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THE STRONG IRREDUCIBILITY OF A CLASS OF COWEN–DOUGLAS OPERATORS ON BANACH SPACES

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Abstract

Let $\mathcal{B}_n(\Omega)$ be the set of Cowen–Douglas operators of index n on a nonempty bounded connected open subset Ω of $\mathbb{C}$. We consider the strong irreducibility of a class of Cowen–Douglas operators $\mathcal{FB}_n(\Omega)$ on Banach spaces. We show $\mathcal{FB}_n(\Omega) \subseteq \mathcal{B}_n(\Omega)$ and give some conditions under which an operator $T \in \mathcal{FB}_n(\Omega)$ is strongly irreducible. All these results generalise similar results on Hilbert spaces.

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1. Introduction

Cowen and Douglas [1] introduced and researched a class of important operators, the Cowen–Douglas operators, on Hilbert spaces. Cowen–Douglas operators were defined in terms of the notion of holomorphic vector bundles, the first time complex geometry was applied in operator theory.

Gilfeather [2] introduced the concept of strongly irreducible operators and Herrero [3] studied the strongly irreducible Cowen–Douglas operators on Hilbert spaces. Jiang and Sun [8] introduced the concept of completely irreducible operators, which is equivalent to the concept of strongly irreducible operators, and showed that it was an approximate replacement of Jordan blocks on infinite dimensional spaces. A number of questions about the operator structure of Hilbert spaces raised by Herrero and Jiang have since been answered (see the books [9, 10]). For further recent developments relating to strongly irreducible Cowen–Douglas operators, see [4–7].

Zhang and Zhong [12, Theorem 2] showed that a Cowen–Douglas operator of index 1 must be strongly irreducible on Banach spaces. It is obvious that Cowen–Douglas operators of index 2 are not always strongly irreducible. In [5], the authors introduced a class of Cowen–Douglas operators on Hilbert spaces and discussed their

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strong irreducibility. In this paper, we will discuss the strong irreducibility of this class of Cowen–Douglas operators on Banach spaces.

In this paper, all Banach spaces are over the complex field. $B(X, Y)$ denotes the set of bounded linear operators from a Banach space X to a Banach space Y and $B(X, X)$ is abbreviated to $B(X)$. The identity on X is denoted by I_X and often abbreviated to I. For an operator $T \in B(X, Y)$, its kernel is $\ker T := \{x \in X : Tx = 0\}$ and its range is $\operatorname{ran} T := \{Tx : x \in X\}$. For a subset A of X, $\operatorname{span} A$ and $\overline{A}$ denote the linear span and the norm-closure of A, respectively. An operator $T \in B(X)$ is said to be quasinilpotent if $\lim_{n\to\infty} \|T^n\|^{1/n} = 0$.

In the following, if there is no special explanation, X is always a Banach space, Ω is a nonempty bounded connected open subset of $\mathbb{C}$ and n is a positive integer.

Definition 1.1 [1]. An operator $T \in B(X)$ is said to be a Cowen–Douglas operator of index n on Ω (defined on X), if the following statements hold:

(1) $\dim \ker(T - \omega) = n$ for all $\omega \in \Omega$;
(2) $\operatorname{ran}(T - \omega) = X$ for all $\omega \in \Omega$;
(3) $\overline{\operatorname{span}}\{\ker(T - \omega) : \omega \in \Omega\} = X$.

Denote the set of Cowen–Douglas operators of index n on Ω (defined on X) by $\mathcal{B}_n(\Omega)(X)$, abbreviated to $\mathcal{B}_n(\Omega)$ when the meaning is clear.

Definition 1.2 [2]. An operator $T \in B(X)$ is said to be strongly irreducible if there exists no nontrivial idempotent in the commutant algebra of T, that is, if $P \in B(X)$ with $P^2 = P$ and $TP = PT$, then $P = 0$ or $P = I$.

Definition 1.3 [5]. For an operator $T \in B(X)$, if there exists a direct sum decomposition $X = X_1 \oplus X_2 \oplus \cdots \oplus X_n$ such that T can be expressed as

$$T = \begin{pmatrix} T_1 & S_{12} & \cdots & S_{1n} \\ & T_2 & \ddots & \vdots \\ & & \ddots & S_{n-1,n} \\ 0 & & & T_n \end{pmatrix}, \tag{1.1}$$

where $T_i \in \mathcal{B}_1(\Omega)(X_i)$ for $1 \le i \le n$ and $S_{ij} \in B(X_j, X_i)$ for $1 \le i < j \le n$ with $S_{i,i+1} \ne 0$ and $T_i S_{i,i+1} = S_{i,i+1} T_{i+1}$ for $1 \le i < n$, then we say $T \in \mathcal{FB}_n(\Omega)(X)$ or simply $T \in \mathcal{FB}_n(\Omega)$.

In Section 2, we show $\mathcal{FB}_n(\Omega) \subseteq \mathcal{B}_n(\Omega)$. In Section 3, we give some conditions under which an operator $T \in \mathcal{FB}_n(\Omega)$ is strongly irreducible. These results generalise the results on Hilbert spaces in [5]. The proofs are different: [5] uses the language of holomorphic vector bundles, while we use only operator theory on Banach spaces.

2. $\mathcal{FB}_n(\Omega) \subseteq \mathcal{B}_n(\Omega)$

In this section, we show $\mathcal{FB}_n(\Omega) \subseteq \mathcal{B}_n(\Omega)$. In fact, we obtain a more general result.

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Proposition 2.1. *Let T be a bounded linear operator on X. Suppose X has a direct sum decomposition $X = X_1 \oplus X_2 \oplus \cdots \oplus X_n$ and T can be expressed as in (1.1) where $T_i \in \mathcal{B}_1(\Omega)(X_i)$ for $1 \le i \le n$ and $S_{ij} \in B(X_j, X_i)$ for $1 \le i < j \le n$. Then $T \in \mathcal{B}_n(\Omega)$.*

Proof. For every $\omega \in \Omega$ and for $1 \le i \le n$, since $\dim \ker(T_i - \omega) = 1$, we can write

$$\ker(T_i - \omega) = \operatorname{span}\{e_{i,\omega}\}$$

with $0 \ne e_{i,\omega} \in X_i$. Since $\operatorname{ran}(T_1 - \omega) = X_1$, there exists an $f_{1,2,\omega} \in X_1$ such that $(T_1 - \omega) f_{1,2,\omega} = -S_{12} e_{2,\omega}$. Let

$$g_{2,\omega} = f_{1,2,\omega} + e_{2,\omega}.$$

Since $\operatorname{ran}(T_2 - \omega) = X_2$, there exists an $f_{2,3,\omega} \in X_2$ such that $(T_2 - \omega) f_{2,3,\omega} = -S_{23} e_{3,\omega}$ and then, invoking again $\operatorname{ran}(T_1 - \omega) = X_1$, there exists an $f_{1,3,\omega} \in X_1$ such that $(T_1 - \omega) f_{1,3,\omega} = -S_{13} e_{3,\omega} - S_{12} f_{2,3,\omega}$. Let

$$g_{3,\omega} = f_{1,3,\omega} + f_{2,3,\omega} + e_{3,\omega}.$$

Continuing in the same way, we can obtain $f_{i,j,\omega} \in X_i$ for all $1 \le i < j \le n$ such that $(T_{j-1} - \omega) f_{j-1,j,\omega} = -S_{j-1,j} e_{j,\omega}$ and

$$(T_i - \omega) f_{i,j,\omega} = -S_{ij} e_{j,\omega} - \sum_{k=i+1}^{j-1} S_{ik} f_{k,j,\omega} \quad (1 \le i \le j-2).$$

Let

$$g_{j,\omega} = \sum_{k=1}^{j-1} f_{k,j,\omega} + e_{j,\omega}.$$

By the choice of $g_{j,\omega}$ it is obvious that $\ker(T - \omega) \supseteq \operatorname{span}\{e_{1,\omega}, g_{2,\omega}, \ldots, g_{n,\omega}\}$.

Conversely, if $x_1 + x_2 + \cdots + x_n \in \ker(T - \omega)$ with $x_i \in X_i$ for $1 \le i \le n$, then $(T_n - \omega) x_n = 0$. Thus $x_n = a_n e_{n,\omega}$ for some $a_n \in \mathbb{C}$. Now

$$\begin{aligned} 0 &= (T_{n-1} - \omega) x_{n-1} + S_{n-1,n} x_n = (T_{n-1} - \omega) x_{n-1} + a_n S_{n-1,n} e_{n,\omega} \\ &= (T_{n-1} - \omega) x_{n-1} - a_n (T_{n-1} - \omega) f_{n-1,n,\omega} = (T_{n-1} - \omega)(x_{n-1} - a_n f_{n-1,n,\omega}), \end{aligned}$$

so $x_{n-1} - a_n f_{n-1,n,\omega} = a_{n-1} e_{n-1,\omega}$ for some $a_{n-1} \in \mathbb{C}$, that is, $x_{n-1} = a_n f_{n-1,n,\omega} + a_{n-1} e_{n-1,\omega}$. Again,

$$\begin{aligned} 0 &= (T_{n-2} - \omega) x_{n-2} + S_{n-2,n-1} x_{n-1} + S_{n-2,n} x_n \\ &= (T_{n-2} - \omega) x_{n-2} + a_n S_{n-2,n-1} f_{n-1,n,\omega} + a_{n-1} S_{n-2,n-1} e_{n-1,\omega} + a_n S_{n-2,n} e_{n,\omega} \\ &= (T_{n-2} - \omega) x_{n-2} - a_{n-1} (T_{n-2} - \omega) f_{n-2,n-1,\omega} - a_n (T_{n-2} - \omega) f_{n-2,n-1,\omega} \\ &= (T_{n-2} - \omega)(x_{n-2} - a_{n-1} f_{n-2,n-1,\omega} - a_n f_{n-2,n-1,\omega}), \end{aligned}$$

and hence $x_{n-2} - a_{n-1} f_{n-2,n-1,\omega} - a_n f_{n-2,n-1,\omega} = a_{n-2} e_{n-2,\omega}$ for some $a_{n-2} \in \mathbb{C}$. Thus $x_{n-2} = a_n f_{n-2,n,\omega} + a_{n-1} f_{n-2,n-1,\omega} + a_{n-2} e_{n-2,\omega}$. Continuing in this way, we conclude

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that $x_i = \sum_{k=i+1}^{n} a_k f_{i,k,\omega} + a_i e_{i,\omega}$ for some $a_i \in \mathbb{C}$ for $1 \le i < n$ and so

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= \sum_{k=2}^{n} a_k f_{1,k,\omega} + a_1 e_{1,\omega} + \sum_{k=3}^{n} a_k f_{2,k,\omega} + a_2 e_{2,\omega} + \cdots + a_n e_{n,\omega} \\ &= a_1 e_{1,\omega} + a_2(e_{2,\omega} + f_{1,2,\omega}) + \cdots + a_n\Big(e_{n,\omega} + \sum_{k=1}^{n-1} f_{k,n,\omega}\Big) \\ &= a_1 e_{1,\omega} + a_2 g_{2,\omega} + \cdots + a_n g_{n,\omega}. \end{aligned}$$

Therefore, $\ker(T - \omega) = \mathrm{span}\{e_{1,\omega}, g_{2,\omega}, \ldots, g_{n,\omega}\}$.

If $a_i \in \mathbb{C}$ for $1 \le i \le n$ are such that $0 = a_1 e_{1,\omega} + a_2 g_{2,\omega} + \cdots + a_n g_{n,\omega}$, then

$$\begin{aligned} 0 &= a_1 e_{1,\omega} + a_2(e_{2,\omega} + f_{1,2,\omega}) + \cdots + a_n\Big(e_{n,\omega} + \sum_{k=1}^{n-1} f_{k,n,\omega}\Big) \\ &= a_1 e_{1,\omega} + \sum_{k=2}^{n} a_k f_{1,k,\omega} + a_2 e_{2,\omega} + \sum_{k=3}^{n} a_k f_{2,k,\omega} + \cdots + a_n f_{n-1,n,\omega} + a_n e_{n,\omega}. \end{aligned}$$

Since $a_i e_{i,\omega} + \sum_{k=i+1}^{n} a_k f_{i,k,\omega} \in X_i$ $(1 \le i < n)$ and $a_n e_{n,\omega} \in X_n$, each of these quantities is 0 and so $a_n = 0$. Again, $a_{n-1} e_{n-1,\omega} = a_{n-1} e_{n-1,\omega} + a_n f_{n-1,n,\omega} = 0$, so $a_{n-1} = 0$. In the same way, we conclude that $a_i = 0$ for all $1 \le i \le n$. Thus, $e_{1,\omega}, g_{2,\omega}, \ldots, g_{n,\omega}$ are linear independent. Therefore $\dim \ker(T - \omega) = n$.

If $\omega \in \Omega$ and $y_1 + y_2 + \cdots + y_n \in X$ with $y_i \in X_i$ for $1 \le i \le n$, since $\mathrm{ran}(T_n - \omega) = X_n$, there exists an $x_n \in X_n$ such that $(T_n - \omega)x_n = y_n$. Since $\mathrm{ran}(T_{n-1} - \omega) = X_{n-1}$, there exists an $x_{n-1} \in X_{n-1}$ such that $(T_{n-1} - \omega)x_{n-1} = y_{n-1} - S_{n-1,n}x_n$. In the same way, we obtain $x_i \in X_i$ for $1 \le i < n$ such that $(T_i - \omega)x_i = y_i - \sum_{k=i+1}^{n} S_{i,k}x_k$. By the choice of the x_i we have $(T - \omega)(x_1 + x_2 + \cdots + x_n) = y_1 + y_2 + \cdots + y_n$ and so $\mathrm{ran}(T - \omega) = X$.

From the first part of the proof,

$$X_1 = \overline{\mathrm{span}}\{\ker(T_1 - \omega) : \omega \in \Omega\} = \overline{\mathrm{span}}\{e_{1,\omega} : \omega \in \Omega\} \subseteq \overline{\mathrm{span}}\{\ker(T - \omega) : \omega \in \Omega\}.$$

For $\omega \in \Omega$, since $e_{2,\omega} = g_{2,\omega} - f_{1,2,\omega} \in \ker(T - \omega) + X_1 \subseteq \overline{\mathrm{span}}\{\ker(T - \omega) : \omega \in \Omega\}$,

$$\ker(T_2 - \omega) = \mathrm{span}\{e_{2,\omega}\} \subseteq \overline{\mathrm{span}}\{\ker(T - \omega) : \omega \in \Omega\}.$$

Thus $X_2 = \overline{\mathrm{span}}\{\ker(T_2 - \omega) : \omega \in \Omega\} \subseteq \overline{\mathrm{span}}\{\ker(T - \omega) : \omega \in \Omega\}$. Again, for $\omega \in \Omega$, since $e_{3,\omega} = g_{3,\omega} - f_{1,3,\omega} - f_{2,3,\omega} \in \ker(T - \omega) + X_1 + X_2 \subseteq \overline{\mathrm{span}}\{\ker(T - \omega) : \omega \in \Omega\}$,

$$\ker(T_3 - \omega) = \mathrm{span}\{e_{3,\omega}\} \subseteq \overline{\mathrm{span}}\{\ker(T - \omega) : \omega \in \Omega\}.$$

Thus $X_3 = \overline{\mathrm{span}}\{\ker(T_3 - \omega) : \omega \in \Omega\} \subseteq \overline{\mathrm{span}}\{\ker(T - \omega) : \omega \in \Omega\}$. Continuing in this way, we conclude that $X_i \subseteq \overline{\mathrm{span}}\{\ker(T - \omega) : \omega \in \Omega\}$ for all $1 \le i \le n$. Therefore,

$$X = X_1 \oplus X_2 \oplus \cdots \oplus X_n = \overline{\mathrm{span}}\{\ker(T - \omega) : \omega \in \Omega\}.$$

This completes the proof that $T \in \mathcal{B}_n(\Omega)$. □

Corollary 2.2. $\mathcal{F}\mathcal{B}_n(\Omega) \subseteq \mathcal{B}_n(\Omega)$.

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Although we have assumed only that $S_{i,i+1}$ $(1 \le i < n)$ is nonzero in Definition 1.3, its range must be dense, as is shown below.

Proposition 2.3. *Let X_1, X_2 be Banach spaces. If $T_1 \in \mathcal{B}_1(\Omega)(X_1)$, $T_2 \in \mathcal{B}_1(\Omega)(X_2)$ and $0 \ne S \in B(X_2, X_1)$ with $T_1 S = S T_2$ then $\overline{\operatorname{ran} S} = X_1$.*

Proof. For $\omega \in \Omega$, since $\dim\ker(T_1 - \omega) = \dim\ker(T_2 - \omega) = 1$, we can write

$$\ker(T_1 - \omega) = \operatorname{span}\{e_{1,\omega}\}, \quad \ker(T_2 - \omega) = \operatorname{span}\{e_{2,\omega}\}$$

for some $0 \ne e_{1,\omega} \in X_1$ and $0 \ne e_{2,\omega} \in X_2$. Since

$$X_2 = \overline{\operatorname{span}}\{\ker(T_2 - \omega) : \omega \in \Omega\} = \overline{\operatorname{span}}\{e_{2,\omega} : \omega \in \Omega\}$$

and $S \ne 0$, there exists an $\omega_0 \in \Omega$ such that $S e_{2,\omega_0} \ne 0$. By [12, Lemma 2], there exist some neighbourhood $\Lambda \subseteq \Omega$ of ω_0 and a holomorphic X_2-valued function h defined on Λ such that, for each $\omega \in \Lambda$, $\ker(T_2 - \omega) = \operatorname{span}\{h(\omega)\}$. Hence $h(\omega) = a_{2,\omega} e_{2,\omega}$ for some $0 \ne a_{2,\omega} \in \mathbb{C}$. Let

$$k : \Lambda \to X_1 : k(\omega) = S(h(\omega)).$$

Then k is a continuous X_1-valued function defined on Λ and

$$k(\omega_0) = S(h(\omega_0)) = S(a_{2,\omega_0} e_{2,\omega_0}) = a_{2,\omega_0} S e_{2,\omega_0} \ne 0.$$

Therefore, there exists a nonempty bounded connected open subset Δ of $\mathbb{C}$ with $\omega_0 \in \Delta \subseteq \Lambda \subseteq \Omega$ such that for each $\omega \in \Delta$, $0 \ne k(\omega) = a_{2,\omega} S e_{2,\omega}$; so $S e_{2,\omega} \ne 0$. But $(T_1 - \omega) S e_{2,\omega} = S(T_2 - \omega) e_{2,\omega} = 0$, and hence $S e_{2,\omega} = a_{1,\omega} e_{1,\omega}$ for some $0 \ne a_{1,\omega} \in \mathbb{C}$. Thus,

$$e_{1,\omega} = \frac{1}{a_{1,\omega}} S e_{2,\omega} \in \operatorname{ran} S.$$

Since $\mathcal{B}_1(\Omega)(X_1) \subseteq \mathcal{B}_1(\Delta)(X_1)$ by [12, Theorem 1], $T_1 \in \mathcal{B}_1(\Delta)(X_1)$. Therefore,

$$X_1 = \overline{\operatorname{span}}\{\ker(T_1 - \omega) : \omega \in \Delta\} = \overline{\operatorname{span}}\{e_{1,\omega} : \omega \in \Delta\} \subseteq \overline{\operatorname{ran} S}.$$

Thus $\overline{\operatorname{ran} S} = X_1$. □

3. The strong irreducibility of operators in $\mathcal{FB}_n(\Omega)$

In this section, we give some conditions under which an operator $T \in \mathcal{FB}_n(\Omega)$ is strongly irreducible. We need two lemmas about Rosenblum operators.

Let X_i be Banach spaces and let $T_i \in B(X_i)$ for $i = 1, 2$. Define the Rosenblum operator τ_{T_1,T_2} by

$$\tau_{T_1,T_2} : B(X_2, X_1) \to B(X_2, X_1) : \tau(S) = T_1 S - S T_2 \quad S \in B(X_2, X_1).$$

We abbreviate τ_{T_1,T_1} to $\tau_{T_1} : B(X_1) \to B(X_1)$.

Lemma 3.1. *Let $T \in \mathcal{B}_1(\Omega)$. If $S \in \ker\tau_T$ and S is quasinilpotent, then $S = 0$.*

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PROOF. For $\omega \in \Omega$, since $\dim \ker(T-\omega) = 1$, we can write

$$\ker(T-\omega) = \operatorname{span}\{e_\omega\}$$

for some $0 \neq e_\omega \in X$. Since $S \in \ker \tau_T$, $TS = ST$ and $(T-\omega)Se_\omega = S(T-\omega)e_\omega = 0$. Thus $Se_\omega = a_\omega e_\omega$ for some $a_\omega \in \mathbb{C}$. Therefore, for all $n \in \mathbb{N}$, $S^n e_\omega = a_\omega^n e_\omega$. Now

$$|a_\omega|^n \|e_\omega\| = \|a_\omega^n e_\omega\| = \|S^n e_\omega\| \leq \|S^n\| \, \|e_\omega\|,$$

which gives

$$|a_\omega| \leq \|S^n\|^{1/n} \to 0 \quad (n \to \infty).$$

Thus $a_\omega = 0$ and $Se_\omega = a_\omega e_\omega = 0$. But

$$X = \overline{\operatorname{span}}\{\ker(T-\omega) : \omega \in \Omega\} = \overline{\operatorname{span}}\{e_\omega : \omega \in \Omega\},$$

and so we have $S = 0$. □

LEMMA 3.2 [11]. *Let $T \in B(X)$. If $S \in \ker \tau_T \cap \operatorname{ran} \tau_T$, then S is quasinilpotent.*

PROPOSITION 3.3. *For $k = 1, 2$, let $T^{(k)} \in \mathcal{FB}_2(\Omega)$ and, under the direct sum decomposition $X = X_1^{(k)} \oplus X_2^{(k)}$, write $T^{(k)}$ as*

$$T^{(k)} = \begin{pmatrix} T_1^{(k)} & S^{(k)} \\ 0 & T_2^{(k)} \end{pmatrix},$$

where $T_1^{(k)} \in \mathcal{B}_1(\Omega)(X_1^{(k)})$, $T_2^{(k)} \in \mathcal{B}_1(\Omega)(X_2^{(k)})$ and $0 \neq S^{(k)} \in B(X_2^{(k)}, X_1^{(k)})$ with $T_1^{(k)} S^{(k)} = S^{(k)} T_2^{(k)}$. If $P \in \ker \tau_{T^{(1)},T^{(2)}}$ and P is invertible, then P is a block upper triangular operator, that is, if $P = (P_{ij})_{2\times 2}$, where $P_{ij} \in B(X_j^{(2)}, X_i^{(1)})$ for $i, j = 1, 2$, then $P_{21} = 0$.

PROOF. Let $Q = P^{-1} = (Q_{ij})_{2\times 2}$, where $Q_{ij} \in B(X_j^{(1)}, X_i^{(2)})$ for $i, j = 1, 2$. Since $P \in \ker \tau_{T^{(1)},T^{(2)}}$, it follows that $T^{(1)}P = PT^{(2)}$ and $QT^{(1)} = T^{(2)}Q$. Since $T^{(1)}P = PT^{(2)}$,

$$\begin{aligned} \begin{pmatrix} T_1^{(1)}P_{11} + S^{(1)}P_{21} & T_1^{(1)}P_{12} + S^{(1)}P_{22} \\ T_2^{(1)}P_{21} & T_2^{(1)}P_{22} \end{pmatrix} &= \begin{pmatrix} T_1^{(1)} & S^{(1)} \\ 0 & T_2^{(1)} \end{pmatrix} \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \\ &= \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} T_1^{(2)} & S^{(2)} \\ 0 & T_2^{(2)} \end{pmatrix} = \begin{pmatrix} P_{11}T_1^{(2)} & P_{11}S^{(2)} + P_{12}T_2^{(2)} \\ P_{21}T_1^{(2)} & P_{21}S^{(2)} + P_{22}T_2^{(2)} \end{pmatrix} \end{aligned}$$

and, since $= QT^{(1)} = T^{(2)}Q$,

$$\begin{aligned} \begin{pmatrix} Q_{11}T_1^{(1)} & Q_{11}S^{(1)} + Q_{12}T_2^{(1)} \\ Q_{21}T_1^{(1)} & Q_{21}S^{(1)} + Q_{22}T_2^{(1)} \end{pmatrix} &= \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} T_1^{(1)} & S^{(1)} \\ 0 & T_2^{(1)} \end{pmatrix} \\ &= \begin{pmatrix} T_1^{(2)} & S^{(2)} \\ 0 & T_2^{(2)} \end{pmatrix} \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} = \begin{pmatrix} T_1^{(2)}Q_{11} + S^{(2)}Q_{21} & T_1^{(2)}Q_{12} + S^{(2)}Q_{22} \\ T_2^{(2)}Q_{21} & T_2^{(2)}Q_{22} \end{pmatrix}. \end{aligned}$$

Since $P_{21}S^{(2)} = T_2^{(1)}P_{22} - P_{22}T_2^{(2)}$ and $T_2^{(2)}Q_{21} = Q_{21}T_1^{(1)}$,

$$\begin{aligned} P_{21}S^{(2)}Q_{21}S^{(1)} &= T_2^{(1)}P_{22}Q_{21}S^{(1)} - P_{22}T_2^{(2)}Q_{21}S^{(1)} \\ &= T_2^{(1)}P_{22}Q_{21}S^{(1)} - P_{22}Q_{21}T_1^{(1)}S^{(1)} \\ &= T_2^{(1)}P_{22}Q_{21}S^{(1)} - P_{22}Q_{21}S^{(1)}T_2^{(1)} = \tau_{T_2^{(1)}}(P_{22}Q_{21}S^{(1)}) \in \operatorname{ran}\tau_{T_2^{(1)}}. \end{aligned}$$

Since $T_2^{(1)}P_{21} = P_{21}T_1^{(2)}$ and $T_2^{(2)}Q_{21} = Q_{21}T_1^{(1)}$,

$$\begin{aligned} T_2^{(1)}P_{21}S^{(2)}Q_{21}S^{(1)} &= P_{21}T_1^{(2)}S^{(2)}Q_{21}S^{(1)} = P_{21}S^{(2)}T_2^{(2)}Q_{21}S^{(1)} \\ &= P_{21}S^{(2)}Q_{21}T_1^{(1)}S^{(1)} = P_{21}S^{(2)}Q_{21}S^{(1)}T_2^{(1)}. \end{aligned}$$

Thus $P_{21}S^{(2)}Q_{21}S^{(1)} \in \ker\tau_{T_2^{(1)}} \cap \operatorname{ran}\tau_{T_2^{(1)}}$. By Lemma 3.2, $P_{21}S^{(2)}Q_{21}S^{(1)}$ is quasinilpotent.

Since $T_2^{(1)} \in \mathcal{B}_1(\Omega)(X_2^{(1)})$, Lemma 3.1 shows $P_{21}S^{(2)}Q_{21}S^{(1)} = 0$. Also, since $\overline{\operatorname{ran} S^{(1)}} = X_1^{(1)}$, we have $P_{21}S^{(2)}Q_{21} = 0$. If $Q_{21} \neq 0$, since $T_2^{(2)}Q_{21} = Q_{21}T_1^{(1)}$, then $\overline{\operatorname{ran} Q_{21}} = X_2^{(2)}$ by Proposition 2.3. Since $\overline{\operatorname{ran} S^{(2)}} = X_1^{(2)}$, this yields $P_{21} = 0$ and so P is a block upper triangular operator. If, on the other hand, $Q_{21} = 0$, then $Q_{11} \neq 0$ since Q is invertible. Since

$$Q_{11}T_1^{(1)} = T_1^{(2)}Q_{11} + S^{(2)}Q_{21} = T_1^{(2)}Q_{11},$$

it follows that $\overline{\operatorname{ran} Q_{11}} = X_1^{(2)}$ by Proposition 2.3. Since $PQ = I$,

$$\begin{pmatrix} P_{11}Q_{11} & P_{11}Q_{12} + P_{12}Q_{22} \\ P_{21}Q_{11} & P_{21}Q_{12} + Q_{22}P_{22} \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}\begin{pmatrix} Q_{11} & Q_{12} \\ 0 & Q_{22} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

and so $P_{21}Q_{11} = 0$. Thus $P_{21} = 0$ and again P is a block upper triangular operator. □

Proposition 3.4. *For $k = 1, 2$, let $T^{(k)} \in \mathcal{FB}_n(\Omega)$ and, under the direct sum decomposition $X = X_1^{(k)} \oplus X_2^{(k)} \oplus \cdots \oplus X_n^{(k)}$, write $T^{(k)}$ as*

$$T^{(k)} = \begin{pmatrix} T_1^{(k)} & S_{12}^{(k)} & \cdots & S_{1n}^{(k)} \\ & T_2^{(k)} & \ddots & \vdots \\ & & \ddots & S_{n-1,n}^{(k)} \\ 0 & & & T_n^{(k)} \end{pmatrix},$$

where $T_i^{(k)} \in \mathcal{B}_1(\Omega)(X_i^{(k)})$ for $1 \le i \le n$ and $S_{ij}^{(k)} \in B(X_j^{(k)}, X_i^{(k)})$ for $1 \le i < j \le n$ with $S_{i,i+1}^{(k)} \neq 0$ and $T_i^{(k)}S_{i,i+1}^{(k)} = S_{i,i+1}^{(k)}T_{i+1}^{(k)}$ for $1 \le i < n$. If $P \in \ker\tau_{T^{(1)},T^{(2)}}$ and P is invertible, then P is a block upper triangular operator, that is, if $P = (P_{ij})_{n\times n}$, where $P_{ij} \in B(X_j^{(2)}, X_i^{(1)})$ for $1 \le i, j \le n$, then $P_{ij} = 0$ for $1 \le j < i \le n$.

Proof. The proof can be given by induction on n. From Proposition 3.3, the result holds for $n = 2$. Suppose that it holds for $n < m$. As in the proof of [5, Proposition 3.2], we can obtain the result for $n = m$ to complete the proof. □

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Proposition 3.5. *Let $T \in \mathcal{FB}_n(\Omega)(X)$ and express T in block upper triangular form as in Definition 1.3. If $P \in \ker \tau_T$, then P is a block upper triangular operator, that is, if $P = (P_{ij})_{n\times n}$, where $P_{ij} \in B(X_j, X_i)$ for $1 \le i, j \le n$, then $P_{ij} = 0$ for $1 \le j < i \le n$.*

Proof. The proof is similar to the proof of [5, Proposition 3.3]. □

Theorem 3.6. *Let $T \in \mathcal{FB}_n(\Omega)(X)$ and express T in block upper triangular form as in Definition 1.3. If $S_{i,i+1} \notin \operatorname{ran} \tau_{T_i,T_{i+1}}$ for $1 \le i \le n-1$, then T is strongly irreducible.*

Proof. Suppose that $P \in B(X)$ with $P^2 = P$ and $TP = PT$. Let $P = (P_{ij})_{n\times n}$ with $P_{ij} \in B(X_j, X_i)$ for $1 \le i, j \le n$. Then $P_{ij} = 0$ for $1 \le j < i \le n$ by Proposition 3.5. Now write

$$(U_{ij})_{n\times n} = TP = PT = (V_{ij})_{n\times n}, \quad (W_{ij})_{n\times n} = P^2 = P = (P_{ij})_{n\times n},$$

where $U_{ij} = V_{ij} = W_{ij} = 0$ for $1 \le j < i \le n$ and

$$U_{ij} = \sum_{k=i}^{j} S_{ik}P_{kj}, \quad V_{ij} = \sum_{k=i}^{j} P_{ik}S_{kj}, \quad W_{ij} = \sum_{k=i}^{j} P_{ik}P_{kj}$$

for $1 \le i \le j \le n$, where $S_{ii} = T_i$. For $1 \le i \le n$, these equations yield

$$T_iP_{ii} = U_{ii} = V_{ii} = P_{ii}T_i, \quad P_{ii}^2 = W_{ii} = P_{ii}.$$

Since $T_i \in \mathcal{B}_1(\Omega)(X_i)$, T_i is strongly irreducible by [12, Theorem 2]. Thus $P_{ii} = 0$ or I.

Suppose $P_{ll} = 0$ and $P_{l+1,l+1} = I$ for some l with $1 \le l \le n-1$. Since

$$\begin{aligned} T_lP_{l,l+1} + S_{l,l+1} &= T_lP_{l,l+1} + S_{l,l+1}P_{l+1,l+1} = U_{l,l+1} \\ &= V_{l,l+1} = P_{ll}S_{l,l+1} + P_{l,l+1}T_{l+1} = P_{l,l+1}T_{l+1}, \end{aligned}$$

it follows that

$$S_{l,l+1} = T_l(-P_{l,l+1}) - (-P_{l,l+1})T_{l+1} = \tau_{T_l,T_{l+1}}(-P_{l,l+1}) \in \operatorname{ran} \tau_{T_l,T_{l+1}},$$

which is a contradiction. If $P_{ll} = I$ and $P_{l+1,l+1} = 0$ for some $1 \le l \le n-1$, similarly we reach a contradiction. Thus $P_{ii} = 0$ for all $1 \le i \le n$ or $P_{ii} = I$ for all $1 \le i \le n$.

If $P_{ii} = 0$ for all $1 \le i \le n$, then

$$P_{i,i+1} = W_{i,i+1} = P_{ii}P_{i,i+1} + P_{i,i+1}P_{i+1,i+1} = 0$$

and

$$P_{i,i+2} = W_{i,i+2} = P_{ii}P_{i,i+2} + P_{i,i+1}P_{i+1,i+2} + P_{i,i+2}P_{i+2,i+2} = 0.$$

In the same way, we can conclude that $P_{ij} = 0$ for all $1 \le i < j \le n$. Hence $P = 0$. If $P_{ii} = I$ for all $1 \le i \le n$, in the same way we can prove $I - P = 0$. Thus $P = I$. Therefore T is strongly irreducible by Definition 1.2. □

Corollary 3.7. *Suppose $T \in \mathcal{FB}_2(\Omega)(X)$ and, under the direct sum decomposition $X = X_1 \oplus X_2$, express T as*

$$T = \begin{pmatrix} T_1 & S \\ 0 & T_2 \end{pmatrix},$$

where $T_1 \in \mathcal{B}_1(\Omega)(X_1)$, $T_2 \in \mathcal{B}_1(\Omega)(X_2)$ and $0 \ne S \in B(X_2, X_1)$ with $T_1S = ST_2$. Then T is strongly irreducible if and only if $S \notin \operatorname{ran} \tau_{T_1,T_2}$.

Proof. If $S \notin \operatorname{ran} \tau_{T_1,T_2}$, then T is strongly irreducible by Theorem 3.6.

Conversely, if $S \in \operatorname{ran} \tau_{T_1,T_2}$, then $S = \tau_{T_1,T_2}(P_{12}) = T_1P_{12} - P_{12}T_2$ for some $P_{12} \in B(X_2, X_1)$. Let

$$P = \begin{pmatrix} I & P_{12} \\ 0 & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} TP &= \begin{pmatrix} T_1 & S \\ 0 & T_2 \end{pmatrix}\begin{pmatrix} I & P_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T_1 & T_1P_{12} \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} T_1 & S + P_{12}T_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & P_{12} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} T_1 & S \\ 0 & T_2 \end{pmatrix} = PT \end{aligned}$$

and

$$P^2 = \begin{pmatrix} I & P_{12} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I & P_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & P_{12} \\ 0 & 0 \end{pmatrix} = P.$$

Thus T is not strongly irreducible. □

Corollary 3.8. *Let $T \in \mathcal{B}_1(\Omega)(X)$ and let*

$$U = \begin{pmatrix} T & I & \cdots & S_{1n} \\ & T & \ddots & \vdots \\ & & \ddots & I \\ 0 & & & T \end{pmatrix},$$

where $S_{ij} \in B(X)$ for $1 \le i, j \le n$ with $j \ge i + 2$. Then U is a strongly irreducible operator on X^n.

Proof. By Definition 1.3, $U \in \mathcal{FB}_n(\Omega)(X^n)$. If $I \in \operatorname{ran} \tau_T$, because $I \in \ker \tau_T$, it follows that I is quasinilpotent by Lemma 3.2, which is a contradiction. Thus $I \notin \operatorname{ran} \tau_T$. Therefore U is strongly irreducible by Theorem 3.6. □

Theorem 3.9. *Let $T \in \mathcal{FB}_n(\Omega)(X)$ be expressed in upper block triangular form as in Definition 1.3. If $S_{i,i+1}$ is invertible for $1 \le i \le n-1$, then T is strongly irreducible.*

Proof. For $1 \le i \le n-1$, let $R_i \in B(X)$ be the block diagonal operator with $I, \ldots, I, S_{12}S_{23} \cdots S_{i-1,i}, I, \ldots, I$ on its diagonal. Let

$$U = (U_{ij})_{n\times n} = R_{n-1} \cdots R_2R_1TR_1^{-1}R_2^{-1} \cdots R_{n-1}^{-1}.$$

Then U is similar to T and U is a block upper triangular operator. For $1 \le i \le n$,

$$\begin{aligned} U_{ii} &= (S_{12}S_{23} \cdots S_{i-1,i})T_i(S_{12}S_{23} \cdots S_{i-1,i})^{-1} \\ &= S_{12}S_{23} \cdots T_{i-1}S_{i-1,i}(S_{12}S_{23} \cdots S_{i-1,i})^{-1} = \ldots \\ &= T_1S_{12}S_{23} \cdots S_{i-1,i}(S_{12}S_{23} \cdots S_{i-1,i})^{-1} = T_1, \end{aligned}$$

and

$$U_{i,i+1} = (S_{12}S_{23} \cdots S_{i-1,i})S_{i,i+1}(S_{12}S_{23} \cdots S_{i,i+1})^{-1} = I.$$

Thus U satisfies the conditions of Corollary 3.8. Hence U is strongly irreducible. Since strong irreducibility is a similarity invariant, T is strongly irreducible. □

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