \dots, n\}$;
- *the set Γ is a finite set with cardinality $n - 1$;*

In this case, the divisor D is a *tree of* $\mathbf{P}^1_k$.

Remark 2.3 Under the assumptions of Proposition 2.2, it is equivalent to require that every log-smooth completion has the mentioned properties or that there exists such a log-smooth completion.

The proof of such a statement is based on the use of important classical ingredients coming from piecewise algebraic geometry, which we mention here for the convenience of the reader.

Theorem 2.4 (*e.g.*, see [6, §4]) *There exists a unique ring morphism*

$$P(\,\cdot\,, T)\colon K_0(\mathrm{Var}_k) \longrightarrow \mathbf{Z}[[T]],$$

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sending $[X]$ *to the polynomial*

$$P(X,T) := \sum_{i=0}^{2\dim(X)} \dim_{\mathbf{Q}_\ell}\big(H^i_{\acute{e}t}\big(\overline{X}, \mathbf{Q}_\ell\big)\big)\, T^i$$

for every connected, smooth, projective, k-variety X.

When $k \subset \mathbf{C}$, the cohomological comparison theorems allows us to use singular cohomology instead of étale cohomology. In that case, one can remark that

$$P(X,T) = \sum_{i=0}^{2\dim(X)} \dim_{\mathbf{Q}}\big(H^i(X^{\mathrm{an}}, \mathbf{Q})\big)\, T^i$$

for every connected, smooth, projective k-variety X. The polynomial $P(X,T)$ is called the *Poincaré polynomial* of X.

Theorem 2.5 ([5]) *Let k be an algebraically closed field of characteristic* 0*. There exists a unique surjective ring morphism*

$$\mathrm{SB}\colon K_0(\mathrm{Var}_k) \longrightarrow \mathbf{Z}[\mathrm{SB}],$$

which sends $[X]$ *to the equivalence class of X under the stably birational equivalence for every connected smooth projective k-variety X. Furthermore, the kernel of the morphism* SB *is the ideal of the ring* $K_0(\mathrm{Var}_k)$ *generated by* $\mathbf{L}$*.*

Recall that two integral k-varieties X, X' are *stably birational* if there exist two integers $m, n \in \mathbf{N}$ such that the k-varieties $X \times_k \mathbf{P}^m_k, X' \times_k \mathbf{P}^n_k$ are birationally equivalent. This definition gives rise to an equivalence relation called *stably birational equivalence*. We denote by $\mathbf{Z}[\mathrm{SB}]$ the free abelian group generated by the equivalence classes of connected, smooth, projective k-varieties under the stably birational equivalence. It is endowed with a ring structure induced by the fiber product over $\mathrm{Spec}(k)$.

Theorem 2.6 ([6, Proposition 6]) *Let k be an algebraically closed field of characteristic* 0*. Let* X, X' *be two k-varieties. Let us assume that* $\dim(X) \le 1$ *and* $[X] = [X']$ *in the ring* $K_0(\mathrm{Var}_k)$*. Then* X, X' *are piecewise isomorphic.*

Proof of Proposition 2.2 (i) ⇒ (ii). The assertion can be deduced from the definition of $K_0(\mathrm{Var}_k)$.

(ii) ⇒ (i). Let (V, D) be a smooth completion of X. Then we have the following relation in the ring $K_0(\mathrm{Var}_k)$:

$$[V] = [D] + \mathbf{L}^2 = [D_{\mathrm{red}}] + \mathbf{L}^2 = \mathbf{L}^2 + \sum_{i=1}^{n} [D_j].$$

For every $j \in \{1, \dots, n\}$, let us fix a projective smooth model D'_j of D_j. Then there exists an integer $m \in \mathbf{Z}$ such that

$$[V] = \mathbf{L}^2 + m + \sum_{j=1}^{n} [D'_j]. \tag{2.2}$$

By applying the morphism SB to equation (2.2), we conclude that either V is rational or there exists $i \in \{1, \dots, n\}$ such that $\mathrm{SB}(V) = \mathrm{SB}(D'_i)$. In this last case, V is

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birationally equivalent to $D'_i \times_k \mathbf{P}^1_k$. From the elimination of the indeterminacies and equation (2.2), we deduce that there exists an integer $r \in \mathbf{Z}$ such that

$$[D'_i](\mathbf{L}+1) = \mathbf{L}^2 + r\mathbf{L} + \sum_{j=1}^{n} [D'_j]. \tag{2.3}$$

Now, by applying the morphism $P(\,\cdot\,, T)$ to (2.3), we conclude that

$$T^2\big(T^2 + 2g(D'_i)T + 1\big) = T^4 + rT^2 + \sum_{j=1, j\neq i}^{n} \big(T^2 + 2g(D'_j)T + 1\big).$$

So $g(D'_i) = 0$ for every $i \in \{1, \dots, n\}$, and all the curves D_i are rational. It follows that the surface V is rational (hence X is rational).

Let us construct a piecewise isomorphism between X and $\mathbf{A}^2_k$. Since X is rational, there exists a closed subscheme C_X (resp. C) of X (resp. $\mathbf{A}^2_k$) of dimension at most one, and an isomorphism of k-schemes $\varphi_0\colon X\backslash C_X \to \mathbf{A}^2_k\backslash C$ such that we have

$$[X] - [C_X] = \mathbf{L}^2 - [C]$$

in the ring $K_0(\mathrm{Var}_k)$. By assumption, $[C_X] = [C]$. Theorem 2.6 proves the existence of a piecewise isomorphism $(\varphi_i)_{i\in I}$ between C_X and C. Then we deduce that the k-surfaces $X, \mathbf{A}^2_k$ are piecewise isomorphic via the family of isomorphisms $(\varphi_i)_{i\in I\cup\{0\}}$.

(iii) $\Rightarrow$ (ii). Let $b_2 := \dim_{\mathbf{Q}_\ell}(H^2_{\text{ét}}(V, \mathbf{Q}_\ell))$. From [6, Lemma 12], it follows that

$$[X] = [V] - [D] = (\mathbf{L}^2 + b_2\mathbf{L} + 1) - b_2(\mathbf{L}+1) + (b_2 - 1) = \mathbf{L}^2$$

in the ring $K_0(\mathrm{Var}_k)$.

(ii) $\Rightarrow$ (iii). Let (V, D) be a log-smooth completion of X. We have shown that the k-surface X is piecewise isomorphic to $\mathbf{A}^2_k$, hence it is rational; so is V. In the ring $K_0(\mathrm{Var}_k)$, we have the following relation:

$$[V] = [D] + \mathbf{L}^2 = [D_{\text{red}}] + \mathbf{L}^2 = \mathbf{L}^2 + \sum_{i=1}^{n} [D_j] - m, \tag{2.4}$$

where $m := |\Gamma|$. By applying the morphism SB to equation (2.4), we deduce , for every integer $i \in \{1, \dots, n\}$, that $D_i \cong \mathbf{P}^1_k$, and that $m = n - 1$. So we have

$$[V] = \mathbf{L}^2 + n(\mathbf{L}+1) - m. \tag{2.5}$$

Now, by computing Poincaré polynomials in equation (2.5) and comparing the coefficients of the terms of degree 2 in the resulting equation, we conclude that

$$\dim_{\mathbf{Q}_\ell}\big(H^2_{\text{ét}}(V, \mathbf{Q}_\ell)\big) = n. \qquad \blacksquare$$

Remark 2.7 One can find connected, smooth, affine, complex surfaces that satisfy no condition in $(\star)$, but verify one of the equivalent conditions of Proposition 2.2. (See [1, §8.26].) Furthermore, there exist non-affine (resp. non-smooth, resp. non-connected) k-varieties whose class in the ring $K_0(\mathrm{Var}_k)$ is $\mathbf{L}^2$.

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3 Proof of Theorem 1.2

From Proposition 2.2, we conclude that (ii) $\Leftrightarrow$ (iii). From Proposition 2.2 and [1, Corollary 2.5, Theorem 2.8], we deduce that (i) $\Rightarrow$ (ii).

Let us prove (ii) $\Rightarrow$ (i). From Alexander's duality theorem (see [2, (27.5)]) applied in the long exact sequence of the pair (V, D) (for cohomology), we obtain an exact sequence of **Q**-vector spaces:

$$
\begin{aligned}
&0 \longrightarrow H_4(X^{\mathrm{an}}, \mathbf{Q}) \longrightarrow H^0(V^{\mathrm{an}}, \mathbf{Q}) \longrightarrow H^0(D^{\mathrm{an}}, \mathbf{Q}) \longrightarrow \\
&\quad\longrightarrow H_3(X^{\mathrm{an}}, \mathbf{Q}) \longrightarrow H^1(V^{\mathrm{an}}, \mathbf{Q}) \longrightarrow H^1(D^{\mathrm{an}}, \mathbf{Q}) \longrightarrow \\
&\quad\longrightarrow H_2(X^{\mathrm{an}}, \mathbf{Q}) \longrightarrow H^2(V^{\mathrm{an}}, \mathbf{Q}) \longrightarrow H^2(D^{\mathrm{an}}, \mathbf{Q}) \longrightarrow \\
&\quad\longrightarrow H_1(X^{\mathrm{an}}, \mathbf{Q}) \longrightarrow H^3(V^{\mathrm{an}}, \mathbf{Q}) \longrightarrow 0 \longrightarrow \\
&\quad\longrightarrow H_0(X^{\mathrm{an}}, \mathbf{Q}) \longrightarrow H^4(V^{\mathrm{an}}, \mathbf{Q}) \longrightarrow 0
\end{aligned}
\tag{3.1}
$$

Since X is assumed to be affine and smooth, it follows, *e.g.*, from [7, Theorem 7.1], that $H_i(X^{\mathrm{an}}, \mathbf{Q}) = 0$ for $i \in \{3, 4\}$. So we only have to prove that $H_i(X^{\mathrm{an}}, \mathbf{Q}) = 0$ for $i \in \{1, 2\}$.

Since $n = b_2$ by Proposition 2.2, it follows from the analysis of diagram (3.1) and Lemma 2.1 that it is enough to prove that the morphism

$$H^2(V^{\mathrm{an}}, \mathbf{Q}) \longrightarrow H^2(D^{\mathrm{an}}, \mathbf{Q})$$

is injective or surjective. Then the next paragraph concludes the proof.

Let X be a smooth affine complex surface with $[X] = \mathbf{L}^2$. Let us prove that all the conditions in the family $(\star)$ are mutually equivalent. By [2, (23.13)], we conclude that $H_i(X^{\mathrm{an}}, \mathbf{Q}) = 0 \Leftrightarrow H^i(X^{\mathrm{an}}, \mathbf{Q}) = 0$. The conditions $H_1(X^{\mathrm{an}}, \mathbf{Q}) = 0$, $H_2(X^{\mathrm{an}}, \mathbf{Q}) = 0$ are equivalent, since by the arguments above they are both equivalent to assuming that X is a **Q**-homological plane. The equivalence $\mathrm{Pic}(X)_{\mathbf{Q}} = 0 \Leftrightarrow H^2(X^{\mathrm{an}}, \mathbf{Q}) = 0$ directly follows from the identification of the **Q**-vector spaces $\mathrm{Pic}(X)_{\mathbf{Q}} \cong H^2(X^{\mathrm{an}}, \mathbf{Q})$.

4 Further Comments

We recall that an *exotic* $\mathbf{C}^n$ is a smooth connected affine **C**-variety of dimension n, non-isomorphic to $\mathbf{A}^n_{\mathbf{C}}$, but diffeomorphic to $\mathbf{R}^{2n}$.

Question 1 *Does there exist a simple characterization of exotic $\mathbf{C}^n$ (especially for $n = 3$) in the Grothendieck ring of complex varieties $K_0(\mathrm{Var}_{\mathbf{C}})$?*

Example 4.1 Let us consider Russell's exotic $\mathbf{C}^3$. Precisely, let us consider the polynomial $x + x^2y + z^3 + t^2 \in k[x, y, z, t]$. This datum defines a smooth complex variety R of dimension 3, which is diffeomorphic to $\mathbf{R}^6$, but not isomorphic to $\mathbf{A}^3_{\mathbf{C}}$. (*e.g.*, see [3] and the references cited here for details and complements). Furthermore, an elementary computation in the ring $K_0(\mathrm{Var}_{\mathbf{C}})$ gives rise to the equality $[R] = \mathbf{L}^3$.

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