Magnitude and Holmes–Thompson intrinsic volumes of convex bodies

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Abstract. Magnitude is a numerical invariant of compact metric spaces, originally inspired by category theory and now known to be related to myriad other geometric quantities. Generalizing earlier results in ℓ_1^n and Euclidean space, we prove an upper bound for the magnitude of a convex body in a hypermetric normed space in terms of its Holmes–Thompson intrinsic volumes. As applications of this bound, we give short new proofs of Mahler's conjecture in the case of a zonoid and Sudakov's minoration inequality.

1 Introduction

Magnitude is a numerical isometric invariant of metric spaces recently introduced by Leinster [19] based on category-theoretic considerations. It has rapidly found connections with a large and growing number of areas of mathematics (see [22] for a survey as of 2017, Sections 6.4 and 6.5 of [20] for a more recent succinct account, and [21] for a more complete bibliography). In appropriate contexts, magnitude encodes a number of classical geometric quantities, including volume [4, 22, 39], Minkowski dimension [28], surface area [9], and other curvature integrals [9–11, 39].

The main result of this paper, Theorem 1.2, provides an upper bound on the magnitude of a convex body in a hypermetric normed space in terms of its Holmes–Thompson intrinsic volumes, generalizing the main result of [29] for convex bodies in Euclidean spaces. (All these terms will be defined in the following paragraphs.) In addition to generalizing some known results about magnitude from ℓ_1^n and Euclidean (or Hilbert) spaces to more general normed spaces, we will see that this upper bound on magnitude can be used to quickly deduce some important known results in convex geometry, namely Mahler's conjecture in the case of a zonoid and Sudakov's minoration inequality. Finally, the proof of Theorem 1.2 helps elucidate the relationship between Holmes–Thompson intrinsic volumes and Leinster's theory of ℓ_1 integral geometry [18], which was developed largely in order to state and prove the result from [22] on which Theorem 1.2 is based.

A metric space $X = (X, d)$ is called **positive definite**, if the matrix $\left[e^{-d(x_i, x_j)} \right]_{1 \leq i, j \leq k}$ is positive definite for every finite collection of distinct points $x_1, \ldots, x_k \in X$. If X is a compact positive definite metric space, the **magnitude** of X can be defined as

$$\text{(1.1)} \qquad \text{Mag}\,(X, d) = \sup \left\{ \frac{\mu(X)^2}{\int \int e^{-d(x, y)} \, d\mu(x) d\mu(y)} \ \middle| \ \mu \in M(X) \right\},$$

where $M(X)$ is the set of finite signed Borel measures on X [27]. It follows immediately from this definition that $\text{Mag}\,(X, d) \in [1, \infty]$, and that magnitude is monotone with respect to set containment for positive definite spaces.

It is a classical fact that for a normed space E, positive definiteness is equivalent both to the property of being isometrically isomorphic to a subspace of L_1, and being hypermetric (which for general metric spaces is a stronger property than positive definiteness; see, e.g., [7, Section 6.1]). We will refer to these spaces as hypermetric normed spaces below. Examples include $\ell_p^n = (\mathbb{R}^n, \|\cdot\|_p)$ for $1 \leq p \leq 2$, in particular ℓ_2^n, which is $\mathbb{R}^n$ with its usual Euclidean norm.

Let $\mathcal{K}^n$ be the class of compact, convex subsets of $\mathbb{R}^n$, equipped with the Hausdorff distance. Recall that a **convex valuation** on $\mathbb{R}^n$ is a function $V : \mathcal{K}^n \to \mathbb{R}$ such that

$$\text{(1.2)} \qquad V(K \cup L) = V(K) + V(L) - V(K \cap L),$$

whenever $K, L, K \cup L \subseteq \mathcal{K}^n$. A convex valuation V is said to be m-homogeneous for $m \in \mathbb{N}$ if $V(tK) = t^m V(K)$ for every $K \in \mathcal{K}^n$ and $t > 0$.

A consequence of Hadwiger's classical theorem (see, e.g., [16, 33, 34]) is that up to scalar multiples, there is a unique continuous, m-homogeneous, rigid motion-invariant convex valuation V_m on $\mathbb{R}^n$ for each $0 \leq m \leq n$. With an appropriate normalization $V_m(K) = \text{vol}_m(K)$ whenever $K \in \mathcal{K}^n$ is m-dimensional and V_m is then called the mth **intrinsic volume**. These quantities, under various normalization and indexing conventions, play a central role in integral geometry.

There are multiple natural choices for the normalization of the volume (i.e., Lebesgue measure) on a finite-dimensional normed space E. For the purposes of integral geometry, it turns out that the most convenient normalization is the **Holmes–Thompson volume** (see, e.g., pages 207–209 of [34] for discussion and references). If E is identified with $(\mathbb{R}^n, \|\cdot\|)$ and $\mathbb{R}^n$ is also given its usual Euclidean structure, then the **Holmes–Thompson volume** of $A \subseteq E$ is given, up to a factor depending only on n, by

$$\text{(1.3)} \qquad \text{vol}_{\text{HT}}^E(A) = \text{vol}_{2n}(A \times B^\circ),$$

where B is the unit ball of E, $B^\circ = \{ y \in \mathbb{R}^n \mid \langle x, y \rangle \leq 1 \text{ for every } x \in B \}$ is its polar body, and vol_{2n} is the standard Liouville volume on $\mathbb{R}^n \times (\mathbb{R}^n)^*$ (equal to the usual normalization of Lebesgue measure on $\mathbb{R}^{2n}$ under the standard identification $(\mathbb{R}^n)^* \cong \mathbb{R}^n$). The Holmes–Thompson volume is invariant under linear changes of coordinates, and thus independent of the precise identification of E with $\mathbb{R}^n$ or Euclidean structure. If $F \subseteq E$ is an affine subspace, then we further define $\text{vol}_{\text{HT}}^F(A) = \text{vol}_{\text{HT}}^{F_0}(A_0)$ for $A \subseteq F$, where F_0 is the linear subspace of E which is parallel to F and A_0 is a translate of A lying in F_0. (The definition can be further extended to Finsler manifolds, but we will not make use of that level of generality here.)

With the normalization defined by (1.3),

(1.4) $$\mathrm{vol}_{\mathrm{HT}}^{\ell_2^n}(A) = \mathrm{vol}_{2n}(A \times B_2^n) = \omega_n \, \mathrm{vol}_n(A)$$

and

(1.5) $$\mathrm{vol}_{\mathrm{HT}}^{\ell_1^n}(A) = \mathrm{vol}_{2n}(A \times B_\infty^n) = 2^n \, \mathrm{vol}_n(A).$$

Here and below, B_p^n denotes the unit ball of ℓ_p^n for $1 \le p \le \infty$ and $\omega_n = \mathrm{vol}_n(B_2^n)$. In contrast to this, it is standard practice to introduce a factor of ω_n^{-1} in the definition (1.3) of $\mathrm{vol}_{\mathrm{HT}}^E$ in order to force the equality $\mathrm{vol}_{\mathrm{HT}}^{\ell_2^n} = \mathrm{vol}_n$. This convention reflects the central role of the Euclidean space ℓ_2^n in integral geometry. However, in the theory of magnitude, ℓ_1^n plays a more central role, and the convention adopted above is more convenient for the statement and proof of our main result.

The following partial analog of Hadwiger's theorem for normed spaces is proved in [32, 35], although it is not given a self-contained statement.

Proposition 1.1   *Let $E = (\mathbb{R}^n, \|\cdot\|)$ be a finite-dimensional hypermetric normed space. For each $1 \le m \le n$, there is a unique even, continuous, m-homogeneous, translation-invariant convex valuation μ_m^E such that $\mu_m^E(K) = \mathrm{vol}_{\mathrm{HT}}^F(K)$ whenever $K \in \mathcal{K}^n$ is m-dimensional and $F \subseteq E$ is the affine subspace spanned by K.*

As discussed in the introduction of [8], results in [2, 5] imply that Proposition 1.1 holds for certain more general normed spaces. However, it is also shown in [32] that the valuations μ_m^E necessarily have some pathological properties (in particular, failure of monotonicity) if E is not hypermetric.

We set $\mu_0^E = 1$. The valuations μ_m^E for $0 \le m \le n$ are referred to as the **Holmes–Thompson intrinsic volumes** on E. Note that with our normalization convention, $\mu_m^{\ell_2^n} = \omega_m V_m$ by (1.4). We will denote by $\widetilde{\mu}_m^E = \omega_m^{-1} \mu_m^E$ the usual normalization of the Holmes–Thompson intrinsic volumes, so that $\widetilde{\mu}_m^{\ell_2^n} = V_m$.

We are now ready to state the main theorem of this paper.

Theorem 1.2   *Suppose that $E = (\mathbb{R}^n, \|\cdot\|)$ is a hypermetric normed space, and let $K \in \mathcal{K}^n$. Then,*

$$\mathrm{Mag}\,(K, \|\cdot\|) \le \sum_{m=0}^{n} 4^{-m} \mu_m^E(K) \le e^{\mu_1^E(K)/4}.$$

For reference, in terms of the the usual convention for Holmes–Thompson intrinsic volumes, the conclusion of Theorem 1.2 states that

$$\mathrm{Mag}\,(K, \|\cdot\|) \le \sum_{m=0}^{n} \frac{\omega_m}{4^m} \widetilde{\mu}_m^E(K) \le e^{\widetilde{\mu}_1^E(K)/2}.$$

Theorem 1.2 will be deduced from [22, Theorem 4.6], stated as Theorem 2.1, which is essentially the special case of Theorem 1.2 for $E = \ell_1^n$. The case of $E = \ell_2^n$ was previously deduced from Theorem 2.1 in [29].

In Sections 3 and 4, we will combine the upper bound from Theorem 1.2 with easy lower bounds on magnitude to obtain new proofs of results in convex geometry

which are not obviously related to magnitude. In the rest of this section, we will state some immediate consequences of Theorem 1.2 about magnitude itself, and in particular for the behavior of the **magnitude function** $t \mapsto \mathrm{Mag}\left(tK, \|\cdot\|\right)$ when $t \to 0$. In contrast to this, in Section 3, we will consider the limit of the magnitude function as $t \to \infty$, and in Section 4, we will use a specific finite value of t. This demonstrates that the upper bound in Theorem 1.2, although not necessarily sharp at any value of $t > 0$, is strong enough to yield useful consequences over the entire range of rescalings of K.

Our first consequence of Theorem 1.2 is already known. It follows by applying the theorem to the convex hull of X, using the monotonicity of magnitude.

Corollary 1.3   *If $E = \left(\mathbb{R}^n, \|\cdot\|\right)$ is a finite-dimensional hypermetric normed space and $X \subseteq E$ is compact, then $\mathrm{Mag}\left(X, \|\cdot\|\right) < \infty$, and*

$$\lim_{t \to 0^+} \mathrm{Mag}\left(tX, \|\cdot\|\right) = 1.$$

The finiteness statement in Corollary 1.3 was first proved in this generality in [22, Proposition 4.13] using Fourier analysis. In the recent paper [23], a different proof was given of the finiteness statement which also yielded the limit statement (called the **one-point property** in [23]). Like the proof of Theorem 1.2, the proof of Corollary 1.3 given in [23] is based on [22, Theorem 4.6].

Theorem 1.2 allows Corollary 1.3 to be generalized to certain infinite-dimensional sets. If $K \subseteq L_1$ is compact and convex, we define

$$\mu_1^{L_1}(K) = \sup \left\{ \mu_1^F\left(K \cap F\right) \,\middle|\, F \subseteq L_1 \text{ is a finite-dimensional affine subspace} \right\}.$$

This definition is natural for subsets of L_1 since the Holmes–Thompson intrinsic volumes are monotone with respect to set containment in hypermetric normed spaces (and in fact, only in the hypermetric case [32]). Together with the facts that magnitude is monotone and lower semicontinuous for compact positive definite spaces [27, Theorem 2.6], Theorem 1.2 yields the following.

Corollary 1.4   *If $X \subseteq L_1$ is compact and $\mu_1^{L_1}(\mathrm{conv}\, X) < \infty$, then $\mathrm{Mag}\left(X, \|\cdot\|_1\right) < \infty$, and*

$$\lim_{t \to 0^+} \mathrm{Mag}\left(tX, \|\cdot\|_1\right) = 1.$$

A version of Corollary 1.4 was proved in Corollaries 2 and 3 of [29] for subsets of a Hilbert space, where the relevant class of sets are known as GB-bodies. Since a separable Hilbert space embeds isometrically in L_1, Corollary 1.4 generalizes those results. The following conjecture is motivated by both Corollary 1.4 and the proof of [23, Theorem 2.1], which gives the first known example of a compact positive definite metric space with infinite magnitude.

Conjecture 1.5   *Suppose that $K \subseteq L_1$ is compact and convex. Then $\mathrm{Mag}\left(K, \|\cdot\|_1\right) < \infty$ if and only if $\mu_1^{L_1}(K) < \infty$.*

The one-point property can be sharpened to the following first-order bound on the magnitude function for small t, generalizing part of [29, Corollary 6] for the Euclidean case.

Corollary 1.6   *Suppose that $E = (\mathbb{R}^n, \|\cdot\|)$ is a hypermetric normed space, and let $K \in \mathcal{K}^n$. Then*

$$\limsup_{t \to 0^+} \frac{\mathrm{Mag}(tK, \|\cdot\|) - 1}{t} \le \frac{1}{4} \mu_1^E(K).$$

The following conjecture, which generalizes [29, Conjecture 5], posits that the upper bounds in Theorem 1.2 are sharp to first order for small convex sets.

Conjecture 1.7   *Suppose that $E = (\mathbb{R}^n, \|\cdot\|)$ is a hypermetric normed space, and let $K \in \mathcal{K}^n$. Then*

$$\lim_{t \to 0^+} \frac{\mathrm{Mag}(tK, \|\cdot\|) - 1}{t} = \frac{1}{4} \mu_1^E(K).$$

When $E = \ell_1^n$, Conjecture 1.7 holds whenever K has nonempty interior, by Theorem 2.1. When $E = \ell_2^n$ the conjecture holds if n is odd and $K = B_2^n$, by [29, Theorem 4]. In all other cases, the conjecture is open, although the results of [9] imply that if $E = \ell_2^n$, n is odd, and K has smooth boundary, then the limit exists.

The rest of this paper is organized as follows: in Section 2, we will prove Theorem 1.2. In Sections 3 and 4, we will see how Theorem 1.2 quickly yields, respectively, Mahler's conjecture for zonoids and Sudakov's minoration inequality. Finally, in Section 5, we will make some remarks about Leinster's ℓ_1 integral geometry, which underlies Theorem 2.1, and its relationships to both the theory of Holmes–Thompson intrinsic volumes and the Wills functional.

2   Proof of Theorem 1.2

To state the theorem on which the proof of Theorem 1.2 is based, we first define some additional notation. Following [18], for $0 \le m \le n$, we define the **ℓ_1 intrinsic volumes** of $K \in \mathcal{K}^n$ by

$$V'_m(K) = \sum_{P \in \mathrm{Gr}'_{n,m}} \mathrm{vol}_m(K|P),$$

where $\mathrm{Gr}'_{n,m}$ denotes the set of m-dimensional coordinate subspaces of $\mathbb{R}^n$ and $K|P$ denotes the orthogonal projection of K onto P. (The natural class of sets to consider here is actually larger than convex bodies, a point that we will return to in Section 5.) Note that if K lies in a d-dimensional subspace of ℓ_1^n, then $V'_m(K) = 0$ for $m > d$. It follows that

$$V'_m(K) = \frac{1}{m!} \sum_{i_1,\ldots,i_m=1}^{n} \mathrm{vol}_m\big(P_{i_1,\ldots,i_m}(K)\big),$$

where $P_{i_1,\ldots,i_m} : \mathbb{R}^n \to \mathbb{R}^m$ is the linear map represented by the matrix whose rows are the standard basis vectors $e_{i_1}, \ldots, e_{i_m} \in \mathbb{R}^n$.

The following result is part of Theorem 4.6 of [22].

Theorem 2.1 *If $K \in \mathcal{K}^n$, then*

$$\text{Mag}\,(K, \|\cdot\|_1) \le \sum_{m=0}^{n} 2^{-m} V_m'(K),$$

with equality when K has nonempty interior.

To deduce Theorem 1.2 from Theorem 2.1, we approximate a hypermetric normed space E by a sequence of n-dimensional subspaces $E_N \subseteq \ell_1^N$. To do this, we use the following fact, which goes back to Lévy (see, e.g., [17, Section 6.1]).

Proposition 2.2 *A finite-dimensional normed space $E = (\mathbb{R}^n, \|\cdot\|)$ is hypermetric if and only if, there exists an even nonnegative measure ρ on S^{n-1} such that*

$$\|x\| = \int |\langle x, y\rangle|\; d\rho(y),$$

for all $x \in \mathbb{R}^n$.

From the perspective of convex geometry, Proposition 2.2 implies that E is hypermetric if and only if B°, the polar body of the unit ball of E, is a zonoid (see [33, Theorem 3.5.3]). In that setting ρ is referred to as the **generating measure** of B°; we will also refer to it as the generating measure for E. In [35], Schneider and Wieacker investigated Holmes–Thompson intrinsic volumes for hypermetric normed spaces with the help of generating measures. We will use the following expression they derived (see [35, formula (64)]).

Proposition 2.3 *Suppose that $E = (\mathbb{R}^n, \|\cdot\|)$ is a hypermetric normed space with generating measure ρ. Then*

$$\mu_m^E(K) = c_m \int_{(S^{n-1})^m} \text{vol}_m\big(K\,|\,\text{lin}(x_1, \ldots, x_m)\big)\sqrt{\det(AA^t)}\; d\rho(x_1)\cdots d\rho(x_m)$$

$$= c_m \int_{(S^{n-1})^m} \text{vol}_m(AK)\; d\rho(x_1)\cdots d\rho(x_m),$$

where c_m depends only on m. Here, $\text{lin}(x_1, \ldots x_n)$ denotes the linear span of $x_1, \ldots, x_m \in \mathbb{R}^n$ and A is the $m \times n$ matrix with rows $x_1, \ldots, x_m$.

Since we are using a different normalization convention than [35], the value of c_m here differs from the one stated in [35]. The proof of the following corollary shows that for our normalization, $c_m = \frac{2^m}{m!}$.

Corollary 2.4 *For $K \in \mathcal{K}^n$, $\mu_m^{\ell_1^n}(K) = 2^m V_m'(K)$.*

Proof The generating measure for ℓ_1^n is $\rho = \frac{1}{2}\sum_{i=1}^{n}(\delta_{e_i} + \delta_{-e_i})$. If $x_1, \ldots, x_m \in \{\pm e_1, \ldots, \pm e_n\}$, then $\sqrt{\det(AA^t)} = 1$ if $\text{lin}(x_1, \ldots, x_m)$ is m-dimensional, and is 0 otherwise. Proposition 2.3 therefore implies that

$$\mu_m^{\ell_1^n}(K) = c_m \int_{(S^{n-1})^m} \mathrm{vol}_m\big(K|\mathrm{lin}(x_1,\ldots,x_m)\big)\sqrt{\det(AA^t)}\, d\rho(x_1)\cdots d\rho(x_m)$$

$$= 2^{-m} c_m \sum_{j_1,\ldots,j_m=1}^{n} \sum_{\varepsilon_1,\ldots,\varepsilon_m \in \{1,-1\}} \mathrm{vol}_m\big(K|\mathrm{lin}(\varepsilon_1 e_{j_1},\ldots,\varepsilon_m e_{j_m})\big)$$

$$= c_m \sum_{j_1,\ldots,j_m=1}^{n} \mathrm{vol}_m\big(K|\mathrm{lin}(e_{j_1},\ldots,e_{j_m})\big)$$

$$= m! c_m V_m'(K).$$

Now, if $E \subseteq \ell_1^n$ is an m-dimensional coordinate subspace, then E is isometrically isomorphic to ℓ_1^m, and when $K \subseteq E$ we have

$$\mu_m^{\ell_1^n}(K) = \mathrm{vol}_{\mathrm{HT}}^E(K) = 2^m\, \mathrm{vol}_m(K) = 2^m V_m'(K)$$

by (1.5) and [18, Lemma 5.2], and so $c_m = \frac{2^m}{m!}$.   ∎

Corollary 2.4 shows in particular that when $E = \ell_1^n$, the first inequality in Theorem 1.2 reduces to Theorem 2.1. It also implies the following generalization of [18, Lemma 5.2].

Corollary 2.5   *If $K \in \mathcal{K}^n$ lies in an m-dimensional subspace $E \subseteq \mathbb{R}^n$, then*

$$V_m'(K) = \frac{\mathrm{vol}_m\big((B_1^n \cap E)^\circ\big)}{2^m}\,\mathrm{vol}_m(K),$$

where the polar body $(B_1^n \cap E)^\circ$ is considered in the subspace E.

For the second inequality in Theorem 1.2, we will need the following generalization of [23, Lemma 3.2].

Lemma 2.6   *If $E = (\mathbb{R}^n, \|\cdot\|)$ is a hypermetric normed space and $K \in \mathcal{K}^n$, then $\mu_{i+j}^E(K) \le \frac{i!j!}{(i+j)!}\mu_i^E(K)\mu_j^E(K)$ for all $i,j \ge 0$. Consequently, $\mu_m^E(K) \le \frac{1}{m!}(\mu_1^E(K))^m$ for each $1 \le m \le n$.*

Proof   Let $x_1,\ldots,x_{i+j} \in \mathbb{R}^n$. Writing A for the matrix with rows $x_1,\ldots,x_{i+j}$, A_1 for the matrix with rows $x_1,\ldots,x_i$, and A_2 for the matrix with rows $x_{i+1},\ldots,x_j$, we have $AK \subseteq A_1K \times A_2K$. Proposition 2.3 then implies that

$$\mu_{i+j}^E(K) \le \frac{2^{i+j}}{(i+j)!} \int_{(S^{n-1})^{i+j}} \mathrm{vol}_i(A_1K)\,\mathrm{vol}_j(A_2K)\, d\rho(x_1)\cdots d\rho(x_{i+j})$$

$$= \frac{i!j!}{(i+j)!}\mu_i^E(K)\mu_j^E(K).$$

This implies in particular that $\mu_{j+1}^E(K) \le \frac{1}{j+1}\mu_1^E(K)\mu_j^E(K)$ for each j, and the second claim now follows by induction.   ∎

We are now ready to prove the main result.

Proof of Theorem 1.2   Let ρ be the generating measure for E. We can approximate ρ by a sequence of discrete measures

$$\rho_N = \sum_{j=1}^{N} w_{N,j} \delta_{\theta_{N,j}}$$

with $w_{N,j} > 0$ and $\theta_{N,j} \in S^{n-1}$. For each N, we get a seminorm, which for sufficiently large $N \geq n$ will be a norm, given by

$$\|x\|_{E_N} = \int_{S^{n-1}} |\langle x, y \rangle| \, d\rho_N(y) = \sum_{j=1}^{m} w_{N,j} |\langle x, \theta_{N,j} \rangle| = \sum_{j=1}^{N} |\langle x, w_{N,j}\theta_{N,j} \rangle|.$$

We write $E_N = (\mathbb{R}^n, \|\cdot\|_{E_N})$. Define $T_N : \mathbb{R}^N \to \mathbb{R}^n$ by $T_N(e_j) = w_{N,j}\theta_{N,j}$. Then

$$\|T_N^*(x)\|_1 = \sum_{j=1}^{N} |\langle T_N^*(x), e_j \rangle| = \sum_{j=1}^{N} |\langle x, T_N(e_j) \rangle| = \|x\|_{E_N},$$

so T_N^* is an isometric embedding of E_N into ℓ_1^N.

We have

$$\|x\|_{E_N} = \int_{S^{n-1}} |\langle x, y \rangle| \, d\rho_N(y) \xrightarrow{N\to\infty} \int_{S^{n-1}} |\langle x, y \rangle| \, d\rho(y) = \|x\|.$$

This implies that $(K, \|\cdot\|_{E_N}) \to (K, \|\cdot\|)$ in the Gromov–Hausdorff metric (see [13, Section 3.A]). Since magnitude is lower semicontinuous with respect to the Gromov–Hausdorff metric on the class of compact positive definite metric spaces [27, Theorem 2.6], it follows that

(2.1) $\mathrm{Mag}\,(K, \|\cdot\|) \leq \liminf_{N\to\infty} \mathrm{Mag}\,(K, \|\cdot\|_{E_N}) = \liminf_{N\to\infty} \mathrm{Mag}\,(T_N^*(K), \|\cdot\|_1).$

Now,

(2.2)
$$
\begin{aligned}
V_m'(T_N^*(K)) &= \frac{1}{m!} \sum_{i_1,\ldots,i_m=1}^{N} \mathrm{vol}_m\big(P_{i_1,\ldots,i_m} T_N^*(K)\big) \\
&= \frac{1}{m!} \sum_{i_1,\ldots,i_m=1}^{N} w_{N,i_1}\cdots w_{N,i_N} \, \mathrm{vol}_m\left(\begin{bmatrix}\theta_{N,i_1} \\ \vdots \\ \theta_{N,i_m}\end{bmatrix} K\right) \\
&= \frac{1}{m!} \int \cdots \int \mathrm{vol}_m(AK) \, d\rho_N(x_1)\cdots d\rho_N(x_m) \\
&\xrightarrow{N\to\infty} \frac{1}{m!} \int \cdots \int \mathrm{vol}_m(AK) \, d\rho(x_1)\cdots d\rho(x_m) \\
&= 2^{-m} \mu_m^E(K),
\end{aligned}
$$

where the last equality follows from Proposition 2.3. Here, A as before stands for the matrix with rows $x_1, \ldots, x_m$, and the matrix in the second row has rows $\theta_{N,i_1}, \ldots, \theta_{N,i_m}$. The first inequality in Theorem 1.2 now follows by combining (2.1), Theorem 2.1, and (2.2). The second inequality then follows by Lemma 2.6.   ∎

3 Application: Mahler's conjecture for zonoids

In this and the following section, we will see how two important results in convex geometry which make no reference to magnitude quickly follow by combining the upper bound on magnitude from Theorem 1.2 with easy lower bounds.

Mahler conjectured in 1939 [25] that if $K \in \mathcal{K}^n$ is symmetric with nonempty interior, then

$$\text{vol}_n(K)\,\text{vol}_n(K^\circ) \geq \frac{4^n}{n!}.$$

Equality is attained (nonuniquely) for $K = B_1^n$ or $K = B_\infty^n$. This has been proved in various special cases and in asymptotic forms (see [15] for a proof when $n = 3$ and further references), but the general case remains open.

In the proof of the following result, we compare the top-order behavior of the first upper bound on magnitude in Theorem 1.2, which is typically not sharp for large convex bodies, with a lower bound that is known to be asymptotically sharp. This comparison immediately implies Mahler's conjecture for zonoids, first proved in [30, 31] (see also [12]).

Corollary 3.1 *If $Z \in \mathcal{K}^n$ is an n-dimensional zonoid, then*

$$\text{vol}_n(Z)\,\text{vol}_n(Z^\circ) \geq \frac{4^n}{n!}.$$

Proof Let $E = (\mathbb{R}^n, \|\cdot\|)$ be the hypermetric normed space with unit ball $B = Z^\circ$. Theorem 1.2 implies that for $t \to \infty$,

$$\text{Mag}(tB, \|\cdot\|) \leq 4^{-n}\mu_n^E(B)t^n + O(t^{n-1}) = 4^{-n}\,\text{vol}_n(B)\,\text{vol}_n(B^\circ)t^n + O(t^{n-1}).$$

On the other hand, for each $t > 0$, we have the lower bound

$$\text{Mag}(tB, \|\cdot\|) \geq \frac{t^n}{n!}$$

(see [19, Theorem 3.5.6] or [22, Proposition 4.13]). Combining these, we obtain

$$\frac{4^n}{n!} \leq \text{vol}_n(B)\,\text{vol}_n(B^\circ) + O(t^{-1}) = \text{vol}_n(Z)\,\text{vol}_n(Z^\circ) + O(t^{-1}),$$

and letting $t \to \infty$ proves the claim. ∎

Remark Although we considered $\text{Mag}(tB, \|\cdot\|)$ in the above proof for convenience, we could equally well consider $\text{Mag}(tK, \|\cdot\|)$ for any n-dimensional convex body K (with the norm still corresponding to $B = Z^\circ$) and obtain the same result.

4 Application: Sudakov minoration

Our last application of Theorem 1.2 is a new proof of Sudakov's minoration inequality, which is a key tool in both high-dimensional convex geometry and the theory of Gaussian processes (see, e.g., [3, 37], respectively). This application uses only the special case of Theorem 1.2 when $E = \ell_2^n$, proved previously in [29]. In that case, the

first upper bound in Theorem 1.2 can be combined with a stronger counterpart to Lemma 2.6 in Euclidean space to deduce the following sharper version of the second upper bound.

Corollary 4.1   *If $K \in \mathcal{K}^n$, then*

$$\mathrm{Mag}\,(K, \|\cdot\|_2) \leq e^{C V_1(K)^{2/3}}.$$

Throughout this section, c, C, and C' refer to absolute positive constants whose values may differ from one instance to another.

Remark   Corollary 4.1 can be extended to infinite-dimensional Hilbert spaces, similarly to Corollary 1.4, but for simplicity, we will restrict attention to finite dimensions in this section.

Proof of Corollary 4.1   It was independently shown by Chevet [6, Lemma 4.2] and McMullen [26, Theorem 2] that the Alexandrov–Fenchel inequalities imply that $V_m \leq \frac{1}{m!} V_1^m$ for every $m \geq 1$. As observed in formula (17) of [29], Theorem 1.2 then implies that

$$\mathrm{Mag}\,(K, \|\cdot\|_2) \leq \sum_{m=0}^n \frac{\omega_m}{m!} \left(\frac{V_1(K)}{4} \right)^m = \sum_{m=0}^n \frac{1}{\Gamma\!\left(1 + \frac{m}{2}\right) m!} \left(\frac{\sqrt{\pi} V_1(K)}{4} \right)^m.$$

We now consider the function

$$f(x) = \sum_{m=0}^\infty \frac{x^m}{\Gamma\!\left(1 + \frac{m}{2}\right) m!},$$

a special case of Wright's generalized hypergeometric function. We claim that $f(x) \leq e^{c x^{2/3}}$ for $x > 0$, which will complete the proof.

If $x \leq 1$, then since $\Gamma\!\left(1 + \frac{m}{2}\right) \geq \frac{\sqrt{\pi}}{2}$ for every $m \geq 0$, we have

$$f(x) \leq \exp\left(\frac{2}{\sqrt{\pi}} x \right) \leq \exp\left(\frac{2}{\sqrt{\pi}} x^{2/3} \right).$$

For $x \geq 1$, Stirling's approximation implies that

$$f(x) = 1 + \sum_{k=1}^\infty \frac{x^{2k}}{(2k)! k!} + \sum_{k=1}^\infty \frac{x^{2k-1}}{(2k-1)! \Gamma\!\left(1 + \frac{2k-1}{2}\right)} \leq 1 + \sum_{k=1}^\infty \frac{c^k x^{2k}}{(3k)!}$$

$$= 1 + \sum_{k=1}^\infty \frac{(c^{1/3} x^{2/3})^{3k}}{(3k)!} \leq e^{c^{1/3} x^{2/3}}. \qquad \blacksquare$$

Since V_1 is 1-homogeneous, Corollary 4.1 is equivalent to the following.

Corollary 4.2   *If $K \in \mathcal{K}^n$, then*

$$V_1(K) \geq C \sup_{t > 0} t^{-1} \left(\log \mathrm{Mag}\,(tK, \|\cdot\|_2) \right)^{3/2}.$$

Corollary 4.3 (Sudakov's minoration inequality)   *Let $K \in \mathcal{K}^n$, and suppose there exist* $x_1, \dots, x_N \in K$ *such that* $\|x_i - x_j\|_2 \geq \varepsilon$ *whenever $i \neq j$. Then*

$$V_1(K) \geq C\varepsilon\sqrt{\log N}.$$

Proof   Assume without loss of generality that $N \geq 2$, and let $\mu = \sum_{i=1}^N \delta_{x_i}$. By (1.1),

$$\mathrm{Mag}\,(tK, \|\cdot\|_2) \geq \frac{N^2}{\int_K \int_K e^{-t\|x-y\|}\,d\mu(x)d\mu(y)} \geq \frac{N}{1 + Ne^{-t\varepsilon}}.$$

If $t = \frac{\log(2N)}{\varepsilon}$, this implies that $\mathrm{Mag}\,(tK, \|\cdot\|_2) \geq \frac{2}{3}N$, and so Corollary 4.2 implies that

$$V_1(K) \geq C\varepsilon\frac{\left[\log\left(\frac{2}{3}N\right)\right]^{3/2}}{\log(2N)} \geq C'\varepsilon\sqrt{\log N}.\qquad\blacksquare$$

Remark   If the supremum in our definition (1.1) of magnitude is restricted to positive measures μ, we obtain a quantity called the **maximum diversity** of (X, d), denoted $D_{\max}(X, d)$ (see [24, 27]). The above proof of Corollary 4.3 shows that

$$\sup_{t>0} t^{-1}\bigl(\log D_{\max}(tK, \|\cdot\|_2)\bigr)^{3/2} \geq C \sup_{\varepsilon>0} \varepsilon\sqrt{\log N(K, \varepsilon)},$$

where $N(K, \varepsilon)$ is the maximum size of a collection of ε-separated points in K. It can similarly be shown that

$$\sup_{t>0} t^{-1}\bigl(\log D_{\max}(tK, \|\cdot\|_2)\bigr)^{3/2} \leq C' \sup_{\varepsilon>0} \varepsilon\sqrt{\log N(K, \varepsilon)}$$

(cf. the proof of [28, Theorem 7.1]). It follows that Sudakov's minoration inequality is equivalent, up to the value of the constant C, to the inequality

$$D_{\max}(K, \|\cdot\|_2) \leq e^{CV_1(K)^{2/3}},$$

a weaker counterpart of Corollary 4.1.

This observation suggests trying to prove sharper lower bounds on $V_1(K)$ than provided by Sudakov's inequality by using Corollary 4.1 and bounding $\mathrm{Mag}\,(K, \|\cdot\|_2)$ from below by leveraging the fact that the supremum in (1.1) is over a space of *signed* measures. We recall that optimal lower bounds on $V_1(K)$ are given by Talagrand's celebrated majorizing measure theorem [36] and its more recent reformulations [37], but those bounds are not easy to apply in practice (see, e.g., [38] for discussion of this). In general, the supremum in (1.1) is not achieved even in the space of signed measures, but the definition of magnitude can be reformulated in several ways that invite consideration from the perspective of distributions and partial differential equations [28]. This perspective has led to the sharpest known results on magnitude in Euclidean spaces [4, 9–11, 40, 41], and may be similarly fruitful in this setting.

5   Some remarks on ℓ_1 integral geometry

The Holmes–Thompson intrinsic volumes were introduced in order to find natural generalization of results from integral geometry in Euclidean spaces to more general

normed spaces (or still more generally, Finsler manifolds). In particular, Schneider and Wieacker [35] showed that in hypermetric normed spaces, the Holmes–Thompson intrinsic volumes satisfy versions of the classical Crofton formula (see [2, 5] for versions in more general settings).

In [18], Leinster similarly proved a suite of results involving the ℓ_1 intrinsic volumes which are counterparts of classical integral geometric theorems. Since Corollary 2.4 shows that up to scaling, Leinster's ℓ_1 intrinsic volumes V'_m applied to convex sets are precisely the Holmes–Thompson intrinsic volumes for ℓ_1^n, one might guess that Leinster's theory is subsumed by Holmes–Thompson integral geometry. However, there are at least two major parts of Euclidean integral geometry for which Leinster proved ℓ_1 analogs in [18], but for which no general Holmes–Thompson version is known.

The first is Hadwiger's theorem (see, e.g., [16, 33, 34]), which states that every continuous, rigid motion-invariant convex valuation on ℓ_2^n is a linear combination of the Euclidean intrinsic volumes. In general normed spaces, Proposition 1.1 classifies only homogeneous valuations with a normalization condition that serves as a proxy for rigid motion-invariance, whereas Hadwiger's theorem also implies that invariant convex valuations are linear combinations of these homogeneous valuations. In ℓ_1^n, Leinster proved an exact analog of Hadwiger's theorem assuming invariance only under the isometry group for the ℓ_1^n norm [18, Theorem 5.4]. To compensate for this smaller isometry group, Leinster assumes the valuations are defined and satisfy (1.2) on the larger class of $\boldsymbol{\ell_1}$-**convex** sets, i.e., sets that are geodesic with respect to the ℓ_1^n metric. (Indeed, the fact that Leinster's ℓ_1 intrinsic volumes satisfy (1.2) for all ℓ_1-convex sets is crucial to the proof of Theorem 2.1, even when that theorem is restricted to convex sets.) This suggests the possibility of stronger Hadwiger-like theorems in normed spaces than Proposition 1.1 for valuations with suitably chosen domains. As discussed in [18], however, the most naive generalization of the ℓ_1 and Euclidean versions of Hadwiger's theorem is typically false.

Second, in [18, Theorem 6.2] Leinster proved the following ℓ_1 version of Steiner's formula (see, e.g., [33, equation (4.1)]): if $X \subseteq \mathbb{R}^n$ is ℓ_1-convex, then

$$(5.1) \qquad \mathrm{vol}_n\big(X + t[0,1]^n\big) = \sum_{m=0}^{n} V'_m(X) t^{n-m}.$$

This formula implies in particular that the ℓ_1 intrinsic volumes, like the Euclidean intrinsic volumes, are particular instances of mixed volumes [33, Section 5.1]. Holmes–Thompson intrinsic volumes are not known to have representations as mixed volumes in general; furthermore, a Steiner-like formula such as (5.1), which would require the intrinsic volumes on the right-hand side to be mixed volumes of a particularly simple form, can only hold under additional restrictions on the normed space E. See [32, Section 5] for some partial results and discussion of these issues.

We end with a simple observation related to (5.1). As noted in [23], the quantity

$$\mathcal{W}'(X) = \sum_{m=0}^{n} V'_m(X)$$

is an ℓ_1 analog of the **Wills functional** (see, e.g., [1]), which can be defined by

$$\mathcal{W}(K) = \sum_{m=0}^{n} V_m(K).$$

The Wills functional was introduced in [42], where it was conjectured that

$$\#(K \cap \mathbb{Z}^n) \leq \mathcal{W}(K)$$

for any $K \in \mathcal{K}^n$. This was shown by Hadwiger [14] to be false for sufficiently large n. However, (5.1) implies that an ℓ_1 version of this conjecture is true in all dimensions.

Proposition 5.1 *Suppose that $X \subseteq \mathbb{R}^n$ is compact and ℓ_1-convex. Then*

$$\#(X \cap \mathbb{Z}^n) \leq \mathcal{W}'(X).$$

Proof By (5.1),

$$\#(X \cap \mathbb{Z}^n) = \mathrm{vol}_n\big((X \cap \mathbb{Z}^n) + [0,1]^n\big) \leq \mathrm{vol}_n\big(X + [0,1]^n\big) = \mathcal{W}'(X). \qquad \blacksquare$$

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