ON REDUCIBILITY OF ULTRAMETRIC ALMOST PERIODIC LINEAR REPRESENTATIONS

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Abstract. Let G be a group and K be a complete ultrametric valued field. Let $AP(G,K)$ be the algebra of the generalized almost periodic functions of G in K. We have shown in a previous paper that when $AP(G,K)$ has an invariant mean, then any almost periodic linear representation is quasi-reducible. Here, we show that with the same hypothesis, any topologically irreducible almost periodic linear representation is finite dimensional; also, any almost periodic linear representation is the topological sum of irreducible representations. Furthermore, we obtain a Peter-Weyl theorem for the algebra $AP(G,K)$.

We use the technical tools of Hopf algebra theory.

I. Notations and definitions.

I.1. Almost periodic functions; almost periodic linear representations. Let G be a group and K be a complete ultrametric valued field. The Banach algebra of bounded functions $f:G\to K$ with the supremum norm $\|f\| = \sup_{s\in G}|f(s)|$ is denoted by $\mathscr{B}(G,K)$. If $f\in\mathscr{B}(G,K)$ we write $\gamma_s f(t) = f(s^{-1}t)$, $\delta_s f(t) = f(ts)$ and $\eta(f)(s) = f(s^{-1})$ the left (resp. right) translation operator and the inversion operator.

Let us recall the extension of the notion of almost periodic functions given by Schikhof [8], [9]. A function $f\in\mathscr{B}(G,K)$ is called *almost periodic* if the set $\Gamma_f = \{\gamma_s f, s\in G\}$ is a *compactoid* of $\mathscr{B}(G,K)$: that is for $\varepsilon>0$, there exist $f_1,\ldots,f_n$ in $\mathscr{B}(G,K)$ and if $s\in G$, there exist $\lambda_1,\ldots,\lambda_n\in K$, $|\lambda_j|\le 1$, such that $\left\|\gamma_s f - \sum_{j=1}^{n}\lambda_j f_j\right\| < \varepsilon$. The space $AP(G,K)$ of almost periodic functions is a closed subalgebra of $\mathscr{B}(G,K)$ and is invariant with respect to the left (right) translation and the inversion.

If E and F are ultrametric Banach spaces over K, we denote by $E\hat{\otimes}F$ the complete tensor product; that is the completion of $E\otimes F$ with respect to the norm $\|z\| = \operatorname{Inf}_{z=\sum x_j\otimes y_j}\left(\max_j \|x_j\|\,\|y_j\|\right)$. In the sequel, all Banach spaces are ultrametric.

One defines as above, the space $AP(G,E)$ of almost periodic functions of G with values in the Banach space E. Furthermore, $AP(G,K)\hat{\otimes}E$ is isometrically isomorphic to $AP(G,E)$ via the linear map Π_E defined by $\Pi_E(f\otimes x)(s) = f(s)\cdot x$ (cf. [3]).

We say that a linear representation $U:G\to\mathscr{L}(E)$ is *almost periodic* if

(i) $\sup_{s\in G}\|U_s\| < +\infty$,

(ii) for any $x\in E$, the function $T_x:G\to E$ defined by $T_x(s) = U_s(x)$ is almost periodic.

I.2. Complete Hopf algebras: Banach comodules. Let (H,m,c,η,σ) be a complete ultrametric Hopf algebra over K, e the unit of H and k the canonical map of K in H. In other words, H is a Banach algebra with multiplication $m:H\hat{\otimes}H\to H$; coproduct

$c: H \hat{\otimes} H \to H$ a continuous algebra homomorphism; inversion or antipode $\eta: H \to H$ a continuous linear map and the counit $\sigma: H \to K$ a continuous algebra homomorphism. The coassociativity and counitary axioms hold, and $m \circ (1_H \otimes \eta) \circ c = k \circ \sigma = m \circ (\eta \otimes 1_H) \circ c$.

One sees that η is an anti-endomorphism of the algebra (resp. coalgebra) H.

EXAMPLE. The algebra $AP(G, K)$ is a complete Hopf algebra with coproduct c such that $\Pi \circ c(f)\ (s, t) = f(st)$; inversion η defined by $\eta(f)\ (s) = f(s^{-1})$ and counit σ defined by $\sigma(f) = f(e)$, where e is the neutral element of G. In fact, $AP(G, K)$ is a complete dual Hopf algebra (cf. [**2**]). □

Let H' be the Banach space dual of H; if we set for μ, $\nu \in H'$, $\mu * \nu = (\mu \otimes \nu) \circ c$, then H' becomes a complete normed algebra with unit σ.

A Banach space E is said a *left Banach H-comodule* if there exists a continuous linear map $\Delta: E \to H \hat{\otimes} E$, called a coproduct, such that $(c \otimes 1_E) \circ \Delta = (1_H \otimes \Delta) \circ \Delta$ and $(\sigma \otimes 1_E) \circ \Delta = 1_E$. A closed linear subspace M of E is a Banach *subcomodule* if $\Delta(M) \subset H \hat{\otimes} M$.

Notice that $\|x\| \le \|\Delta(x)\| \le \|\Delta\|\ \|x\|$, $x \in E$.

EXAMPLE. Let E be a left Banach $AP(G, K)$-comodule of coproduct Δ. If we set $\varepsilon_s(f) = f(s)$ the evaluation map at $s \in G$, then $U_s^{\Delta} = (\varepsilon_{s^{-1}} \otimes 1_E) \circ \Delta$ defines an almost periodic (a.p.) linear representation of G in E. Conversely, let $U: G \to \mathscr{L}(E)$ be an a.p. linear representation. If d_U denotes the linear map of E in $AP(G, E)$ defined by $d_U(x)(s) = U_{s^{-1}}(x)$, then $\Delta_U = \Pi_E^{-1} \circ d_U$ is a coproduct of E and E is a left Banach $AP(G, K)$-comodule. These correspondences are reciprocical (cf. [**3**]). □

If E is a left Banach H-comodule, and we set $\mu \cdot x = (\mu \otimes 1_E) \circ \Delta(x)$, for $\mu \in H'$, $x \in E$, one induces on E a complete normed *right H'-module* structure.

Let us recall that a Banach space V over K is *pseudo-reflexive* if the canonical map of V into its bidual space V'' is isometric. It is well known that any linear subspace of a pseudo-reflexive space is pseudo-reflexive.

Also, any Banach space which is a dual space is pseudo-reflexive. It follows that $\mathscr{B}(G, K)$, $AP(G, K)$ and its linear subspaces are pseudo-reflexive. Furthermore let D be a finite dimensional subspace of the pseudo-reflexive space V and $0 < \alpha < 1$; then for every $d' \in D'$ there exists $v' \in V'$ such that $v'_{|D} = d'$ and $\|v'\| \le \dfrac{1}{\alpha} \|d'\|$ (cf. [**5**] or [**7**]).

THEOREM 1. *Let H be a complete ultrametric Hopf algebra that is a pseudo-reflexive Banach space and let E be a left Banach H-comodule of coproduct Δ.*

(i) *A closed linear subspace M of E is a left Banach H-subcomodule of E if and only if M is a complete right H'-submodule of E.*

(ii) *Let $x \in E$; the closure M_x of $H' \cdot x$ in E is a Banach H-subcomodule of E that contains x and is a Banach space of countable type.*

Proof. (i) If M is a left Banach subcomodule of E, then for $x \in M$, $\Delta(x) \in H \hat{\otimes} M$, and if $\mu \in H'$, then $\mu \cdot x = (\mu \otimes 1_E) \circ \Delta(x) \in K \hat{\otimes} M = M$.

On the other hand, if M is a complete right H'-submodule of E, then $\mu \cdot x =$

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$(\mu \otimes 1_E) \circ \Delta(x) \in M$ for all $\mu \in H'$ and $x \in M$. Since $\Delta(x) \in H \hat{\otimes} E$, we can write $\Delta(x) = \sum_{j \geq 1} a_j \otimes x_j$ where $(a_j)_{j \geq 1}$ is an α-orthogonal set of H, $(x_j)_{j \geq 1} \subset E$ and $\alpha \sup_{j \geq 1} \|a_j\| \, \|x_j\| \leq \|\Delta(x)\| \leq \sup_{j \geq 1} \|a_j\| \, \|x_j\|$. Hence $\mu \cdot x = \sum_{j \geq 1} \langle \mu, a_j \rangle x_j \in M$ for all $\mu \in H'$.

Let ℓ be an integer ≥ 1; for $n \geq l+1$ the subspace of H of dimension n, $H_n = \bigoplus_{j=1}^{n} K a_j$ contains a_ℓ. Let $a'_{n\ell}$ be the linear form on H_n defined by $\langle a'_{n\ell}, a_j \rangle = \delta_{\ell j}$, $1 \leq j \leq n$; then $\frac{1}{\|a_\ell\|} \leq \|a'_{n\ell}\| \leq \frac{1}{\alpha \, \|a_\ell\|}$. Since H is a pseudo-reflexive Banach space, there exists $\mu_{n\ell} \in H'$ such that the restriction of $\mu_{n\ell}$ to H_n is $a'_{n\ell}$ and $\|\mu_{n\ell}\| \leq \frac{1}{\alpha} \|a'_{n\ell}\| \leq \frac{1}{\alpha^2} \frac{1}{\|a_\ell\|}$. Therefore, for every $n \geq \ell + 1$, $\mu_{n\ell} \cdot x = \sum_{j \geq 1} \langle \mu_{n\ell}, a_j \rangle x_j = x_\ell + \sum_{j \geq n+1} \langle \mu_{n\ell}, a_j \rangle x_j \in M$. However

$$\left\| \sum_{j \geq n+1} \langle \mu_{n\ell}, a_j \rangle x_j \right\| \leq \sup_{j \geq n+1} \|\mu_{n\ell}\| \, \|a_j\| \, \|x_j\| \leq \frac{1}{\alpha^2} \frac{1}{\|a_\ell\|} \sup_{j \geq n+1} \|a_j\| \, \|x_j\|$$

and $\lim_{n \to +\infty} \sup_{j \geq n+1} \|a_j\| \, \|x_j\| = 0$. It follows that $x_\ell = \lim_{n \to +\infty} \mu_{n\ell} \cdot x \in M$ and $\Delta(x) = \sum_{l \geq 1} a_\ell \otimes x_\ell \in H \hat{\otimes} M$. That is M is a Banach subcomodule of E.

(ii) Let $x \in E$; it is clear that $H' \cdot x$ contains x, furthermore $\mu \cdot (\nu \cdot x) = (\nu * \mu) \cdot x \in H' \cdot x$ for all $\mu, \nu \in H'$; hence the closure $\overline{H' \cdot x} = M_x$ is a complete right H'-submodule of E.

With the same notations as in (i) we have $\Delta(x) = \sum_{j \geq 1} a_j \otimes x_j$. Let $E_0 = E[x_1, \ldots, x_j, \ldots]$ be the closed linear subspace of E spanned by $(x_j)_{j \geq 1}$. First, it is clear that E_0 is a Banach space of countable type. On the other hand, if $\mu \in H'$, we have $\mu \cdot x = \sum_{j \geq 1} \langle \mu, a_j \rangle x_j \in E_0$; hence $H' \cdot x \subset E_0$ and $M_x = \overline{H' \cdot x} \subset E_0$.

Since M_x is a closed right H'-submodule of E, we deduce from (i) that M_x is a left Banach H-subcomodule of E and that $x_j \in M_x$, $j \geq 1$. Hence $M_x = E_0$ is a Banach space of countable type. □

NOTE. The theorem, applied to an a.p. linear representation U of G in E, shows that if $x \in E$, the closed linear subspace of E spanned by $C_x = \{U_s(x), s \in G\}$ is of countable type. As observed in [3], this also follows from the fact that C_x is a compactoid of E.

II. Banach comodule morphisms.

II.1. Definition. Let E and F be two left Banach H-comodules of coproducts Δ_E and Δ_F respectively. A continuous linear map $u: E \to F$ is a *Banach comodule morphism* if $\Delta_F \circ u = (1_H \otimes u) \circ \Delta_E$.

LEMMA 1. *Let $u: E \to F$ be a Banach comodule morphism.*
(i) *If V is a Banach subcomodule of F, then $u^{-1}(V)$ is a Banach subcomodule of E.*
(ii) *The closure $\overline{u(E)}$ of $u(E)$ is a Banach subcomodule of F.*

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Proof. (i) Indeed, for $x \in u^{-1}(V)$, $\Delta_E(x) = \sum_{j\geq 1} a_j \otimes x_j$ where $(a_j)_{j\geq 1}$ is an α-orthogonal set of H and $(x_j)_{j\geq 1} \subset E$. Since $\Delta_F(V) \subset H \hat{\otimes} V$ and $u(x) \in V$, we have $\Delta_F(u(x)) = \sum_{\ell\geq 1} b_\ell \otimes y_\ell$, where $(b_\ell)_{\ell\geq 1} \subset H$ and $(y_\ell)_{l\geq 1} \subset V$.

Let $H_0 = E[a_1, \ldots, a_j, \ldots; b_1, \ldots, b_\ell, \ldots]$ be the closed subspace of H spanned by $\{a_j, j \geq 1; b_\ell, \ell \geq 1\}$. This Banach space is of countable type. If $H_1 = E[a_1, \ldots, a_j, \ldots]$ is the closed subspace of H spanned by $(a_j)_{j\geq 1}$, then there exists a continuous linear projection p of H_0 onto H_1 such that $\|p\| \leq \frac{1}{\alpha}$ (cf. [7]). Let $a'_j \in H'_1$ be defined by $\langle a'_j, a_t\rangle = \delta_{jt}$, $t \geq 1$ and put $\tilde{a}'_j = a'_j \circ p \in H'_0$. Then $(\tilde{a}'_j \otimes 1_F) \circ \Delta_F(u(x)) = \sum_{\ell\geq 1} \langle \tilde{a}'_j, b_\ell\rangle y_\ell = (\tilde{a}'_j \otimes 1_F) \circ (1_H \otimes u) \circ \Delta_E(x)$ $= \sum_{t\geq 1} \langle \tilde{a}'_j, a_t\rangle u(x_t) = \sum_{t\geq 1} \delta_{jt} u(x_t) = u(x_j)$. Therefore $u(x_j) = \sum_{l\geq 1} \langle \tilde{a}'_j, b_\ell\rangle y_\ell \in V$; hence $x_j \in u^{-1}(V)$ and $\Delta_E(x) = \sum_{j\geq 1} a_j \otimes x_j \in H \hat{\otimes} u^{-1}(V)$.

(ii) For $z = u(x)$ in $u(E)$, $\Delta_F(z) = \Delta_F(u(x)) = (1_H \otimes u) \circ \Delta_E(x) = \sum_{j\geq 1} a_j \otimes u(x_j) \in H \hat{\otimes} \overline{u(E)}$. Therefore $\Delta_F(u(E)) \subset H \hat{\otimes} \overline{u(E)}$. However Δ_F is a homeomorphism of F onto $\Delta_F(F)$; hence $\Delta_F(\overline{u(E)}) = \overline{\Delta_F(u(E))} \subset H \hat{\otimes} \overline{u(E)}$.

Corollary. *Let V and W be Banach subcomodules of the left Banach H-comodule E; then $V \cap W$ is a Banach subcomodule of E.*

Proof. (a)- Although a direct proof of this corollary is easy, we have the opportunity to define the *direct sum of a finite family* $(E_i, \Delta_i)_{1\leq i\leq n}$ *of left Banach H-comodules* as follows. Let $E = \bigoplus_{i=1}^{n} E_i$, equipped with a norm equivalent to the norm $\left\|\sum_{i=1}^{n} x_i\right\| = \max_{1\leq i\leq n} \|x_i\|$. Put $\Delta = \bigoplus_{i=1}^{n} \Delta_i$; i.e. $\Delta\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} \Delta_i(x_i)$. It is readily seen that (E, Δ) is a left Banach H-comodule.

(b)- Put $F = E \oplus E$ and $\Delta_F = \Delta_E \oplus \Delta_E$; then $V \oplus W$ is a Banach subcomodule of F if V and W are Banach subcomodules of E. The continuous linear injective map u of E into F defined by $u(x) = x \oplus x$ is a Banach comodule morphism. Thus $V \cap W = u^{-1}(V \oplus W)$ is a Banach subcomodule of E.

II.2. Spaces of Banach comodule morphisms. Let us recall that a continuous linear operator $u: E \to F$ is *completely continuous* if $u = \lim_{n\to+\infty} u_n$, where $u_n: E \to F$ is linear continuous of finite rank. Furthermore, the space $C(E, F)$ of completely continuous operators is closed in $\mathscr{L}(E, F)$ and is isometrically isomorphic to $E' \hat{\otimes} F$.

If E and F are left Banach H-comodules we denote by $\mathrm{Hom}_{\mathrm{com}}(E, F)$ the set of Banach comodule morphisms. We set $C_{\mathrm{com}}(E, F) = C(E, F) \cap \mathrm{Hom}_{\mathrm{com}}(E, F)$ and $\mathrm{End}_{\mathrm{com}}(E) = \mathrm{Hom}_{\mathrm{com}}(E, E)$; $C_{\mathrm{com}}(E) = C_{\mathrm{com}}(E, E)$.

Proposition 1. *Let E, F and L be three left Banach H-comodules.*

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(i) *If* $u:E\to F$ *and* $v:F\to L$ *are Banach comodule morphisms, then* $v\circ u:E\to L$ *is a Banach comodule morphism.*

(ii) $\mathrm{Hom}_{\mathrm{com}}(E,F)$ [*resp.* $\mathrm{End}_{\mathrm{com}}(E)$] *is a Banach space* [*resp. a unitary Banach algebra*]. *Furthermore* $C_{\mathrm{com}}(E,F)$ [*resp.* $C_{\mathrm{com}}(E)$] *is a closed linear subspace* [*resp. a closed two-sided ideal*] *of* $\mathrm{Hom}_{\mathrm{com}}(E,F)$ [*resp.* $\mathrm{End}_{\mathrm{com}}(E)$].

Proof. It is easy. For instance $\Delta_L\circ(v\circ u)=(\Delta_L\circ v)\circ u=(1_H\otimes v)\circ\Delta_F\circ u=(1_H\otimes v)\circ(1_H\otimes u)\circ\Delta_E=[1_H\otimes(v\circ u)]\circ\Delta_E$. Also, if $u=\lim_{n\to+\infty}u_n$ with $u_n\in\mathrm{Hom}_{\mathrm{com}}(E,F)$, then $\Delta_F\circ u=\Delta_F\circ\left(\lim_{n\to+\infty}u_n\right)=\lim_{n\to+\infty}\Delta_F\circ u_n=\lim_{n\to+\infty}(1_H\otimes u_n)\circ\Delta_E=(1_H\otimes u)\circ\Delta_E$.

COROLLARY. *Let* $u\in\mathrm{End}_{\mathrm{com}}(E)$ [*resp.* $C_{\mathrm{com}}(E)$] *if* $S=\sum_{n\geq0}\lambda_nX^n$ *is a formal power series with coefficients in* K [*resp. and* $\lambda_0=0$] *such that* $S(u)=\sum_{n\geq0}\lambda_nu^n$ *is converging in* $\mathscr{L}(E)$, *then* $S(u)\in\mathrm{End}_{\mathrm{com}}(E)$ [*resp.* $C_{\mathrm{com}}(E)$].

II. 3. When H admits a left integral.

II.3.1. Banach comodule morphism associated with a linear map. By definition, a *left integral* for the complete Hopf algebra is an element ν of H' such that $\mu*\nu=\langle\mu,e\rangle\nu$, for all $\mu\in H'$.

Assume that the duality $\langle H',H\rangle$ is separated; then $\nu\in H'$ is a left integral for H if and only if $(1_H\otimes\nu)\circ c=k\circ\nu$.

In the sequel, we suppose that H admits a left integral ν such that $\langle\nu,e\rangle=1$. Hence the continuous linear form $\varphi=\nu\circ m\circ(1_H\otimes\eta):H\hat{\otimes}H\to K$ satisfies: (i) $\varphi\circ c=\sigma$ and (ii) $(\varphi\otimes1_H)\circ(1_H\otimes c)=(1_H\otimes\varphi)\circ(c\otimes1_H)$. Furthermore $\varphi(a\otimes e)=\nu(a)$ and $\|\nu\|\leq\|\varphi\|\leq\|\nu\|\,\|\eta\|$.

Let E and F be two left Banach H-comodules of coproducts Δ_E and Δ_F. If $u:E\to F$ is a continuous linear map, we put as in group representations theory

$$u^{\#}=(\varphi\otimes1_F)\circ(1_H\otimes\Delta_F)\circ(1_H\otimes u)\circ\Delta_E;$$

hence $u^{\#}:E\to F$ is linear and continuous.

PROPOSITION 2. *Let* E *and* F *be two left Banach comodules and* $u:E\to F$ *be a continuous linear map.*

(i) $u^{\#}$ *is a Banach comodule morphism.*

(ii) *The map* $u\to u^{\#}$ *of* $\mathscr{L}(E,F)$ *into* $\mathrm{Hom}_{\mathrm{com}}(E,F)$ *is linear and continuous. Moreover,* $u^{\#\#}=u^{\#}$ *and* u *is a Banach comodule morphism if and only if* $u^{\#}=u$.

Proof: (i) One verifies that

$$1_H\otimes[(\varphi\otimes1_F)\circ(1_H\otimes\Delta_F)\circ(1_H\otimes u)\circ\Delta_E]=$$
$$(1_H\otimes\varphi\otimes1_F)\circ(1_H\otimes1_H\otimes\Delta_F)\circ(1_H\otimes1_H\otimes u)\circ(1_H\otimes\Delta_E)$$

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Hence, one sees that

$$(1_H\otimes u^{\#})\circ\Delta_E=(1_H\otimes[(\varphi\otimes 1_F)\circ(1_H\otimes\Delta_F)\circ(1_H\otimes u)\circ\Delta_E])\circ\Delta_E=$$
$$=(1_H\otimes\varphi\otimes 1_F)\circ(1_H\otimes 1_H\otimes\Delta_F)\circ(1_H\otimes 1_H\otimes u)\circ(1_H\otimes\Delta_E)\circ\Delta_E=$$
$$=(1_H\otimes\varphi\otimes 1_F)\circ(1_H\otimes 1_H\otimes\Delta_F)\circ(1_H\otimes 1_H\otimes u)\circ(c\otimes 1_E)\circ\Delta_E=$$
$$=(1_H\otimes\varphi\otimes 1_F)\circ(c\otimes 1_H\otimes 1_F)\circ(1_H\otimes\Delta_F)\circ(1_H\otimes u)\circ\Delta_E=$$
$$=(\varphi\otimes 1_H\otimes 1_F)\circ[1_H\otimes(c\otimes 1_F)\circ\Delta_F]\circ(1_H\otimes u)\circ\Delta_E=$$
$$=(\varphi\otimes 1_H\otimes 1_F)\circ(1_H\otimes 1_H\otimes\Delta_F)\circ(1_H\otimes\Delta_F)\circ(1_H\otimes u)\circ\Delta_E.$$

However $(\varphi\otimes 1_H\otimes 1_F)\circ(1_H\otimes 1_H\otimes\Delta_F)=\Delta_F\circ(\varphi\otimes 1_F)$. Therefore, $(1_H\otimes u^{\#})\circ\Delta_E=\Delta_F\circ(\varphi\otimes 1_F)\circ(1_H\otimes\Delta_F)\circ(1_H\otimes u)\circ\Delta_E=\Delta_F\circ u^{\#}$; i.e. $u^{\#}$ is a comodule morphism.

(ii) It is readily seen that $u\to u^{\#}$ is linear and continuous with norm $\le\|\varphi\|\,\|\Delta_E\|\,\|\Delta_F\|$.

If u is a comodule morphism, one sees that

$$u^{\#}=(\varphi\otimes 1_F)\circ(1_H\otimes\Delta_F)\circ(1_H\otimes u)\circ\Delta_E=(\varphi\otimes 1_F)\circ(1_H\otimes\Delta_F)\circ\Delta_F\circ u$$
$$=(\varphi\otimes 1_F)\circ(c\otimes 1_F)\circ\Delta_F\circ u=(\varphi\circ c\otimes 1_F)\circ\Delta_F\circ u=[(\sigma\otimes 1_F)\circ\Delta_F]\circ u=u.$$

Conversely, if $u=u^{\#}$, from (i) it follows that u is a comodule morphism. Hence, for any continuous linear map $u:E\to F$, one has $u^{\#\#}=u^{\#}$.

Lemma 2. *Let E, F and L be three left Banach H-comodules, $u:E\to F$ and $v:F\to L$ be continuous linear maps. Then $(v\circ u^{\#})^{\#}=v^{\#}\circ u^{\#}$*

Proof. Obviously, $(v\circ u^{\#})^{\#}=(\varphi\otimes 1_L)\circ(1_H\otimes\Delta_L)\circ(1_H\otimes v\circ u^{\#})\circ\Delta_E=(\varphi\otimes 1_L)\circ(1_H\otimes\Delta_L)\circ(1_H\otimes v)\circ(1_H\otimes u^{\#})\circ\Delta_E=(\varphi\otimes 1_L)\circ(1_H\otimes\Delta_L)\circ(1_H\otimes v)\circ\Delta_F\circ u^{\#}=v^{\#}\circ u^{\#}$.

Corollary. *Let E be a left Banach H-comodule. If $p:E\to E$ is a continuous linear projection of E onto $M=p(E)$ and if M is a Banach subcomodule of E; then $p^{\#}$ is a projection of E onto M and $E=M\oplus N$, a direct sum of Banach comodules, where $N=\ker p^{\#}$.*

Proof. Put $\Delta_E=\Delta$. By hypothesis, for any $y\in M$, $\Delta(y)=\sum_{\ell\ge1}b_\ell\otimes y_\ell\in H\hat{\otimes}M$. Let $x\in E$; setting $\Delta(x)=\sum_{j\ge1}a_j\otimes x_j\in H\hat{\otimes}E$, it follows that

$$p^{\#}(x)=(\varphi\otimes 1_E)\circ(1_H\otimes\Delta)\circ(1_H\otimes p)\circ\Delta(x)$$
$$=(\varphi\otimes 1_E)\Big(\sum_{j\ge1}a_j\otimes\Delta\circ p(x_j)\Big)=\sum_{j\ge1}\sum_{\ell\ge1}\varphi(a_j\otimes b_{\ell j})y_{\ell j}$$

with $y_{\ell j}\in M$. Since $y_{\ell j}=p(x_{\ell j})$, we have

$$p^{\#}(x)=p\Big(\sum_{j\ge1}\sum_{\ell\ge1}\varphi(a_j\otimes b_{\ell j})x_{\ell_j}\Big)=p(z)\in p(E)=M;$$

i.e. $p^{\#}(E)\subset M$.

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On the other hand, since for any $y \in M$, $p(y) = y$, one has

$$p^{\#}(y) = (\varphi \otimes 1_E)\Big(\sum_{l \geq 1} b_\ell \otimes \Delta(y_\ell)\Big) = (\varphi \otimes 1_E) \circ (1_H \otimes \Delta) \circ \Delta(y)$$
$$= (\varphi \otimes 1_E) \circ (c \otimes 1_E) \circ \Delta(y) = (\varphi \circ c \otimes 1_E) \circ \Delta(y) = (\sigma \otimes 1_E) \circ \Delta(y) = y.$$

Therefore $M \subset p^{\#}(M) \subset p^{\#}(E)$. We have proved that $M = p^{\#}(E)$.

Since for any $x \in E$, $p^{\#}(x) = p(z)$, one has $p \circ p^{\#}(x) = p \circ p(z) = p(z) = p^{\#}(x)$; in other words, $p \circ p^{\#} = p^{\#}$. Hence $p^{\#} = p^{\#\#} = (p \circ p^{\#})^{\#} = p^{\#} \circ p^{\#}$; i.e. $p^{\#}$ is a linear projection as well as a comodule morphism of E onto M. The corollary is proved.

NOTE. This corollary gives a proof of the implication (iv)$\Rightarrow$(i) of the Theorem 3 in [**3**].

PROPOSITION 3. *Let E and F be two left Banach H-comodules and let $u: E \to F$ be a completely continuous operator. Then $u^{\#}$ is completely continuous.*

Proof. (i) Since the map $u \to u^{\#}$ is linear and continuous, if $u = \sum_{n \geq 1} x'_n \otimes z_n \in C(E, F) = E' \hat{\otimes} F$, one has, $u^{\#} = \sum_{n \geq 1} (x'_n \otimes z_n)^{\#}$ in $\mathrm{Hom}_{\mathrm{com}}(E, F)$. But $C(E, F)$ is a Banach space; hence, it suffices to prove that for any $x' \in E'$ and any $z \in F$, one has $(x' \otimes z)^{\#} \in C(E, F)$.

(ii) Put $u = x' \otimes z$. First, $\Delta_F(z) = \sum_{\ell \geq 1} b_\ell \otimes z_\ell$, where $(b_\ell)_{\ell \geq 1} \subset H$ and $(z_\ell)_{\ell \geq 1}$ is an α-orthogonal set of F, with

$$\alpha \sup_{\ell \geq 1} \|b_\ell\| \, \|z_\ell\| \leq \|\Delta_F(z)\| \leq \sup_{\ell \geq 1} \|b_\ell\| \, \|z_\ell\|. \tag{0}$$

Also, for $x \in E$, $\Delta_E(x) = \sum_{j \geq 1} a_j \otimes x_j$, where $(a_j)_{j \geq 1} \subset H$, $(x_j)_{j \geq 1}$ is an α-orthogonal set of E and one has an inequality similar to (0). On the other hand,

$$u^{\#}(x) = (\varphi \otimes 1_F) \circ (1_H \otimes \Delta_F) \circ (1_H \otimes u) \circ \Delta_E(x)$$
$$= (\varphi \otimes 1_F) \circ (1_H \otimes \Delta_F)\Big(\sum_{j \geq 1} a_j \otimes u(x_j)\Big) = (\varphi \otimes 1_F)\Big(\sum_{j \geq 1} a_j \otimes \langle x', x_j \rangle \Delta_F(z)\Big)$$
$$= \sum_{j \geq 1} \sum_{\ell \geq 1} \langle x', x_j \rangle \varphi(a_j \otimes b_\ell) z_\ell.$$

However $\|\langle x', x_j \rangle \varphi(a_j \otimes b_\ell) z_\ell\| \leq \|x'\| \, \|\varphi\| \, \|a_j\| \, \|x_j\| \, \|b_\ell\| \, \|z_\ell\|$ with $\lim_{j \to +\infty} \|a_j\| \, \|x_j\| = 0 = \lim_{l \to +\infty} \|b_\ell\| \, \|z_\ell\|$. Hence, the family $(\langle x', x_j \rangle \varphi(a_j \otimes b_\ell) z_\ell)_{j,\ell}$ is summable. Therefore

$$u^{\#}(x) = \sum_{\ell \geq 1} \sum_{j \geq 1} \langle x', x_j \rangle \varphi(a_j \otimes b_\ell) z_\ell. \tag{1}$$

Let $F_0 = E[z_1, \ldots, z_\ell, \ldots]$ be the closed subspace of F spanned by the α-orthogonal set $(z_\ell)_{\ell \geq 1}$. If $z'_\ell \in F'_0$ is defined by $\langle z'_\ell, z_k \rangle = \delta_{\ell k}$, then $\dfrac{1}{\|z_\ell\|} \leq \|z'_\ell\| \leq \dfrac{1}{\alpha} \dfrac{1}{\|z_\ell\|}$.

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It is clear that for any $x \in E$, $u^{\#}(x) \in F_0$ and we obtain the adjoint map ${}^t u^{\#}: F_0' \to E'$ with ${}^t u^{\#}(z_\ell') \in E'$ and for any $x \in E$, we have

$$\langle {}^t u^{\#}(z_\ell'), x\rangle = \langle z_\ell', u^{\#}(x)\rangle = \sum_{j\geq 1} \langle x', x_j\rangle \varphi(a_j \otimes b_\ell). \tag{2}$$

Moreover,

$$|\langle {}^t u^{\#}(z_\ell'), x\rangle| \leq \sup_{j\geq 1} |\langle x', x_j\rangle|\, |\varphi(a_j \otimes b_\ell| \leq \|x'\|\, \|\varphi\|\, \|b_\ell\| \sup_{j\geq 1} \|a_j\|\, \|x_j\|$$

$$\leq \frac{1}{\alpha} \|x'\|\, \|\varphi\|\, \|b_\ell\|\, \|\Delta_E(x)\| \leq \frac{1}{\alpha} \|x'\|\, \|\varphi\|\, \|\Delta_E\|\, \|b_\ell\|\, \|x\|.$$

Hence, we have

$$\|{}^t u^{\#}(z_\ell')\| \leq \frac{1}{\alpha} \|x'\|\, \|\varphi\|\, \|\Delta_E\|\, \|b_\ell\|. \tag{3}$$

It follows that

$$\|{}^t u^{\#}(z_\ell') \otimes z_\ell\| \leq \frac{1}{\alpha} \|x'\|\, \|\varphi\|\, \|\Delta_E\|\, \|b_\ell\|\, \|z_\ell\|.$$

Since $\lim_{\ell\to+\infty} \|b_\ell\|\, \|z_\ell\| = 0$, we have $\lim_{\ell\to+\infty} \|{}^t u^{\#}(z_\ell') \otimes z_\ell\| = 0$ and $\sum_{l\geq 1} {}^t u^{\#}(z_\ell') \otimes z_\ell \in E' \hat{\otimes} F = C(E, F)$. However for any $x \in E$,

$$\Big(\sum_{\ell\geq 1} {}^t u^{\#}(z_\ell') \otimes z_\ell\Big)(x) = \sum_{\ell\geq 1} \langle {}^t u^{\#}(z_\ell'), x\rangle z_\ell = \sum_{\ell\geq 1} \sum_{j\geq 1} \langle x', x_j\rangle \varphi(a_j \otimes b_\ell) z_\ell = u^{\#}(x),$$

by (1). It follows that $u^{\#} = \sum_{\ell\geq 1} {}^t u^{\#}(z_\ell') \otimes z_\ell \in C(E, F)$.

Remark 1. Recall that ν is a left integral for $AP(G, K)$ such that $\langle \nu, 1\rangle \neq 0$ iff $\nu \neq 0$ and $\langle \nu, \gamma_s f\rangle = \langle \nu, f\rangle = \langle \nu, \delta_s f\rangle$ for all $s \in G$. Moreover, if $\langle \nu, 1\rangle = 1$, ν is called an *invariant mean*. If F is a Banach space, we have an extension $\nu_F = \nu \otimes 1_F$ of ν on $AP(G, F) \simeq AP(G, K) \hat{\oplus} F$ with values in F and $\nu_F(\gamma_s \varphi) = \nu_F(\varphi) = \nu_F(\delta_s \varphi)$, $s \in G$.

For the a.p. linear representations $U: G \to \mathscr{L}(E)$, $V: G \to \mathscr{L}(F)$ and the continuous linear map $u: E \to F$, the intertwining operator (as well as comodule morphism) $u^{\#}$ can be written in the classical form $u^{\#}(x) = \int_G V_s \circ u \circ U_{s^{-1}}(x)\, d\nu_F(s)$.

II.3.2. Comodules which are free Banach spaces. Let E be a *free Banach space*; that is to say, E is isomorphic to a space $c_0(I, K) = \Big\{(\lambda_j)_{j\in I} \subset K / \lim_j \lambda_j = 0\Big\}$. In other words, there exists $(e_j)_{j\in I} \subset E$, called a *base* of E, two real numbers α_0 and $\alpha_1 > 0$, such that any $x \in E$ can be written in the form $x = \sum_{j\in I} \lambda_j e_j$, $\lambda_j \in K$ and $\alpha_0 \sup_{j\in I} |\lambda_j| \leq \|x\| \leq \alpha_1 \sup_{j\in I} |\lambda_j|$. For the continuous linear form $e_j' \in E'$, defined by $\langle e_j', e_\ell\rangle = \delta_{j\ell}$, one has $\dfrac{1}{\alpha_1} \leq \|e_j'\| \leq \dfrac{1}{\alpha_0}$.

Let F be another Banach space. The complete tensor product $F\hat{\otimes}E$ is isomorphic to $c_0(I,F)=\left\{(y_i)_{j\in I}\subset F/\lim_j \|y_j\|=0\right\}$. In fact, each $z\in F\hat{\otimes}E$ is in the form $z=\sum_{j\in I} y_j\otimes e_j$, where $y_j\in F$ and $\lim_j \|y_j\|=0$. Furthermore $\alpha_0 \sup_{j\in I}\|y_j\|\le\|z\|\le\alpha_1\sup_{j\in I}\|y_j\|$.

Assume that the free Banach space E is a left Banach H-comodule with coproduct $\Delta:E\to H\hat{\otimes}E$. For $x\in E$, one has $\Delta(x)=\sum_{j\in I}A_j(x)\otimes e_j$. Hence one defines, for each $j\in I$, a continuous linear map $A_j:E\to H$ and $\alpha_0\sup_{j\in I}\|A_j\|\le\|\Delta\|\le\alpha_1\sup_{j\in I}\|A_j\|$.

Put, for $\ell\in I$, $A_j(e_\ell)=a_{\ell j}\in H$; one has

$$\Delta(e_\ell)=\sum_{j\in I}a_{\ell j}\otimes e_j,\qquad \lim_j a_{\ell j}=0. \tag{4}$$

NOTE. $A_j=(1_H\otimes e_j')\circ\Delta$ and $\bigcap_{j\in I}\ker A_j=(0)$.

More generally, if $x'\in E'$, we put $A_{x'}=(1_H\otimes x')\circ\Delta$. Obviously, H is a left Banach H-comodule with respect to its coproduct c.

LEMMA 3. *For any $x'\in E'$, the linear map $A_{x'}=(1_H\otimes x')\circ\Delta:E\to H$ is a comodule morphism.*

Proof. It is easy to see that $c\circ(1_H\otimes x')=c\otimes x'=(1_H\otimes 1_H\otimes x')\circ(c\otimes 1_E)$, So $c\circ A_{x'}=c\circ(1_H\otimes x')\circ\Delta=(1_H\otimes 1_H\otimes x')\circ(c\otimes 1_E)\circ\Delta=(1_H\otimes 1_H\otimes x')\circ(1_H\otimes\Delta)\circ\Delta$ $=[1_H\otimes(1_H\otimes x')\circ\Delta]\circ\Delta=(1_H\otimes A_{x'})\circ\Delta$.

LEMMA 4. *For all $\ell, j\in I$, one has*

(i) $c(a_{\ell j})=\sum_{n\in I}a_{\ell n}\otimes a_{nj}$,

(ii) $\sigma(a_{\ell j})=\delta_{\ell j}$,

(iii) $\sum_{n\in I}a_{\ell n}\eta(a_{nj})=\delta_{\ell j}\cdot e=\sum_{n\in I}\eta(a_{\ell n})a_{nj}$.

Proof. (i) Since $A_j=(1_H\otimes e_j')\circ\Delta$ is a comodule morphism, we have $c(a_{\ell j})=c\circ A_j(e_\ell)=(1_H\otimes A_j)\circ\Delta(e_\ell)=(1_H\otimes A_j)\left(\sum_{n\in I}a_{\ell n}\otimes e_n\right)=\sum_{n\in I}a_{\ell n}\otimes A_j(e_n)=\sum_{n\in I}a_{\ell n}\otimes a_{nj}$.

(ii) Obviously, from $(\sigma\otimes 1_E)\circ\Delta=1_E$ we get, for $x\in E$, $x=\sum_{j\in I}\sigma(A_j(x))e_j$. Hence $e_\ell=\sum_{j\in I}\sigma(A_j(e_\ell))e_j=\sum_{j\in I}\sigma(a_{\ell j})e_j$ and $\sigma(a_{\ell j})=\delta_{\ell j}$.

(iii) This follows readily from (i), (ii) and $m\circ(1_H\otimes\eta)\circ c=k\circ\sigma=m\circ(1_H\otimes\eta)\circ c$.

LEMMA 5. *Assume that H admits a left integral ν such that $\langle\nu,e\rangle=1$. Put $\varphi=\nu\circ m\circ(1_H\otimes\eta)$. For all $\ell,j\in I$, one has $\sum_{n\in I}\varphi(a_{\ell n}\otimes a_{nj})=\delta_{\ell j}$.*

Proof. Since $\varphi\circ c=\sigma$, from Lemma 4 one deduces that $\delta_{\ell j}=\sigma(a_{\ell j})=\varphi\circ c(a_{\ell j})=\sum_{n\in I}\varphi(a_{\ell n}\otimes a_{nj})$. ☐

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Now assume that the duality $\langle H', H\rangle$ is separated and that H admits a left integral ν such that $\langle \nu, e\rangle = 1$. By Proposition 3, we know that $(e'_\ell \otimes e_j)^{\#} \in C(E)$, for all $\ell, j \in I$.

With (4) and by definition of $(e'_\ell \otimes e_j)^{\#}$ one verifies that

$$(e'_\ell \otimes e_j)^{\#}(e_i) = \sum_{n\in I} \varphi(a_{i\ell} \otimes a_{jn}) e_n \ (i, j, \ell \in I). \tag{5}$$

REMARK 2. One deduces from (5) and Lemma 5 that there exists $\ell \in I$ such that $(e'_\ell \otimes e_\ell)^{\#}$ is different from the null operator.

By a previous result, we know that the space $C(E) = E' \hat{\otimes} E$ is isomorphic to $c_0(I, E')$. Let $u \in C(E)$; then $u = \sum_{\ell\in I} \psi_\ell \otimes e_\ell$ with $\psi_\ell \in E'$, $\lim_\ell \psi_\ell = 0$ and $\alpha_0 \sup_{\ell\in I} \|\psi_\ell\| \le \|u\| \le \alpha_1 \sup_{\ell\in I} \|\psi_\ell\|$. Furthermore one has $u = \sum_{\ell\in I} {}^t u(e'_\ell) \otimes e_\ell$.

It is well known that one has the trace form $\mathrm{Tr}: E' \hat{\otimes} E \to K$ defined by $\mathrm{Tr}(x' \otimes x) = \langle x', x\rangle$, which is linear and continuous with $|\mathrm{Tr}(u)| \le \|u\|$. Here one obtains for $u = \sum_{\ell\in I} \psi_\ell \otimes e_\ell \in C(E) = E' \hat{\otimes} E$, $\mathrm{Tr}(u) = \sum_{\ell\in I} \langle \psi_\ell, e_\ell\rangle = \sum_{\ell\in I} \langle e'_\ell, u(e_\ell)\rangle$.

Hence, for $u = (e'_\ell \otimes e_j)^{\#}$; $\ell, j \in I$; one has

$$\mathrm{Tr}[(e'_\ell \otimes e_j)^{\#}] = \sum_{n\in I} \varphi(a_{n\ell} \otimes a_{jn}) \qquad (j, \ell \in I). \tag{6}$$

DEFINITION. A complete Hopf algebra H is called *supple* if H is a pseudo-reflexive Banach space and if $\eta \circ \eta = 1_H$.

EXAMPLES. (1) $AP(G, K)$ and its complete sub-Hopf-algebras are supple.

(2) Any commutative (resp. cocommutative) complete Hopf algebra which is a pseudo-reflexive Banach space is supple.

LEMMA 6. *Let H be a supple complete Hopf algebra. If H admits a left integral ν such that $\langle \nu, e\rangle = 1$, then the map $\varphi = \nu \circ m \circ (1_H \otimes \eta)$ satisfies $\varphi(a \otimes b) = \varphi(b \otimes a)$, for all a, $b \in H$.*

Proof. Since η is an anti-endomorphism of the algebra H and since $\eta \circ \eta = 1_H$ implies $\nu \circ \eta = \nu$, one has for $a, b \in H$, $\varphi(a \otimes b) = \nu \circ m \circ (1_H \otimes \eta)(a \otimes b) = \nu(a\eta(b)) = \nu \circ \eta(a\eta(b)) = \nu(b\eta(a)) = \varphi(b \otimes a)$.

The following proposition strengthens Remark 2.

PROPOSITION 4. *Let H be a supple complete Hopf algebra that admits a left integral ν such that $\langle \nu, e\rangle = 1$. Let E be a left Banach H-comodule which is a free Banach space with base $(e_j)_{j\in I}$. Then for each $\ell \in I$, the comodule endomorphism $(e'_\ell \otimes e_\ell)^{\#}$ of E is a completely continuous operator such that* $\mathrm{Tr}[(e'_\ell \otimes e_\ell)^{\#}] = 1$.

Proof. Indeed, one deduces from (6), Lemmas 5 and 6 that $\mathrm{Tr}[(e'_\ell \otimes e_\ell)^{\#}] = \sum_{n\in I} \varphi(a_{n\ell} \otimes a_{\ell n}) = \sum_{n\in I} \varphi(a_{\ell n} \otimes a_{n\ell}) = 1$.

REMARK 3. In the same way, if $\ell \ne j$, then $\mathrm{Tr}[(e'_\ell \otimes e_j)^{\#}] = \sum_{n\in I} \varphi(a_{n\ell} \otimes a_{jn}) = \sum_{n\in I} \varphi(a_{jn} \otimes a_{n\ell}) = 0$.

Let $E'_0 = E[e'_j, j \in I]$ be the closed subspace of E' spanned by $(e'_j)_{j\in I}$. The space

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$E_0'\hat{\otimes}E$ is a closed subspace of $E'\hat{\otimes}E = C(E)$. Since $(e_j')_{j\in I}$ is a base of E_0', each $u \in E_0'\hat{\otimes}E$ can be written $u = \sum_{\ell\in I}\sum_{j\in I}\lambda_{\ell j}e_j'\otimes e_\ell$ with $\lambda_{\ell j}\in K$, $\lim_\ell \sup_{j\in I}|\lambda_{\ell j}| = 0$ and for any $\ell\in I$, $\lim_j |\lambda_{\ell j}| = 0$. One has $\mathrm{Tr}(u) = \sum_{\ell\in I}\lambda_{\ell\ell}$.

COROLLARY 1. *For each $u \in E_0'\hat{\otimes}E$, one has* $\mathrm{Tr}(u^{\#}) = \mathrm{Tr}(u)$. *In particular, if* $\dim E$ *is finite, for each $u\in\mathscr{L}(E)$, one has* $\mathrm{Tr}(u^{\#}) = \mathrm{Tr}(u)$.

Proof. Obviously, $\mathrm{Tr}(u^{\#}) = \sum_{\ell\in I}\sum_{j\in I}\lambda_{\ell j}\,\mathrm{Tr}[(e_j'\otimes e_\ell)^{\#}] = \sum_{\ell\in I}\sum_{j\in I}\lambda_{\ell j}\delta_{\ell j} = \mathrm{Tr}(u)$.

COROLLARY 2. *Let $x\in E$, $x\neq 0$; there exists $x'\in E'$ such that $\langle x',x\rangle = 1$ and* $\mathrm{Tr}[(x'\otimes x)^{\#}] = 1$.

Let us recall that if $u\in C(E)$ then the Fredholm determinant of u is $\det(1_E - tu) = 1 + \sum_{q\geq 1}(-1)^q\,\mathrm{Tr}(\Lambda^q u)t^q$ and $\det(1_E - tu)$ is a power series of infinite radius of convergence; (cf. [**4**] and [**10**]). Furthermore $\det(1_E - tu)\cdot 1_E = (1_E - tu)P_1(t,u)$, where $P_1(t,u)$ is the Fredholm resolvent of u. Hence for $\lambda\in K$, $1_E - \lambda u$ is invertible in $\mathscr{L}(E)$ if and only if $\det(1_E - \lambda u)\neq 0$.

With the operators as above, for instance for $u_\ell = (e_\ell'\otimes e_\ell)^{\#}$, $\ell\in I$, one has

$$\det(1_E - tu_\ell) = 1 - t + \sum_{q\geq 2}(-1)^q\,\mathrm{Tr}(\Lambda^q u_\ell)t^q. \tag{7}$$

III. Reducibility of Banach comodules.

III.1.1. Simple Banach comodules.

DEFINITION. A left Banach H-comodule E is called *simple* or *topologically irreducible* if E is not the null space and does not contain any closed subcomodule different from (0) and E.

It follows immediately from Theorem 1 that, when H is a pseudo-reflexive Banach space, any simple left Banach H-comodule is a vector space of countable type.

THEOREM 2. *Let H be a supple complete Hopf algebra that admits a left integral ν such that $\langle\nu, e\rangle = 1$. Then any left Banach H-comodule that is not the null space contains at least a finite dimensional subcomodule different from* (0).

Proof. For this proof, we apply Riesz's decomposition theorem.

(a) Let E be a left Banach comodule over the supple complete Hopf algebra H with $E\neq(0)$. By Theorem 1, if $x\in E$, $x\neq 0$, then $x\in M = H'\cdot x$; hence M is different from (0). Furthermore M is a Banach subcomodule of E and is a Banach space of countable type. Therefore M is a free Banach space (cf. [**7**]).

(b) Assume that H admits a left integral ν such that $\langle\nu,e\rangle = 1$. Hence by Proposition 4, there exists a completely continuous operator u which is an endomorphism of the comodule M and such that $\mathrm{Tr}(u) = 1$. It follows that $\det(1_M - tu) = 1 - t + \sum_{q\geq 2}(-1)^q$

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$\operatorname{Tr}(\Lambda^q u)\, t^q$ is a *non constant* power series with infinite radius of convergence. According to the p-adic Weierstrass' factorization theorem, one has $\det(1_M - tu) = \prod_{q\geq 1} P_q$, where P_q is a polynomical such that $P_q(0) = 1$, $d^\circ P_q \geq 1$ (see for example [**1**]). That is $\det(1_M - tu)$ has its zeros in a subfield of the algebraic closure $\tilde{K}$ of K. Following [**10**] one has the following results.

(b_1) *First*, $\det(1_M - tu)$ *has a zero* $\lambda \in K^*$.

Let $h \geq 1$ be the multiplicity of λ; one has $M = N(\lambda) \oplus F(\lambda)$ (Riesz's decomposition), where $N(\lambda) = \ker(1_M - \lambda u)^h$ and $\dim N(\lambda) = h$. However $(1_M - \lambda u)^h$ is a comodule endomorphism of M; therefore $N(\lambda)$ is a subcomodule of M of finite dimension $h \geq 1$ and $N(\lambda)$ is a non-null finite dimensional subcomodule of E.

(b_2) *Second*, $\det(1_M - tu)$ *has no zero in* K^*.

Let $\zeta \in \tilde{K}$ be a zero of $\det(1_M - tu)$. Let $R(t) = 1 - \sum_{j=1}^{\ell} \gamma_j t^j$ be the polynomial of minimal degree such that $R(\zeta^{-1}) = 0$ and $R(0) = 1$. Setting $R(u) = 1_M - \sum_{j=1}^{\ell} \gamma_j u^j = 1_M - v$, we see that $v = \sum_{j=1}^{\ell} \gamma_j u^j$ is a comodule endomorphism of M as well as a completely continuous operator.

Let $\zeta^{(2)}, \ldots, \zeta^{(\ell)}$ be the conjugates of ζ in $\tilde{K}$. The field $L = K[\zeta, \zeta^{(2)}, \ldots, \zeta^{(\ell)}]$ is a finite extension of K. Put $M_L = L \hat{\otimes}_K M = L \otimes_K M$; hence $u_L = 1_L \otimes u$ is a completely continuous operator on M_L. Moreover, $\zeta \in L$ is a zero of $\det(1_{M_L} - tu_L) = \det(1_M - tu)$; hence $1_{M_L} - \zeta u_L$ is not invertible in $\mathscr{L}_L(M_L)$.

Since $R(t) = (1 - \zeta t)\prod_{j=2}^{\ell} (1 - \zeta^{(j)} t)$ and $R(u) = 1_M - v$, one has

$$R(u)_L = 1_{M_L} - \sum_{j=1}^{\ell} \gamma_j 1_L \otimes u^j = 1_{M_L} - \sum_{j=1}^{\ell} \gamma_j u_L^j = 1_{M_L} - v_L = (1_{M_L} - \zeta u_L)\prod_{j=2}^{\ell} (1_{M_L} - \zeta^{(j)} u_L).$$

It follows that the completely continuous operator v_L is such that $1_{M_L} - v_L = R(u_L)$ is not invertible in $\mathscr{L}_L(M_L)$. Consequently $1_M - v$ is not invertible in $\mathscr{L}(M)$; i.e. 1 is a zero of $\det(1_M - tv)$ with multiplicity $h' \geq 1$. Therefore we have the Riesz decomposition $M = N(R) \oplus F(R)$, where $N(R) = \ker(1_M - v)^{h'} = \ker(R(u)^{h'})$ and $\dim N(R) = h' \geq 1$.

But $R(u)^{h'}$ is a comodule endomorphism of M; hence $N(R) = \ker(R(u)^{h'})$ is a subcomodule of M of finite dimension $h' \geq 1$. Therefore $N(R)$ is a non-null finite dimensional subcomodule of E.

REMARK 4. The subspace $F(\lambda)$ (resp. $F(R)$) of M is also a Banach subcomodule of E.

THEOREM 3. *Let H be a supple complete Hopf algebra that admits a left integral ν such that $\langle \nu, e\rangle = 1$. Then any simple left Banach H-comodule E is finite dimensional.*

Proof. This is obvious from Theorem 2. Indeed, let $x \in E$, $x \neq 0$; one has $\overline{H' \cdot x} = M \neq (0)$; hence $M = E$ and by Theorem 1, E is a free Banach space. With the notations in the proof of Theorem 2, one has in the first case $E = N(\lambda) = \{x \in E / u(x) = \lambda^{-1} x\}$, $\dim E = h$ and $u = \lambda^{-1} 1_E$. In the second case, one has $E = N(R) = \{x \in E / v(x) = x\}$, $\dim E = h'$ and $R(u) = 0$.

COROLLARY. (Schur's Lemma.) *Under the above hypothesis on H, if E is a simple left Banach H-comodule, then* $\operatorname{End}_{\text{com}}(E)$ *is a (skew) field of finite dimension* $\leq (\dim E)^2$. *Moreover, if K is algebraically closed, then* $\operatorname{End}_{\text{com}}(E) = K \cdot 1_E$ *and if K is of characteristic* $p \neq 0$, *then* $(p, \dim E) = 1$.

Proof. It suffices to observe that there exist $u \in \operatorname{End}_{\text{com}}(E)$ such that $\operatorname{Tr}(u) = 1$. If K is algebraically closed, one has $u = \lambda 1_E$, hence $\operatorname{Tr}(u) = 1 = \lambda \dim E$

III.1.2. Reducibility of Banach comodules.

PROPOSITION 5. *Let H be a supple complete Hopf algebra that admits a left integral v such that $\langle v, e\rangle = 1$. Then any left Banach H-comodule E that is a Banach space of countable type is a topological direct sum of simple comodules.*

Proof. Indeed, by Theorem 2, E contains finite subcomodules different from the null space. Hence E contains a simple subcomodule. Let $W = \sum_{\ell \in S} V_\ell$ be the sum of all simple subcomodules of E. As in semi-simple module theory, there exists a subset T of S such that $W = \bigoplus_{\ell \in T} V_\ell$. Put $E_0 = \bar{W} = \hat{\bigoplus}_{\ell \in T} V_\ell$, the closure of W in E. It is clear that E_0 is a Banach subcomodule of E.

On the other hand, since E is a Banach space of countable type and E_0 is a closed subspace of E, for $0 < \alpha < 1$, there exists a linear projection of E onto E_0 such that $\|p\| \leq \frac{1}{\alpha}$ (cf. [**7**]). Therefore by the Corollary of Lemma 2 or Theorem 3 of [**3**] one has the direct sum of Banach comodules $E = E_0 \oplus F_0$. If F_0 is different from (0), F_0 must contain a simple subcomodule V. Clearly V is not contained in E_0; that contradicts the definition of E_0. Consequently $F_0 = (0)$ and $E = E_0 = \hat{\bigoplus}_{\ell \in T} V_\ell$.

THEOREM 4. *Let H be a supple complete Hopf algebra that admits a left integral v such that $\langle v, e\rangle = 1$. Then any left Banach H-module E is a topological direct sum of simple comodules.*

Proof. As above, put $W = \sum_{j \in I} V_j$, the sum of all simple subcomodules of E. There exists $J \subset I$ such that $W = \bigoplus_{j \in J} V_j$ (any simple subcomodule of E is isomorphic to one of the V_j, $j \in J$).

Let $x \in E$, $x \neq 0$; the Banach subcomodule $M_x = \overline{H' \cdot x}$ of E being a Banach space of countable type, one has, by Proposition 5, $x \in M_x = \hat{\bigoplus}_{\ell \in T} V_\ell$, where V_ℓ is a simple subcomodule of M_x and obviously of E. Hence for $\varepsilon > 0$, there exists a finite subset F of T, $x_\ell \in V_\ell$ for $\ell \in F$, such that $\left\| x - \sum_{\ell \in F} x_\ell \right\| < \varepsilon$. Since $\sum_{\ell \in F} x_\ell \in \bigoplus_{\ell \in F} V_\ell \subset W$, one has $x \in \bar{W}$ and $E = \bar{W} = \hat{\bigoplus}_{j \in J} V_j$.

REMARK 5. Let Ω be the family of the isomorphic classes of simple left Banach H-comodules. Let $E(\omega)$ be the isotypical component of E for $\omega \in \Omega$, i.e. the sum of all simple subcomodules of E belonging to ω. It may happen that $E(\omega) = (0)$. One has

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$E(\omega) = \oplus(V_j, V_j \in \omega)$, $W = \bigoplus_{\omega\in\Omega} E(\omega)$ and if H satisfies the hypothesis of Theorem 4, then $E = \hat{\bigoplus}_{\omega\in\Omega} E(\omega)$.

III.2. Application to $H = AP(G, K)$. Let us recall that the complete Hopf algebra $AP(G, K)$ (as well as any of its complete Hopf subalgebras) is a supple Hopf algebra. A left integral ν over $AP(G, K)$, if it exists, such that $\langle \nu, 1\rangle = 1$ is called an *invariant mean.*

From [3] we know that the category of left Banach $AP(G, K)$-comodules is in a bijective correspondence with the category of almost periodic linear representations of G.

The following theorem is a direct application of Theorems 3 and 4. One has an equivalent theorem for any complete Hopf subalgebra of $AP(G, K)$ as $AP_{\mathcal{T}}(G, K)$, $\mathcal{PP}(G, K)$ and $\mathcal{PP}_{\mathcal{T}}(G, K)$: the algebra $\mathcal{PP}(G, K)$ is the subalgebra of the elements f in $AP(G, K)$ such that $\{\gamma_s f, s \in G\}$ is relatively compact in $\mathcal{B}(G, K)$; *if* $\mathcal{T}$ is a group topology on G, $AP_{\mathcal{T}}(G, K)$[resp. $\mathcal{PP}_{\mathcal{T}}(G, K)$] is the subalgebra of the functions f in $AP(G, K)$ [resp. $\mathcal{PP}(G, K)$] such that f is $\mathcal{T}$-continuous.

THEOREM 5. *Assume that $AP(G, K)$ admits an invariant mean.*

(i) *Any topologically irreducible almost periodic linear representation is finite dimensional.*

(ii) *Let $U: G \to \mathcal{L}(E)$ be an almost periodic linear representation of G. The Banach space E is a topological direct sum of irreducible U-invariant subspaces of E.*

NOTE. Let Ω be the family of the classes of topologically irreducible almost linear representations. With the above hypothesis, one has $E = \hat{\bigoplus}_{\omega\in\Omega} E(\omega)$ (cf. Remark 5).

COROLLARY 1. (Peter-Weyl Theorem). *Assume that $AP(G, K)$ admits an invariant mean. Then the space $R_b(G, K)$ of the bounded representative functions of G in K is a dense subspace of $AP(G, K)$.*

Proof. The left regular representation γ of G in $AP(G, K)$ is almost periodic $[\gamma_s f(t) = f(s^{-1}t)]$. It is clear that any γ-invariant finite dimensional subspace of $AP(G, K)$ is contained in $R_b(G, K)$.

Let $f \in AP(G, K)$, $f \neq 0$ and $M_f = \overline{AP(G, K)' \cdot f}$. One has $M_f = \hat{\bigoplus}_{\ell\in T} V_\ell$, where V_ℓ is γ-invariant and topologically irreducible. Hence $\dim V_\ell$ is finite and $V_\ell \subset R_b(G, K)$ for all $\ell \in T$. Moreover, as in the proof of Theorem 4, there exist a finite subset F of T and $\ell \in F$, $f_\ell \in V_\ell$, such that $\left\| f - \sum_{\ell\in F} f_\ell \right\| < \varepsilon$. Since $\sum_{\ell\in F} f_\ell \in \bigoplus_{\ell\in F} V_\ell \subset R_b(G, K)$, we have shown that $R_b(G, K)$ is dense in $AP(G, K)$. □

Let $\omega \in \Omega$ be the class of the topologically irreducible almost periodic linear representation (V, ρ) of G. With the hypothesis of Theorem 5, one has $\dim V = n$ finite. Let $R(\rho)$ be the subspace of $AP(G, K)$ spanned by the coefficient functions of ρ; i.e. the functions $s \to \langle x', \rho(s) \cdot x\rangle$, where $x' \in V'$ and $x \in V$. One has $\dim R(\rho) \leq n^2$ and it is readily seen that $R(\omega) = R(\rho)$ depends only on ω. Let $\check{\omega}$ be the class of $(V', \check{\rho})$, where

$\check{\rho}(s) = {}^t\rho(s^{-1})$; then $(V', \check{\rho})$ is irreducible and $R(\check{\omega}) = \eta(R(\omega))$. Fix a base $(e_j)_{1\le j\le n}$ of V and let $(e'_j)_{1\le j\le n} \subset V'$ be its dual base. Let us consider for $1 \le j \le n$ the linear map $A_j : V \to AP(G, K)$ defined by $A_j(x)(s) = \langle e'_j, \rho(s^{-1}) \cdot x\rangle$. One has $\gamma_s \circ A_j = A_j \circ \rho(s)$ (directly or see Lemma 3). Since $A_j(e_j)(e) = 1$, one has $\ker A_j \neq V$ and since (V, ρ) is irreducible, $\ker A_j = (0)$, i.e. A_j is injective. Put $H_j = A_j(V)$; the linear representations (V, ρ) and (H_j, γ) are equivalent. Hence $(H_j, \gamma) \in \omega$, for $1 \le j \le n$. It is readily seen that $\eta(R(\omega)) = \sum_{j=1}^{n} H_j$ and there exists $J \subset [1, n]$ such that $\eta(R(\omega)) = \bigoplus_{j\in J} H_j$. Moreover $\eta(R(\omega))$ is the isotypical component of $AP(G, K)$ corresponding to ω. Therefore, if $(\omega_1, \ldots, \omega_m)$ is a finite subset of Ω, then $\sum_{r=1}^{m} \eta(R(\omega_r)) = \bigoplus_{r=1}^{m} \eta(R(\omega_\tau))$. It follows that $\sum_{\tau=1}^{m} R(\omega_\tau) = \bigoplus_{\tau=1}^{m} R(\omega_\tau)$.

Since any finite dimensional almost periodic linear representation is reducible, we have proved the following result.

COROLLARY 2. *Assume that $AP(G, K)$ admits an invariant mean. Then*

$$R_b(G, K) = \bigoplus_{\omega\in\Omega} R(\omega) \quad \textit{and} \quad AP(G, K) = \hat{\bigoplus_{\omega\in\Omega}} R(\omega).$$

NOTE. $R(\omega)$ is a subcogebra of $AP(G, K)$ for $\omega \in \Omega$.

COROLLARY 3. *Assume that the group G is commutative and that $AP(G, K)$ admits an invariant mean. If the field K is algebraically closed, then Ω can be identified with* $\mathrm{Hom}_b(G, K^*) = \hat{G}$, *the bounded character group of G and $AP(G, K) = \hat{\bigoplus}_{\chi\in\hat{G}} K \cdot \chi$.*

Proof. The proof runs as in the classical case. Indeed, with the hypothesis on $AP(G, K)$, if (V, ρ) is irreducible and K is algebraically closed then $\mathrm{End}_\rho(V) = K \cdot 1_V$. Since G is commutative, for $s \in G$, one has $\rho(s) \in \mathrm{End}_\rho(V)$, hence $\rho(s) = \chi(s) \cdot 1_V$ and χ is a bounded character of G (which implies $|\chi(s)| = 1$, $s \in G$). It follows that $\Omega = \hat{G}$. Since $R(\chi) = K \cdot \chi$, we have $AP(G, K) = \hat{\bigoplus}_{\chi\in\hat{G}} K \cdot \chi$ (compare with **[8]**, **[9]**).

More generally one can prove the following result.

COROLLARY 4. *Let $CAP(G, K)$ be the closed subalgebra of the central functions $f \in AP(G, K)$; i.e. $f(sts^{-1}) = f(t)$, s, $t \in G$. Assume that $AP(G, K)$ admits an invariant mean. Set for $(V, \rho) \in \omega$, $\chi^u_\omega(s) = \mathrm{Tr}(\rho(s) \circ u)$, where $u \in \mathrm{End}_\rho(V)$. Hence $\{\chi^u_\omega, \omega \in \Omega$, $u \in \mathrm{End}_\rho(V)$ for a fixed $(V, \rho) \in \omega\}$ is a total subset of the Banach space $CAP(G, K)$. Moreover, if K is algebraically closed, setting $\chi_\omega(s) = \mathrm{Tr}(\rho(s))$ for a fixed $(V, \rho) \in \omega$, one has $CAP(G, K) = \hat{\bigoplus}_{\omega\in\Omega} K \cdot \chi_\omega$.*

NOTES. (i) Let us say that (G, K) is *a.p.i.m.* if $AP(G, K)$ admits an invariant mean ν. Schikhof has given in **[8]**, **[9]** the characterization of the a.p.i.m. pairs (G, K) when G is commutative and with the extra condition $\|\nu\| = 1$. It remains to characterize all the a.p.i.m. pairs (G, K).

(ii) If there exists an invariant mean ν on $AP(G, K)$, and putting for f, $g \in AP(G, K)$, $(f * g)(s) = \langle \nu, f \cdot \gamma_s(\eta(g))\rangle$, then $AP(G, K)$ is equipped with a new structure

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of Banach algebra, non unitary if G is infinite. Can one use this algebra structure in the aim to establish the above results? In the case of $\mathscr{P}\mathscr{P}(G, K)$ see [**6**].

REMARK 6. For any complete Hopf algebra H, one can define the representative subalgebra $\mathscr{R}(H)$ of H similar to $R_b(G, K)$. If H is supple and admits a left integral v such that $\langle v, e\rangle = 1$, then one has a translation of Theorem 5 and its Corollaries 1 and 2.

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